

# **Chemical Reactor Analysis and Design Fundamentals**

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*To Cheryl, Melanie and Blake,*

*and*

*Carol, Alice and Barbara,*

*for their love and encouragement.*

## Preface

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Chemical reactors are at the core of the chemical engineering discipline, and chemical reactor analysis and design is one of the distinguishing courses that clearly separates the chemical engineers from the other engineering professionals. Given that chemical reactor analysis and design is a mature and stable topic in the curriculum of chemical engineering, however, it is natural to ask what is the motivation for a new text on this topic.

We offer our motivations here. This book grew out of the combined experience of the two authors teaching this subject to undergraduates for more than 30 years. Given the rapidly changing landscape of scale and type of reactors of interest to practicing chemical engineers (chemical vapor deposition reactors, pharmaceutical fermentors, micro-reactors, as well as traditional catalytic crackers, bulk polymerization reactors, etc.), it seems unwise to emphasize one industrial sector and treat its reactor types in detail. Practicing chemical engineers work in a broad array of industrial sectors, and many will change sectors during their careers. If chemical engineering has any important distinguishing characteristic, it is a set of fundamentals that apply to all scales and all types of reaction and transport processes involving chemical change.

This book is all about reactor fundamentals. Rather than presenting many facts about reactors, we focus on the framework for how to think about reactors — a framework for thinking that enables one, with some experience, to establish any of these facts for oneself, and discover new facts given new situations. All engineering and science textbooks do this to some extent; in this text, we will do it to a rather large extent.

Computations matter in this subject. Reactor fundamentals, like the fundamentals in any subject, are few in number. But the diversity of the consequences of these fundamentals is enormous. Computational approaches provide a powerful and general approach to systematically investigating these consequences without making unrealistic simplifying assumptions. We attempt to exploit the significant advances in computing algorithms, software, and hardware in order to revise and streamline the presentation of reactor fundamentals. We focus on two high-level languages intended for numerical computation, Octave and

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MATLAB, Octave is freely available for a variety of hardware platforms and can be down-loaded from [www.octave.org](http://www.octave.org). MATLAB is commercially available from The MathWorks, Inc., and is becoming a commonly available tool of industrial engineering practice. These languages allow us to focus on essentials and ignore programming details, which is the goal of any "high-level" language.

Students should not feel compelled to recall the detailed information in the figures, but recall only the concepts, principles and main results. Students will have the computational tools to re-create the figures in this or any other textbook on this subject. For example, all calculations required for the figures in this text were performed with Octave. Students should not feel compelled to memorize design equations for reactor types. The goal is to develop sufficient expertise so that students can set up appropriate models from the basic principles for each new problem they encounter. That is the time-tested way to instill confidence that one can analyze a new situation, which we fully expect to be the experience of practicing engineers.

Newly practicing engineers certainly will need to learn the economics of manufacturing their main products, main reactor configurations, detailed energy-recovery schemes, contacting patterns, new catalysts, and reactor monitoring and control systems. Rather than make inadequate efforts to include all of this information as a survey during the reactor design course, we feel this material deserves separate coverage, or can be learned during the early years of engineering practice. We do not regard it as poor preparation that some of this information is not included in university curricula; rather, we regard learning this material on the job as part of the maturation process of the practicing professional chemical engineer.

Finally, we hope the text conveys some of the excitement that chemical engineers feel for chemical reactions and chemical reactors. Understanding the fundamentals prepares you to investigate new situations. Grounding in the fundamentals enables you to explore with confidence and be creative. Enjoy.

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## Acknowledgments

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We benefited greatly from the help and advice of many colleagues during the preparation of this text. Taking Warren Stewart's graduate reactor modeling course greatly influenced the first author's way of thinking about this subject. More recently, Warren also provided many helpful comments on computational issues involving reaction-diffusion, and parameter estimation. Harmon Ray contributed many insightful and helpful comments on a variety of topics, especially on issues of reactor mixing and industrial polymerization reactors. Bob Bird provided many helpful comments on presenting the material related to transport phenomena. He also provided detailed feedback on a review copy of the text.

Marty Feinberg read an early version of the notes, and provided detailed feedback on many issues including reaction rates and presenting computational material. Yannis Kevrekidis used a preliminary version of the notes as part of a graduate reactor modeling course at Princeton. His feedback and encouragement are very much appreciated. John Falconer also provided many helpful comments.

Tom Edgar provided much-needed departmental support and encouragement when the authors began writing. Roger Bonnacaze, Bud-die Mullins and C. D. Rao used early versions of the notes and Octave software in teaching the undergraduate course at The University of Texas. Their feedback was very helpful.

Several graduate students contributed to the software, which is an essential part of this text. Ken Muske developed some examples for the chapter on mixing. John Campbell worked on programs for managing thermochemical data, Sankash Venkatasankar contributed programs on parameter estimation and reaction-diffusion, and Dominik Nagel contributed programs to calculate potential-energy surfaces. Chris Rao provided helpful feedback on mixing issues, and Matt Tenny and Eric Haseltine provided helpful input on the cover design. Eric also helped prepare the solution manual. We have benefited from the feedback of many classes of undergraduates both at the University of Wisconsin and The University of Texas. We appreciate the enthusiasm with which they received some very early and incomplete drafts of the notes.

Our publisher deserves special recognition for providing enough



flexibility so that we could realize our original vision, while still insisting that we actually finish the project.

Finally, we would like to acknowledge the essential contributions of John Eaton. When the two authors first outlined the goals of this writing project, we envisioned creating in parallel a convenient computational environment. The original goal of the computational environment was to enable students to quickly and conveniently define and solve new reactor design problems as a means to gain deeper understanding and appreciation of chemical reactor fundamentals. Starting from this simple idea, John Eaton created the Octave software project. We could not have written this text in the way we planned without Octave. Every figure in the text requiring calculation was prepared with Octave. John contributed in a myriad of other ways. He designed the typesetting macros and the text version control system. We maintained more than 10 versions of the source files on more than 7 different computing systems at various stages of production. With John's expert guidance, we easily managed this complexity; in fact, the computing tools worked so well, that the computational aspects of the book production were enjoyable.

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# 1

## Setting the Stage

### 1.1 Introduction

The chemical reactor lies at the heart of most chemical processes. The design and operation of the reactor often determine the success or failure of the entire process. In the overall process, feedstocks are delivered to the chemical reactor at the appropriate temperature, pressure, and concentrations of species. The chemical reactor is that essential component in which the feed is converted into the desired products. The chemical reactor is the place in the process where the most value is added: lower-value feeds are converted into higher-value products.

The reactor is normally followed by separation processes that separate the products from the unreacted feed and the reactor byproducts. The modern chemical processing plant, whether processing fine or commodity chemicals, manufacturing pharmaceuticals, refining petroleum, or fabricating microelectronic devices, is a highly integrated operation. In addition to meeting production and purity goals, the process design and operation also are influenced by the sometimes conflicting goals of minimizing energy consumption, and minimizing the amount of feed or product that must be kept in storage.

Many aspects of reactor analysis and design are treated in this text, including predicting performance, specifying initial and feed conditions, sizing the reactor, specifying operating conditions, and selecting the reactor type. To accomplish these goals, we construct and illustrate a set of reactor analysis and design principles. As we will see, the principles are small in number, but the consequences of the principles are rich, diverse and complex. These principles can be applied at many size scales to many different types of chemically reacting systems. While the reactor of interest is often a man-made vessel, sometimes the reactor is a living organism. We will illustrate throughout the text, using

many different examples, the diversity of systems and issues addressed by chemical reactor analysis and design.

**Batch, continuous-stirred-tank, and plug-flow reactors.** We model many chemical reactors using three main reactor archetypes: batch, continuous-stirred-tank, and plug-flow reactors. By virtue of their design and the typical operating conditions, many complex chemical reactors can be well approximated by these three simple reactor types. The material and energy balances of these three reactors are sets of first-order, nonlinear ordinary differential equations (ODEs) or nonlinear algebraic equations, or in some situations, differential algebraic equations (DAEs). The great simplification that has become standard practice in introductory reactor design texts is to neglect the momentum balance and a careful treatment of the fluid flow pattern within the reactor.

Concentration, temperature and pressure are therefore the usual dependent variables that are solved as functions of time or distance along the reactor as the independent variable. Sometimes the reactor model is quite simple, such as describing a single reaction in an isothermal reactor, and a single ODE or algebraic equation describes a single species concentration or extent of a single reaction. More often the design involves many reactions and nonisothermal operation, and coupled sets of ODEs or algebraic equations are needed to describe the temperature, pressure and species concentrations. Regardless of the complexity, the design problem is approached in the same manner using the same set of principles. In all but the simplest cases, numerical methods are required to solve the models. Fortunately, high-level programming languages are readily available and easily can be used to solve the complex models. We make extensive use of numerical methods and high-level programming languages in this text to solve complex models.

## 1.2 Some Classifications and Terminology

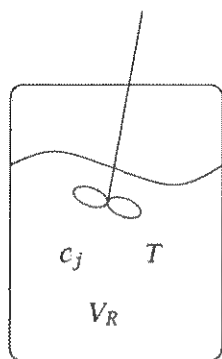
**Ideal mixing and plug flow.** The batch, continuous-stirred-tank, and plug-flow reactors are defined by certain idealized assumptions on the fluid flow. The batch and continuous-stirred-tank reactors are assumed to be ideally well mixed, which means that the temperature, pressure and species concentrations are independent of spatial position within the reactor. The plug-flow reactor describes a special type of flow in a tube in which the fluid is well mixed in the radial direction and varies

only in the axial or tube length direction. Plug flow often describes well the limit of fully developed turbulent flow, i.e., flow in the limit of large Reynolds number. If the flow in a tubular reactor is not turbulent (Reynolds Number less than  $10^3$ – $10^4$ ), then the flow may not be well modeled as ideal plug flow, and other models are needed to describe the reactor.

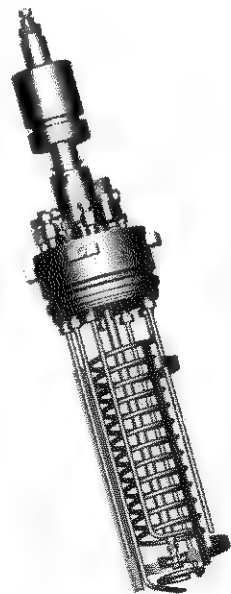
**Homogeneous and heterogeneous reactions.** Another important reactor classification pertains to the phase in which the reaction occurs. The reactants and products of homogeneous chemical reactions are in a single phase. Examples include: steam cracking of ethane to ethylene, and the photochemical reactions of chlorocarbons in the troposphere that lead to ozone destruction for gas-phase reactions; enzymatic isomerization of glucose to fructose, and esterification of an acid and alcohol to produce an ester for liquid-phase reactions; and, the fusing of limestone and charcoal to produce calcium carbide, and the interfacial reaction between strontium oxide and silicon dioxide to form strontium silicate for solid-phase reactions. Sometimes the reactants and products are transported through the reactor in one phase, but the reaction occurs in a different phase, often a solid phase. The heterogeneously catalyzed reactions such as zeolite-catalyzed cracking of high-boiling crude oil and iron-catalyzed ammonia synthesis from hydrogen and nitrogen involve two phases. Sometimes three phases are present simultaneously during the reaction, such as synthesis gas ( $\text{CO} + \text{H}_2$ ) reactions over iron-based catalysts that are suspended in a high molecular weight alkane to help moderate the reaction temperature and dissolve the reaction product. Multiphase reactions need not involve heterogeneous catalysts. The reaction of liquid *p*-xylene and gaseous  $\text{O}_2$  to produce liquid terephthalic acid, occurs via free-radical intermediates in the liquid phase of the reactor. Another two-phase, noncatalytic reaction is the low-pressure epitaxial growth of Si(100) films from gas-phase disilane ( $\text{Si}_2\text{H}_6$ ).

**Batch, semi-batch and continuous operation.** The operation of the reactor can be classified as batch, semi-batch and continuous. In batch operation, the reactor is charged with reactants, the reaction takes place, and after some processing time the contents of the reactor are removed as product. Batch reactors, depicted in Figure 1.1, are often used for liquid-phase reactions, and the manufacture of low-volume, high value-added products, such as specialty fine chemicals, pharmaceuticals and fermentation products. Batch reactors also are used in





**Figure 1.1:** Schematic diagram of a batch reactor.

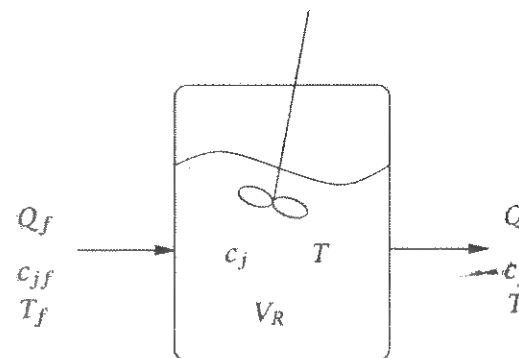


**Figure 1.2:** Expanded view of the internals of a batch reactor. Courtesy of Autoclave Engineers, Division of Snap-tite, Inc.

situations where it is not practical to implement a continuous process. The great flexibility of batch processing allows the reactor to be used for the manufacture of many different products. Figure 1.2 provides an expanded view of the internals used in a batch reactor showing the cooling coils, tubes for adding (sparge tube) and removing (sample tube) fluid, and baffles to ensure complete mixing. This unit is placed in a cylindrical vessel that comprises the exterior of the batch reactor.

The batch cycle begins by charging the reactants to the vessel, and often heating the reactants to the reaction temperature. The cycle often ends by bringing the contents to a discharge temperature, emptying the vessel, and cleaning the vessel before the next charge of reactants. Product may or may not form during these preparation steps and they often involve manual labor, so manufacturing costs can be considerably higher than a corresponding continuous process.

The semi-batch process is similar to the batch process except feed addition occurs during the batch cycle. Products may also be removed during the semi-batch process. The addition/removal policy allows one to control the reaction rate or heat release during reaction. The semi-batch reactor also may provide more complete



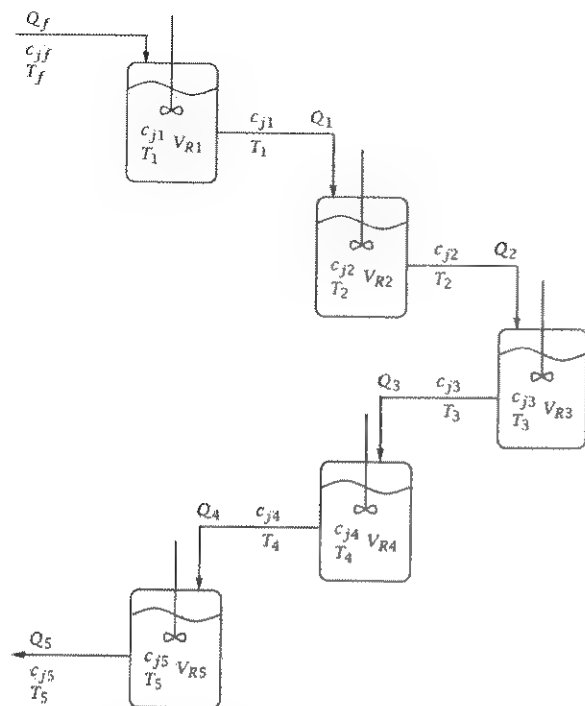
**Figure 1.3:** Schematic diagram of a CSTR; the effluent composition of the CSTR is identical to the conditions that exist in the reactor.

use of the reactor volume in reactions such as polymerizations that convert lower-density reactants to higher-density products during the course of the reaction.

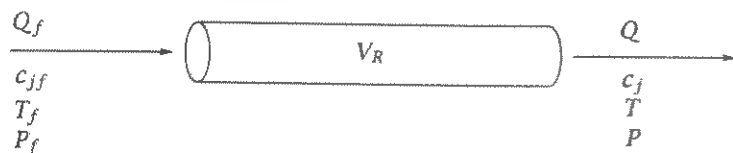
The continuous-stirred-tank reactor (CSTR) is shown in Figure 1.3. Reactants and products flow into and out of the reactor continuously, and the contents of the reactor are assumed to be well mixed. The well-mixed assumption can be realized more easily for liquids than gases, so CSTRs are often used for liquid-phase reactions. The fluid composition and temperature undergo a step change when passing from the feed stream into the interior of the reactor; the composition and temperature of the effluent stream are identical to those of the reactor.

The CSTR is used extensively in situations where intense agitation is required, such as the addition of a gaseous reactant to a liquid by transfer between the bubbles and the continuous liquid, and the suspension of a solid or second liquid within a continuous liquid phase. Polymerization reactions are sometimes conducted in CSTRs. It is common to employ a cascade or series of CSTRs in which the effluent from the first reactor is used as feed to the second and so forth down the cascade (Figure 1.4). The cascade permits one to realize high conversion of reactant, while minimizing total reactor volume.

The plug-flow reactor (PFR) is a constant cross-section, tubular reactor as depicted in Figure 1.5. Under turbulent flow conditions, the velocity profile becomes plug-like, which greatly simplifies the material and energy balances. The velocity, composition and temperature



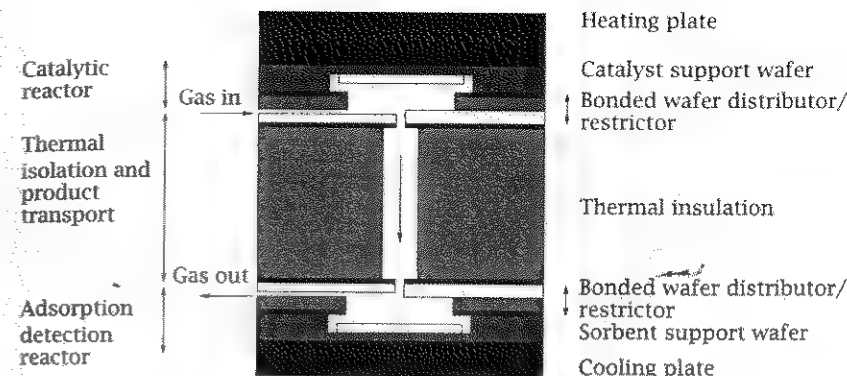
**Figure 1.4:** Schematic diagram of five CSTRs in series; the effluent of each reactor becomes the feed to the next.



**Figure 1.5:** Schematic diagram of a plug-flow reactor.

are functions of only the axial position. In this text, we are usually interested in only the steady-state profile in the tube and neglect the dynamics.

Plug-flow reactors are used for gas-phase and liquid-phase reactions. If the PFR is filled with a porous catalyst and the fluid flowing in the void space is turbulent, the reactor is referred to as a fixed-bed reactor. We will see that the isothermal PFR usually leads to higher con-



**Figure 1.6:** Cross-sectional view of a  $1 \mu\text{l}$  combinatorial screening reactor. Copyright 2001, Symyx Technologies, Inc., used with permission.<sup>1</sup>

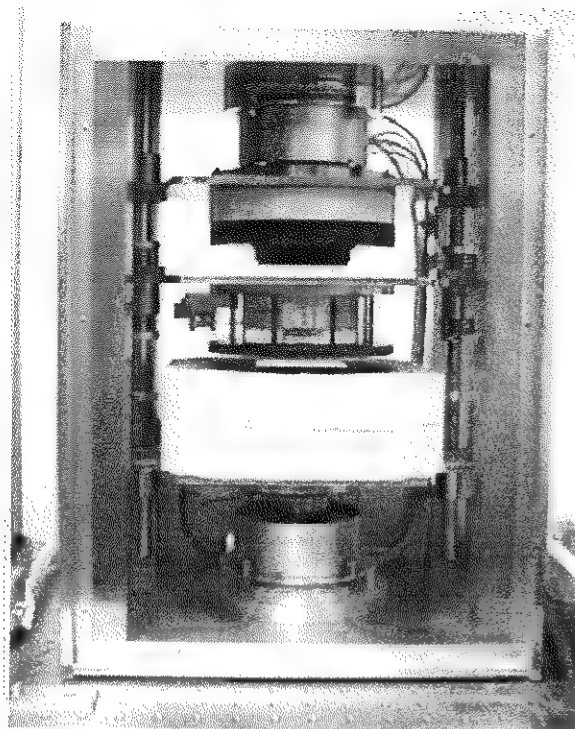
version of reactant per unit volume than the CSTR, i.e., there is more efficient use of volume in a PFR. For this reason, PFRs are employed in situations that require high capacity and high conversion. PFRs also are used in situations involving highly exothermic or endothermic reactions; a bundle of small-diameter tubes can be placed in a furnace for endothermic reactions, or surrounded by a high-temperature boiling fluid for exothermic reactions. PFRs are the reactor of choice for gas-phase reactions due to the problems of mixing gases in a CSTR.

### 1.3 Scale

The size or volume of chemical reactors varies widely. Reactor volumes can range from hundreds of nanoliters for combinatorial, lab-on-a-chip reactor systems, to several hundred thousand liters for certain petroleum refining operations. In the combinatorial reactors, one is interested in determining if a reaction proceeds and in minimizing the scale of the experiment so many combinations or conditions can be screened rapidly. Figure 1.6 presents a schematic view of  $1 \mu\text{l}$  test reactors that are used for combinatorial screening of heterogeneous catalysts.

Figure 1.7 presents an assembled view of these reactors. As many as 256 different catalyst formulations can be deposited into shallow

<sup>1</sup>All rights reserved, U.S. Patent No. 5,985,356, 6,004,617. Additional U.S. and foreign patents pending.

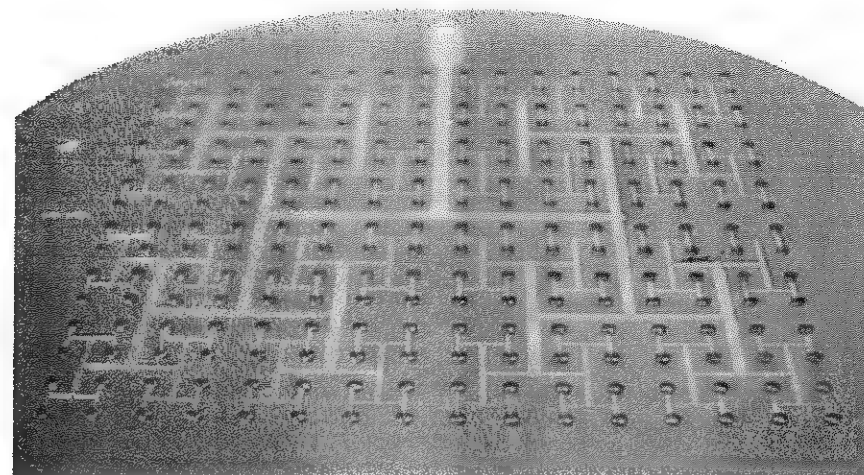


**Figure 1.7:** Assembled view of 256 1- $\mu$ L reactors built into a 3-in by 3-in support wafer. Copyright 2001, Symyx Technologies, Inc., used with permission.<sup>1</sup>

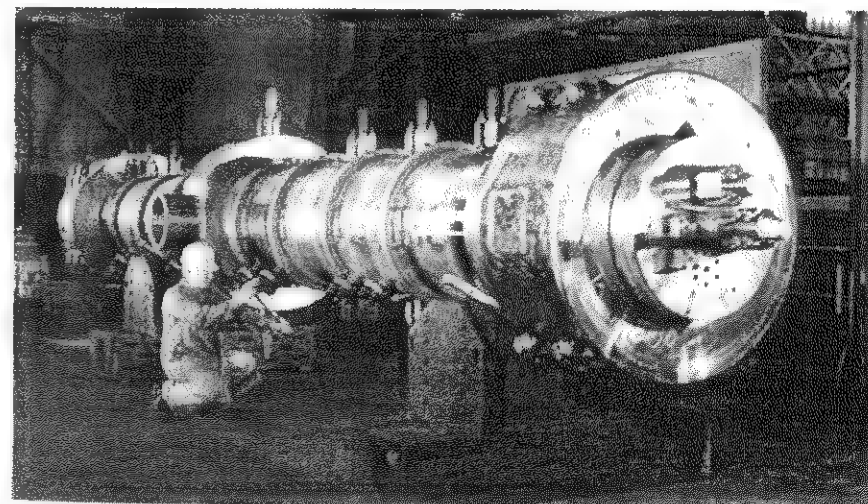
wells in a 3-in by 3-in support wafer. By changing the gas flowrates, the residence time can be varied from 10 ms to 1 s. Uniform flow distribution to each of the reactors is but one of the design challenges for this small-scale, massively parallel device. The B-tree flow distribution network to accomplish this is shown in Figure 1.8.

At the opposite end of the spectrum, the size of petroleum refining reactors is dictated by a desire to maximize production given a fixed reaction rate. These reactors are therefore large. For example, Figure 1.9 displays a 950-L, low-density polyethylene polymerization reactor. This process is discussed further in Section 1.4.3.

The time required for chemical reaction also varies widely. In the batch reactor, the batch time determines the production rate for the reactor. For flow systems we define residence time, which is the reactor



**Figure 1.8:** Top view of a combinatorial reactor gas-distribution network. Copyright 2001, Symyx Technologies, Inc., used with permission.<sup>1</sup>



**Figure 1.9:** Polyethylene reactor; this 16-in inner-diameter reactor is designed to operate at 35,000 psi and 600 F; in operation, this reactor is in a vertical configuration. Courtesy of Autoclave Engineers, Division of Snap-tite, Inc.

volume divided by the feed volumetric flowrate; this quantity has the units of time, and can be thought of as the time required to displace the equivalent of one reactor volume. Batch times and residence times can be as short as milliseconds for high-temperature, gas-phase reactions, such as ammonia oxidation and HCN synthesis in flow reactors, or as long as days for liquid-phase fermentation reactions in batch reactors.

The rate of chemical reaction has units of moles/time-volume and is a function of temperature and composition. If the rate is high, as we would anticipate for a high-temperature oxidation or pyrolysis reaction, the time needed for a desired production rate is small. Similarly, if the rate is low, a long time is needed. The reactor volume is determined by the reaction rate and amount of product to be manufactured.

## 1.4 Some Examples

We live in a chemical world and most of the items you use in daily life result from chemical processes, including the ink on this page and the paper itself. Chances are one or more of the articles of clothing you are wearing are made from synthetic fibers. We are surrounded by and regularly use items that are made from plastics and polymers. The automobiles we drive and the fuels we use to power them depend on products made in chemical reactors. Electronic devices and computer chips require a myriad of chemical reaction steps in their manufacture. Chemical reaction engineering is the core discipline that designs, analyzes and creates the processes to convert natural resources into other intermediate chemicals and final products.

### 1.4.1 Chemical Vapor Deposition of Silicon-Germanium Alloy Films

The demand for faster microelectronic devices with higher reliability, lower power requirements, and lower cost is driving the current silicon-based devices close to their physical limits. One solution that permits the traditional silicon-based circuit design to be retained is to use silicon-germanium alloys in place of silicon in the active region of the device. Silicon-germanium alloys have higher mobilities than silicon and a lower bandgap enabling faster speeds and lower power. Devices such as heterojunction bipolar transistors and field-effect transistors are fabricated by a series of steps that involve masking the semiconductor wafer with a coating, selectively removing regions of the mask with lithography, implanting dopant ions such as boron or phosphorous, and adding layers of metal and dielectrics to build up the

device from the wafer surface [18, 7, 11]. For silicon-germanium alloy-based devices, a thin film (tens of nanometers to microns) is grown onto Si(001) wafers. Typically the reactants are hydride gases, such as disilane ( $\text{Si}_2\text{H}_6$ ) and germane ( $\text{GeH}_4$ ), at pressures on the order of 0.1 to several torr in a process known as chemical vapor deposition (CVD). Device speed and performance are related to the amount of Ge in the alloy, and control of the film composition is critical.

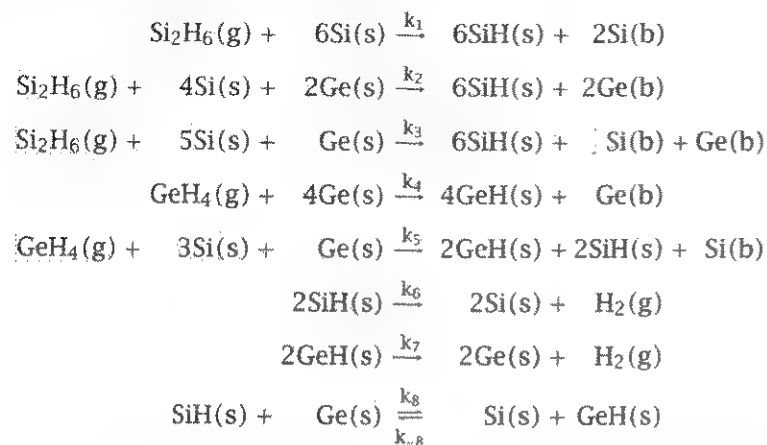
As microelectronics manufacturing moves to ever larger wafers (currently 300-mm diameter wafers are routine), single-wafer reactors are used in the manufacturing processing steps (Figure 1.10). In a single-wafer reactor, the wafer is supported in the horizontal position, a gas distribution system is used to ensure uniform flow of reactants over the wafer surface, and the wafer is heated radiatively to maintain the entire cross section at a constant temperature. When the pressure is low enough, the diffusion lengths of the gas-phase components are long enough that one can assume the gas phase to be well mixed, uniform and independent of position. This means the gas phase can be modeled as a CSTR in which gas enters and leaves the reactor in the flow streams and undergoes reaction at the surface of the wafer.

Much is known about the chemistry that takes place during film growth [5, 19, 15, 2, 12, 13]. This information can be compactly summarized in a fairly small, reduced-order kinetic model that accounts for surface and gas-phase components, and can be used to predict the



**Figure 1.10:** Top view of a single-wafer CVD reactor; single-wafer processing generally employs cluster tools that permit a wafer to be shuttled between the different reaction/process chambers without exposing it to contamination or ambient conditions; the top view of one reactor shows the pins on which a wafer rests during the CVD reaction. Courtesy of Applied Materials.

film growth rate [9]. One such kinetic model consists of eight reactions among nine species. The reactions are



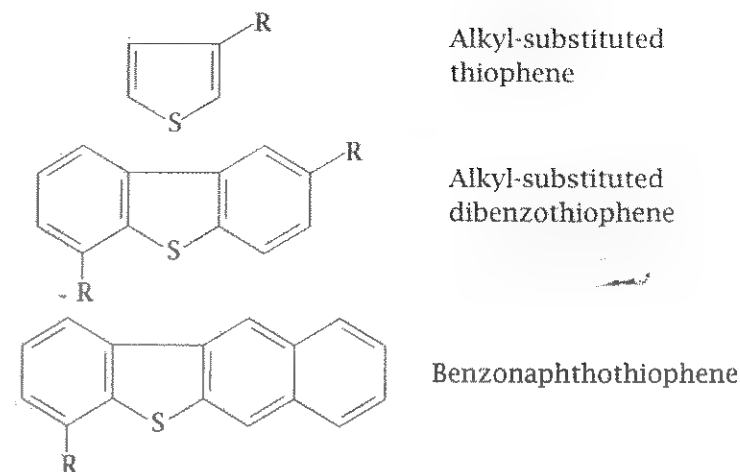
where the symbols (s), (b) and (g) refer to the wafer surface, the wafer bulk, and the gas phase, respectively. The production rates of the nine species follow from the stoichiometry of the eight reactions and the reaction rates as we will see in Chapter 2, and are given by

$$\begin{aligned}
 R_{\text{Si}_2\text{H}_6} &= -r_1 - r_2 - r_3 \\
 R_{\text{GeH}_4} &= -r_4 - r_5 \\
 R_{\text{Si}(\text{s})} &= -6r_1 - 4r_2 - 5r_3 - 3r_5 + 2r_6 + r_8 \\
 R_{\text{Ge}(\text{s})} &= -2r_2 - r_3 - 4r_4 - r_5 + 2r_7 - r_8 \\
 R_{\text{SiH}(\text{s})} &= 6r_1 + 6r_2 + 6r_3 + 2r_5 - 2r_6 - r_8 \\
 R_{\text{GeH}(\text{s})} &= 4r_4 + 2r_5 - 2r_7 + r_8 \\
 R_{\text{Si}(\text{b})} &= 2r_1 + r_3 + r_5 \\
 R_{\text{Ge}(\text{b})} &= 2r_2 + r_3 + r_4 \\
 R_{\text{H}_2} &= r_6 + r_7
 \end{aligned}$$

The production rates enable us to follow the changing composition of the gas phase, the surface, the film growth rate, and the Si/Ge film alloy composition.

#### 1.4.2 Hydrodesulfurization

Crude oil contains a number of organosulfur compounds (RS), including thiols, sulfides, thiophenes and alkyl-substituted thiophenes [8],



**Figure 1.11:** Representative organosulfur compounds illustrating the aromatic character of the molecules; when the alkyl substitution position is near the S atom, it is difficult to desulfurize the molecule.

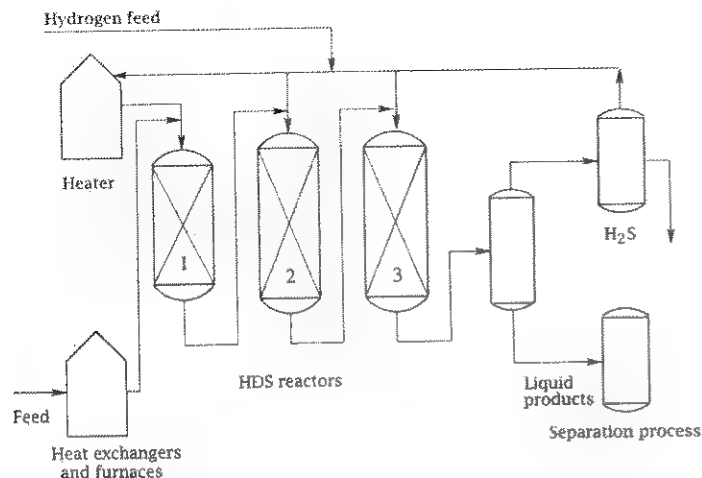
some of which are shown in Figure 1.11. The sulfur must be removed from the petroleum feeds to prevent it from poisoning precious metal catalysts that are used in many refinery processes and to meet the restrictions for sulfur content in fuels. The total sulfur content of crude oil can easily be several percent and it must be reduced to several parts per million (ppm). This sulfur is removed by treating the petroleum feed with hydrogen at high pressures in a fixed-bed or trickle-bed catalytic reactor; the process is referred to as hydrodesulfurization (HDS) [3]. During HDS the carbon-sulfur bonds are broken and the hydrogen replaces the sulfur and  $\text{H}_2\text{S}$  is formed.



The fixed-bed reactors typically operate in the temperature range 525–750 K, and hydrogen pressure range 35–100 atm.

Figure 1.12 presents a simplified process diagram. The feed is heated and mixed with hydrogen in the inlet to a fixed-bed catalytic reactor. HDS is an exothermic reaction and if the reactor temperature becomes too high, undesired side reactions occur, such as the hydrogenation of the unsaturated bonds and hydrogenolysis of the C—C bonds. Multiple beds operating in series and at different temperatures, with inter-





**Figure 1.12:** Simplified hydrodesulfurization process diagram; three separate fixed-bed catalytic reactors in series; reactor 1 is used as a guard bed in which the metals are removed to not poison the HDS catalyst; reactors 2 and 3 permit control over temperature and sulfur removal. Adapted from McCulloch [8].

bed cooling, is one way to control the temperature and to ensure the complete removal of the sulfur. Separate reactors in series permit the reactor sequence to be changed as catalyst activity degrades with time. Figures 1.13 and 1.14 present two views of a multi-bed fixed-bed reactor in which the beds are built into one vessel. Within each section containing catalyst, the reactor is modeled as a fixed bed. After cooling the outflow of one of the beds, the reaction is allowed to proceed in the next section. Multi-bed reactors are generally employed with exothermic reactions, and the fixed-bed sections are operated adiabatically. In this way, the reactor temperature can be controlled by the heat removal. These concepts are discussed in Chapter 6. Figure 1.14 shows an expanded view of the internal construction of a fixed-bed reactor; trays to support the catalyst particles and distribute the flows uniformly across the reactor cross section are shown. Commercial HDS reactors such as shown in Figure 1.14 can hold up to 60,000 kg of catalyst.

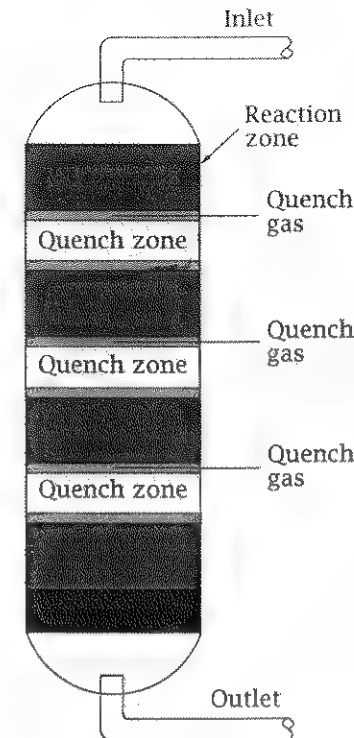
Optimal reactor operation and design are two methods to maximize sulfur removal. In addition, considerable research effort is focused on finding new catalysts to improve activity, selectivity and durability,

and to adapt to the ever more stringent emission standards [10]. The catalytic reaction chemistry is quite complex. As can be seen from Figure 1.11, the organosulfur molecules are highly unsaturated, and in addition to hydrogenation of the C—S bonds, one wants to avoid saturation of all the C=C double bonds. The catalysts have different types of sites, which complicates the kinetic model [10]. Hydrogen must adsorb dissociatively. The organosulfur molecule must adsorb and orient itself so that the sulfur atom is over an active site, such as an exposed molybdenum atom on a molybdenum disulfide catalyst. The reaction rate is influenced by the relative amounts of reactants (adsorbed hydrogen and organosulfur molecule) and the reactivity of the catalyst. Describing the rate of these kinds of complex reactions on catalyst surfaces is covered in Chapter 5, where we show the rate of the HDS reaction can be expressed by

$$r = \frac{-kc_{H_2}^\alpha c_{RS}^\beta}{(1 + K_{H_2}c_{H_2} + K_{RS}c_{RS})^\gamma}$$

#### 1.4.3 Olefin Polymerization

Polyethylene is a major commodity plastic, with more than 33 billion pounds of the resin produced in the United States in 2000 [1]. Polyethylene encompasses a family of semicrystalline polymers with ethylene as the major building block [6]. The resins are loosely grouped into three classes: low-density polyethylene (LDPE), high-density polyethylene (HDPE), and linear-low-density polyethylene (LLDPE). LDPE is a homopolymer of ethylene with side-chain branching at a frequency



**Figure 1.13:** Cross-sectional view of a commercial HDS reactor containing four fixed-bed reactors with interstage cooling/quenching; quenching is necessary to limit the temperature increase for this exothermic reaction. Adapted from UOP LLC.



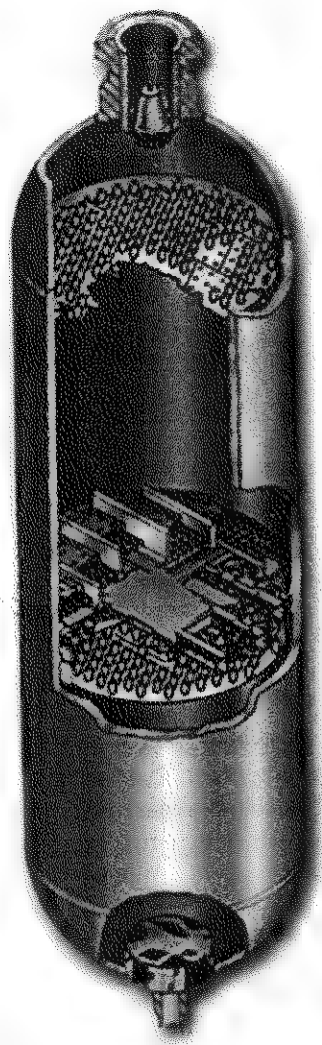


Figure 1.14: Expanded view of a multi-bed HDS reactor showing the gas distribution tray at the top and the catalyst support tray in the middle; these reactors can range in diameter from 1.5–5 m. Courtesy of UOP LLC.

of 2–50 per 1000 carbons in the chain. Its density is in the range 0.915–0.940 g/cm<sup>3</sup>. HDPE can be a homopolymer or, more commonly, a copolymer with butene or hexene, and has a density greater than 0.940 g/cm<sup>3</sup>, and a much lower density of side branches than LDPE, 0.5–10 branches per 1000 carbons in the chain. LLDPE is a copolymer of ethylene and small amounts of  $\alpha$ -olefins. By changing the density, molecular weight, and branching-group frequency and type, the chemical resistance and mechanical properties can be tailored to a variety of uses. Polyethylene is used in films for bags and packaging, ranging from garbage and trash bags to whole blood storage; coatings for paper, metal and electrical wires; containers; and piping and tubing.

Low-density polyethylene is produced in a free-radical initiated, high-pressure process. Free-radical polymerization mechanisms and kinetics are discussed in Chapter 5. The polymerization process begins with an initiation reaction between ethylene and an initiator to form a primary radical. The primary radical undergoes step-wise chain growth (propagation) until it is terminated by a variety of mechanisms that stop chain growth. The LDPE reactors are either stirred autoclaves or tubular reactors. Figure 1.9 displays a commercial LDPE tubular reactor. Typical tubular reactors have diameters of 15–30 in and lengths of 17–25 ft, and operate in the pressure range of 30,000–45,000 psi

and temperature range of 200–300°C. The process temperatures are high enough that the polyethylene is in the melt state, and the pressures are high enough to form a supercritical mixture of ethylene and polyethylene, which acts as the polymerization medium. The polymerization reaction itself is exothermic and the process is operated to prevent excessive temperature increases. The temperature control is accomplished by limiting the amount of ethylene converted within the reactor and/or removing heat through the reactor walls.

High-density polyethylene is produced at lower pressures, from 10–30 atm. HDPE requires a catalyst [22, 17, 6]. Various catalysts have been used including: chromium oxide supported on an inert porous substrate such as silica; a Ziegler catalyst consisting of titanium or vanadium compounds (TiCl<sub>4</sub>, TiCl<sub>3</sub>, VOCl<sub>3</sub>) on a support (silica, MgCl<sub>2</sub>, graphite, carbon black) and an organoaluminum cocatalyst (Al(C<sub>2</sub>H<sub>5</sub>)<sub>3</sub>); and, a metallocene (a complex of zirconium, titanium or hafnium containing cyclopentadienyl rings) and an organoaluminum cocatalyst. HDPE processes can employ a slurry of catalyst, polymer and diluent. One such slurry process involves circulating the reaction mixture at 5–12 m/s around a loop that can range from 0.5–1.0 m in diameter and 200 m in total length. The heat of reaction is removed by a cooling jacket that surrounds the reactor, and the high flowrate ensures turbulent flow within the loop [6]. Ethylene, catalyst and diluent are fed continuously. Polymer particles form and are removed continuously; these particles reside in the reactor loop for 0.5–2.5 hr, and are removed after they settle in a side leg of the loop. Ethylene conversion can reach 95–98%.

Gas-phase polymerization also is used for HDPE in the process represented schematically in Figure 1.15. The cylindrical reactor can be

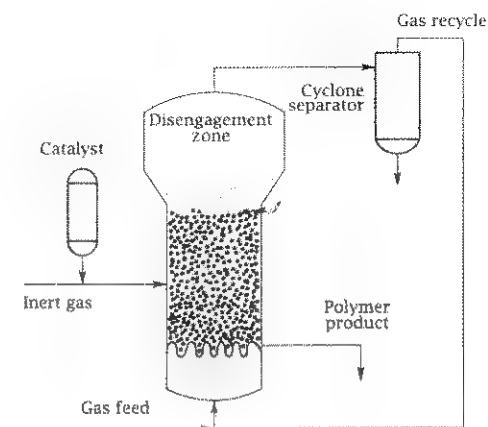


Figure 1.15: Simplified polymerization process. This material is used by permission of John Wiley & Sons, Inc., Copyright © 1996 [6].

25-30 ft in diameter and have a five-to-one length-to-diameter ratio. High linear velocities serve to fluidize the catalyst particles that form; fluidization facilitates removing the heat of reaction. Catalyst is added continuously and polymer particles are removed once they reach a particular size (about 500-1000  $\mu\text{m}$ ). The reactor diameter is larger at the top, which lowers the linear velocity and acts to disengage the polymer particles from the unreacted gases. These same reactors are used for LLDPE. The choice of catalyst and the presence of inert gases and comonomers are used to regulate the polymer resin properties and molecular weight. The reactor typically operates at 220-370 psig and 160-205°F.

#### 1.4.4 Hepatitis B Virus Modeling

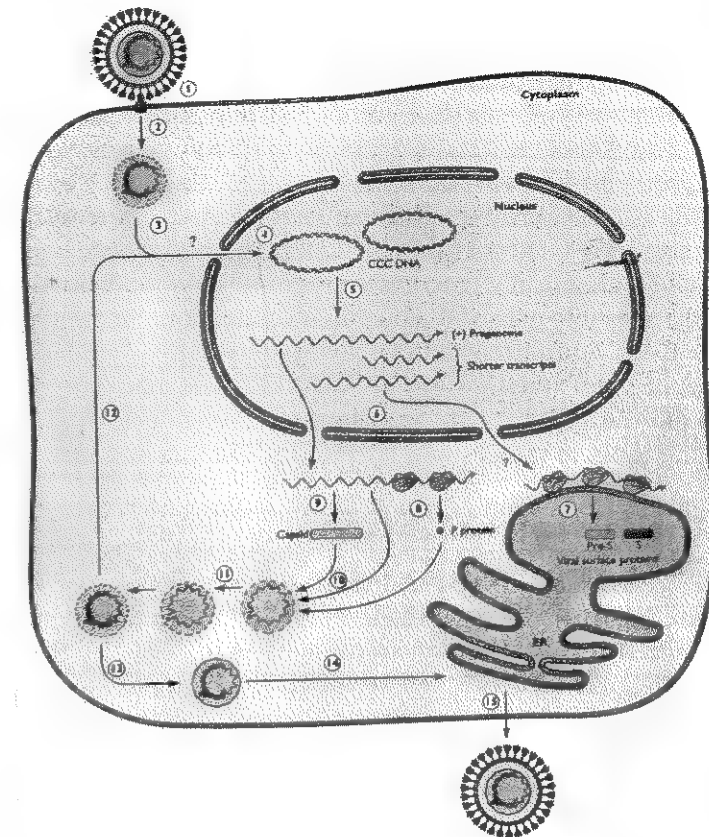
As a final example, we wish to display the wide scope of chemical reaction modeling principles. Consider Figure 1.16, which shows some of the biochemical reaction events that occur in a single cell during the reproduction cycle of the hepatitis B virus [4, p.767]. The understanding and modeling of these biochemical events is an area of current research activity [14, 20, 21, 16].

The following is a simplified but useful model of part of this reproductive system



These reactions correspond to the following steps in Figure 1.16:

1. Reaction 1.1 accounts for Steps 5, 9-11.
2. Reaction 1.2 accounts for Step 12.
3. Reaction 1.3 accounts for Steps 5-7.
4. Reactions 1.4 and 1.5 are not present in Figure 1.16, but may prove useful to explain potential loss of active cccDNA.
5. Reaction 1.6 accounts for Steps 13-15.



**Figure 1.16:** The chemical events comprising the reproduction cycle of the hepatitis B virus. Courtesy of ASM Press [4].

When we change to this context, the “chemical reactor” of interest becomes the living cell or, if we also model the cell population, the human liver. The chemical reaction modeling principles remain valid. In Chapter 4 we use this simple model to make quantitative predictions about the evolution of the viral species concentrations. We also show how to model systems that have small concentrations of species, down to less than a few hundred molecules. In Chapter 9 we explore estimating the rate constants that appear in this virus model given the kinds of laboratory measurements that are available.

Tailoring a model to make successful quantitative predictions of a system of interest is still more of an art than a science. We should not underestimate the complexity of some of the systems of interest to chemical engineers. Because we cannot include all the details, our models are always incomplete, and it is possible to make naive use of modeling approaches, and produce models with little connection to reality and little predictive value. This caution is perhaps especially true for biological systems. On the other hand, if we wish to increase our understanding of a chemically reacting system, skillful model building is often an indispensable part of the overall investigation. Simple models often can explain complex system behavior, especially when feedback mechanisms or autocatalytic steps are involved. The main goal of this text is to build the skill set with which chemical engineers apply reaction modeling tools to understand chemically reacting systems.

## 1.5 An Overview of the Text

**Chapter 2.** The remaining text is divided into eight chapters. We begin in Chapter 2 by discussing stoichiometry or the quantitative relationship between the different chemical species undergoing chemical reaction. We define chemical reaction rate and species production rate, and develop the accounting system for tracking the change in the reaction extent and the species concentration. Since most processes involve multiple chemical reactions, we make free use of matrices and linear algebra to summarize compactly the reaction stoichiometry.

**Chapter 3.** Next, in Chapter 3, we briefly review the important facts concerning the equilibrium state of a system undergoing chemical reaction. Most chemical processes do not reach equilibrium, but knowledge of the equilibrium state of the system allows one to define limits of reactor performance and identify operating conditions to realize desired production rates. The conditions for equilibrium are developed using the Gibbs energy and the chemical potential or species activity. We also briefly review phase equilibrium so that we are prepared for multi-phase reactions. The condition for chemical and phase equilibrium is generally stated as the minimization of an appropriate energy function or maximization of entropy. The use of numerical optimization methods is illustrated for solving complex reaction equilibrium problems involving many reactions.

**Chapter 4.** In Chapter 4 we develop the material balances for the three reactor types: batch (and semi-batch), continuous-stirred-tank,

and plug-flow reactors. We consider homogeneous, single-phase, isothermal reactions at this point. We derive and illustrate the use of the basic material balances for single and multiple reactions using molar concentration or molar flow as the only dependent variable. We also consider the volume change upon reaction and show how to employ the equation of state to complement the species material balances in situations in which the fluid density is not constant. We also introduce the use of stochastic simulation to simulate reaction kinetics in well-mixed reactors.

**Chapter 5.** We feel it works best to introduce the chemical reactor as soon as possible in Chapter 4, so we delay a comprehensive study of reaction rates and reaction rate expressions until Chapter 5. In Chapter 5 we provide a simple, theoretical framework for predicting the rate of a chemical reaction and for relating detailed statements of the reaction chemistry to reaction rates. We treat both homogeneous and heterogeneous reactions. Because chemical reactions occur at extremely different rates, and species are present in extremely different concentrations, we can often reduce the complexity of the reaction mechanism without changing the main features of the model. We develop the two main procedures for reducing the reaction mechanism complexity: the reaction equilibrium assumption, and the quasi-steady-state assumption. We illustrate the use of these assumptions in developing kinetic expressions, such as free-radical polymerization kinetics. The chapter concludes with a discussion of mechanisms for reactions occurring at the surfaces of solid catalysts.

**Chapter 6.** In Chapter 6 we develop the energy balance for the chemical reactor. The combined material and energy balances provide us with a rich description of many chemically reacting systems. The coupling of the material and energy balances also provides some surprises. The reactor behavior can become complex and interesting, and we explore some of the complex behavior such as multiple steady states and sustained oscillations. We explore such issues as how to remove or add heat to the system to overcome equilibrium limitations on conversion, and how to integrate a process to use the heat of reaction to preheat the feed entering the reactor.

**Chapter 7.** Chapter 7 considers the industrially important case of heterogeneous reactions taking place in solid catalyst particles. In the catalyst particle, we must consider the combined reaction-diffusion problem to be able to evaluate the temperature and the species con-

centrations. The material and energy balances for reaction-diffusion in catalyst particles result in fairly challenging boundary-value problems, and we discuss numerical methods for solving them. Packing the solid catalyst particles in a tube and passing a fluid stream over them produces the fixed-bed reactor. We show how to couple the fluid balances to the catalyst particle balances in order to predict the overall fixed-bed reactor behavior.

**Chapter 8.** The tradition in introductory reactor design courses is to neglect a careful treatment of the fluid flow and use the simplified, ideal reactors for modeling. In Chapter 8 we explore what to do when the reactor flow pattern is not well represented by these ideal mixing assumptions. We introduce the reactor residence time distribution and describe the general issues of mixing in chemical reactors. We describe the limitations of these approximate mixing models. In some cases, accurate modeling requires one to solve for the complete velocity profile in the reactor. Although that topic is beyond the scope of this text, computational fluid dynamics software is evolving to the point that this approach is becoming tractable for many problems of interest.

**Chapter 9.** The fundamental reactor modeling principles covered in Chapters 2–8 provide the framework in which we think about chemical reactors. We understand which phenomena cause which observed reactor behaviors, and which design variables should be changed if we wish to alter the reactor performance. But when we want to make *quantitative* predictions of reactor performance, we require values for the model parameters. It is a simple fact that most of the parameters needed for the chemistries and reactor configurations of interest are not available in the literature. To make these models useful in standard industrial practice, therefore, we must be able to conveniently determine or estimate these parameters from experimental data collected on the system of interest. Chapter 9 covers this important topic of parameter estimation, which is not usually addressed in a systematic manner in introductory treatments of reactor analysis and design.

**Appendix A: Computational Methods.** Finally, in Appendix A we summarize the numerical methods that have been necessary to solve the reactor models presented in the text. Linear algebra and matrices are introduced to handle reaction stoichiometry involving multiple reactions. Methods for solving ordinary differential equations and differential-algebraic equations are the real workhorses of reaction engineering; they are required in solving the material and energy balances.

We also make extensive use of methods to compute the sensitivities of solutions of differential equations to the parameters appearing in the model. These sensitivities are needed for two purposes. First, they help in the problem of estimating model parameters from data. Second, they provide one means for solving the challenging boundary-value problems that arise with simultaneous reaction and diffusion in catalyst pellets. Optimization methods are used extensively in science and engineering. In this text, we used optimization for three main purposes: finding the equilibrium state for complex situations involving multiple reactions and multiple phases; estimating model parameters from data; and solving reactor design problems.

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## 2

# The Stoichiometry of Reactions

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## 2.1 Introduction

Stoichiometry is defined as the determination of the proportions in which chemical elements combine or are produced and the weight relations in any chemical reaction.<sup>1</sup> In this chapter we explore and develop these quantitative relations between the different chemical species undergoing chemical reaction.

The next section establishes the accounting procedure for tracking chemical change and introduces the stoichiometric matrix. Section 2.3 introduces the concept of linearly independent reactions and discusses the implications of mass conservation on the stoichiometric matrix. Section 2.4 defines the rates of reactions and rates of production of chemical species due to the reactions. Section 2.5 explores the issues involved in calculating production rates given reaction rates and vice versa. We also formulate and solve the least-squares problem of extracting best estimates of reaction rates given production rate measurements containing errors.

Section 2.6 provides a summary of the important concepts and relationships that may prove useful as a study guide or quick reference. References for further study are provided at the end of the chapter. Exercises are provided in Section 2.7 for reinforcing the concepts and to further develop one's understanding. The last several exercises introduce new material and show how the stoichiometry fundamentals presented in this chapter lead into other interesting topics. The reactor analysis book by Aris [1] influenced several sections of this chapter.

<sup>1</sup>Webster's New World College Dictionary, fourth edition, 2000.



## 2.2 Chemical Reactions and Stoichiometry

We shall consider three primary examples as a means of illustrating the main concepts in this chapter. The first example is one of the reactions responsible for smog formation in the atmosphere. It consists of a single reaction among three species. The second example is the water gas shift reaction, and it illustrates the case of multiple reactions; it consists of three reactions among six chemical species. The third example illustrates the complexity of common industrial reactions of interest, consisting of 20 reactions among 14 species. These reactions have been proposed to describe a silicon chemical vapor deposition process, which is an important step in the production of microelectronic materials.

For the first example, consider two molecules of nitric oxide and one molecule of oxygen reacting to form two molecules of nitrogen dioxide. The stoichiometry of this reaction is



The convention that we follow is reactants appear on the left-hand side of the chemical reaction symbol,  $=$ , and products appear on the right-hand side. In this example, there is a single chemical reaction and three different chemical species taking part in the reaction, NO, O<sub>2</sub>, and NO<sub>2</sub>.

The second example is known as the water gas shift reaction. The overall stoichiometry of this reaction is

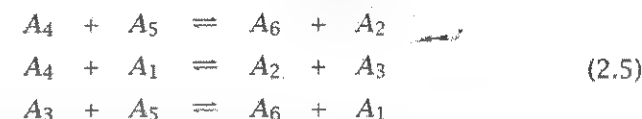


The rate of this reaction is important in determining the CO/CO<sub>2</sub> ratio in exhaust gases from internal combustion engines, and in determining the H<sub>2</sub> content in the feed for fuel cells. It also is known that the following two reactions are needed to describe what is happening at the molecular level,



Reactions 2.2-2.4 comprise a simple reaction network. There are three chemical reactions and six different chemical species taking part in the three reactions, H, H<sub>2</sub>, OH, H<sub>2</sub>O, CO, and CO<sub>2</sub>.

In order to organize the way we discuss chemical reactions, the following notation is convenient. Let the symbol,  $A_j$ , represent the  $j$ th species taking part in a reaction. In the first example, we can choose  $A_1 = \text{NO}$ ,  $A_2 = \text{O}_2$ , and  $A_3 = \text{NO}_2$ . In the water gas shift example, we can choose  $A_1 = \text{H}$ ,  $A_2 = \text{H}_2$ ,  $A_3 = \text{OH}$ ,  $A_4 = \text{H}_2\text{O}$ ,  $A_5 = \text{CO}$ , and  $A_6 = \text{CO}_2$ . Using the  $A_j$  notation, we can express the water gas shift reactions as



Reactions 2.5 suppress the identities of the species for compactness. We can further compress the description by moving all of the *variables* to the right-hand side of the chemical reaction symbol and replacing it with an equality sign,

$$\begin{aligned} -A_4 - A_5 + A_6 + A_2 &= 0 \\ -A_4 - A_1 + A_2 + A_3 &= 0 \\ -A_3 - A_5 + A_6 + A_1 &= 0 \end{aligned} \quad (2.6)$$

Again notice the sign convention that *products* have *positive* coefficients and *reactants* have *negative* coefficients in Equations 2.6.<sup>2</sup> Equations 2.6 now resemble a set of three linear algebraic equations and motivates the use of matrices. Using the rules of matrix multiplication, one can express Equations 2.6 as

$$\begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.7)$$

The matrix appearing in Equation 2.7 provides an efficient description of the stoichiometry for the reaction network, and is appropriately known as the **stoichiometric matrix**. Giving the stoichiometric matrix the symbol  $\mathbf{v}$ , and writing  $\mathbf{A}$  to denote the column vector of the  $A_j$ ,  $j = 1, \dots, 6$ , our final summary of the water gas shift reaction appears as

$$\mathbf{v}\mathbf{A} = \mathbf{0}$$

<sup>2</sup>Boldface letters provide a mnemonic device.

The element  $v_{ij}$  in the stoichiometric matrix is the stoichiometric coefficient for the  $j$ th species in the  $i$ th reaction. The index  $i$  runs from 1 to  $n_r$ , the total number of reactions in the network, and the index  $j$  runs from 1 to  $n_s$ , the total number of species in the network. We say that  $\mathbf{v}$  is an  $n_r \times n_s$  matrix. After piling up this much abstraction to describe what started out as a simple set of three reactions, let us work a few examples to reinforce the concept of the stoichiometric matrix.

### Example 2.1: Stoichiometric matrix for a single reaction

Find the stoichiometric matrix for the nitric oxide example,



#### Solution

The nitric oxide example consists of one reaction and three species. We can assign the species to the  $A$  as follows:  $A_1 = \text{NO}$ ,  $A_2 = \text{O}_2$ ,  $A_3 = \text{NO}_2$ . The reaction can then be written as

$$-2A_1 - A_2 + 2A_3 = \begin{bmatrix} -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0$$

The stoichiometric matrix for a single reaction is a row vector, in this case,

$$\mathbf{v} = \begin{bmatrix} -2 & -1 & 2 \end{bmatrix}$$

□

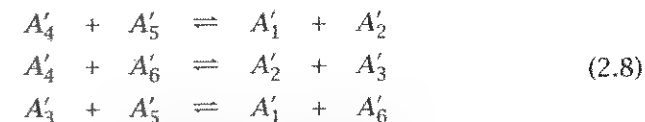
### Example 2.2: Columns of $\mathbf{v}$

Since we are free to assign chemical species to the  $A_j$  in any order we choose, consider what happens if we change the order of the species in the water gas shift example. Instead of using  $A_1 = \text{H}$  and  $A_6 = \text{CO}_2$ , what is the stoichiometric matrix if  $A_1$  is chosen to be  $\text{CO}_2$  and  $A_6$  is chosen to be  $\text{H}$ ?

#### Solution

Switching the identities of the first and sixth species in the  $A$  vector gives us the following modified vector,  $A'_1 = \text{CO}_2$ ,  $A'_2 = \text{H}_2$ ,  $A'_3 = \text{OH}$ ,

$A'_4 = \text{H}_2\text{O}$ ,  $A'_5 = \text{CO}$ , and  $A'_6 = \text{H}$ . Reactions 2.5 are then modified to give



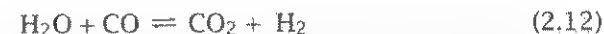
Extracting the modified  $\mathbf{v}$  matrix from Equations 2.8 gives

$$\mathbf{v}' = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 1 \end{bmatrix} \quad (2.9)$$

It is clear from examining Equations 2.9 and 2.7 that switching the identities of species one and six in the  $A$  vector has necessitated switching the first and sixth columns in the stoichiometric matrix. Therefore one can make the connection between the columns of  $\mathbf{v}$  and the species taking part in the reactions. More precisely, the  $j$ th column of the  $\mathbf{v}$  matrix supplies the stoichiometric numbers of the  $j$ th species in all of the reactions. □

### Example 2.3: Rows of $\mathbf{v}$

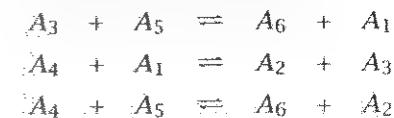
Just as we are free to assign the species to the  $A$  vector in any order we choose, we are also free to express the reactions in any order we choose. We now explore what happens if instead of expressing the water gas shift reaction as it appears in Reactions 2.2–2.4, we express it as



Notice we have written the original third reaction first and the original first reaction third. What is the impact of this change on the  $\mathbf{v}$  matrix?

#### Solution

Using the original ordering of the  $A$  vector, we can ascribe the following elements of a third stoichiometric matrix,

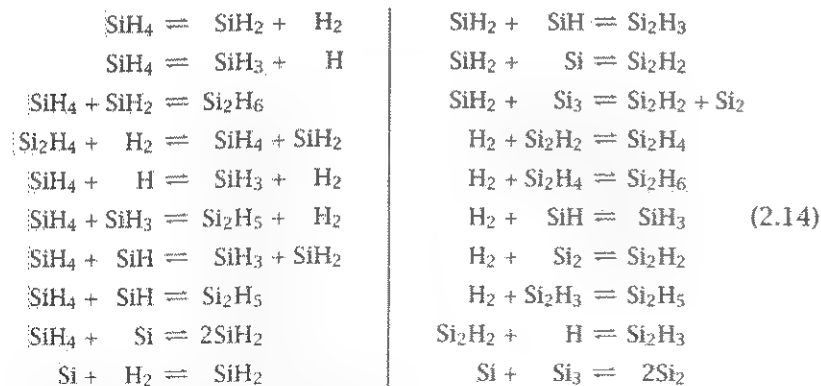


$$\mathbf{v}'' = \begin{matrix} \rightarrow & \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \end{bmatrix} & \leftarrow \end{matrix} \quad (2.13)$$

As one might expect, exchanging the orders of the first and third reactions causes us to exchange the first and third rows in the  $\mathbf{v}$  matrix as evidenced by comparing the matrices in Equations 2.13 and 2.7. We can therefore make the connection between the rows of  $\mathbf{v}$  and the reactions.<sup>3</sup> The  $i$ th row of the stoichiometric matrix contains the stoichiometric numbers of all species in the  $i$ th reaction.  $\square$

From the previous two examples it is clear that one could develop a large number of stoichiometric matrices to describe the same set of chemical reactions. Since there is no reason to prefer one ordering of species and reactions over another, one may permute the columns and rows into any order and maintain a valid stoichiometric matrix.

We now introduce the third example, which is a more complicated reaction network. The following chemistry has been proposed to describe a silicon chemical vapor deposition (CVD) reaction, which is an important process in the production of microelectronic materials.



(2.14)

The student should not be dismayed by the complexity of this reaction network. Indeed the principles for analyzing a CVD reactor with this chemistry are exactly the same as the principles for analyzing the

simpler nitric oxide and water gas shift chemistries presented previously. The only difference is that Reactions 2.14 are complex enough that we use a computer to keep track of the algebra for us. In fact, Reactions 2.14 are a simplified version of 120 reactions that were originally postulated for this reaction network [3].

Chemical engineers should also bear in mind that Reactions 2.14 are quite simple compared to many mechanisms that have been proposed for combustion problems in which it is not uncommon to have several hundred reactions. Polymerizations and long-chain-producing reactions consist of thousands of species and associated reactions. Obviously the stoichiometry of these complex problems is intractable if we do not develop a systematic, automated procedure. Developing and understanding that procedure is the topic of the next several sections.

#### Example 2.4: Stoichiometric matrix for CVD chemistry

Determine the stoichiometric matrix corresponding to Reactions 2.14.

#### Solution

The first thing we notice is that there are 20 reactions or  $n_r = 20$ . We then look through all of the reactions and identify the different species taking part. After writing this out we notice that there are 14 different species,  $n_s = 14$ . A possible assignment to the  $\mathbf{A}$  vector is: H, H<sub>2</sub>, Si, SiH, SiH<sub>2</sub>, SiH<sub>3</sub>, SiH<sub>4</sub>, Si<sub>2</sub>, Si<sub>2</sub>H<sub>2</sub>, Si<sub>2</sub>H<sub>3</sub>, Si<sub>2</sub>H<sub>4</sub>, Si<sub>2</sub>H<sub>5</sub>, Si<sub>2</sub>H<sub>6</sub>, Si<sub>3</sub>. With an  $\mathbf{A}$  chosen, it is a simple matter to look through Reactions 2.14 and find the stoichiometric coefficients of each species in each reaction. Do not forget the convention that species appearing as products in a given reaction have positive coefficients and those appearing as reactants have negative coefficients. Practice filling out a few rows of the  $\mathbf{v}$  matrix and check it with the values given in Equation 2.15.

Notice that for this example  $\mathbf{v}$  is a  $20 \times 14$  matrix, and it contains many zero entries. A matrix with many zero entries is called sparse. The large number of zeros simply reflects the physical fact that very few molecules can take part in a particular reaction. All of the reactions in the CVD chemistry, for example, are unimolecular or bimolecular. More will be said about this issue in the discussion of mechanisms in Chapter 5.

<sup>3</sup>Another mnemonic.

$$\mathbf{v} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad (2.15)$$

□

**Example 2.5: Conservation of mass**

Show that conservation of mass in a chemical reaction can be stated as

$$\mathbf{v}\mathbf{M} = 0$$

in which  $M_j$  is the molecular weight of species  $j$ .

**Solution**

In a chemical reaction, the number of molecules is not conserved in general. For example in the nitric oxide reaction,  $2\text{NO} + \text{O}_2 \rightleftharpoons 2\text{NO}_2$ , three reactant molecules react to form two product molecules. The mass, however, is conserved in chemical (i.e., not nuclear) reactions. It is clear in the above example that the atoms (N and O) are conserved, so the mass is conserved. Another way to state conservation of mass involves molecular weights of the species. In the nitric oxide reaction, the molecular weights of reactants and products are related by  $2M_{\text{NO}} +$

$M_{\text{O}_2} = 2M_{\text{NO}_2}$ . Equivalently

$$-2M_{\text{NO}} - M_{\text{O}_2} + 2M_{\text{NO}_2} = \begin{bmatrix} -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} M_{\text{NO}} \\ M_{\text{O}_2} \\ M_{\text{NO}_2} \end{bmatrix} = 0$$

If we put the molecular weights of the species in a vector,  $\mathbf{M}$ , conservation of mass for this single reaction can be written as,

$$\mathbf{v}\mathbf{M} = 0$$

For the water gas shift reaction, using the ordering of the species,  $A_1 = \text{H}$ ,  $A_2 = \text{H}_2$ ,  $A_3 = \text{OH}$ ,  $A_4 = \text{H}_2\text{O}$ ,  $A_5 = \text{CO}$ , and  $A_6 = \text{CO}_2$ ,

$$\mathbf{M} = \begin{bmatrix} M_{\text{H}} & M_{\text{H}_2} & M_{\text{OH}} & M_{\text{H}_2\text{O}} & M_{\text{CO}} & M_{\text{CO}_2} \end{bmatrix}^T$$

in which the superscript  $T$  means the transpose of the matrix. The transpose of the matrix means to exchange the rows for columns and vice versa. For the first reaction,  $\text{H}_2\text{O} + \text{CO} \rightleftharpoons \text{CO}_2 + \text{H}_2$ , we know

$$M_{\text{CO}_2} + M_{\text{H}_2} - M_{\text{H}_2\text{O}} - M_{\text{CO}} = 0$$

or

$$\begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} M_{\text{H}} \\ M_{\text{H}_2} \\ M_{\text{OH}} \\ M_{\text{H}_2\text{O}} \\ M_{\text{CO}} \\ M_{\text{CO}_2} \end{bmatrix} = 0$$

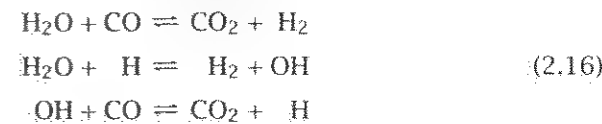
which is the first row of  $\mathbf{v}$  in Equation 2.7 multiplied by  $\mathbf{M}$ . The second and third reactions simply fill out the second and third rows of  $\mathbf{v}$  so that again, for multiple reactions

$$\mathbf{v}\mathbf{M} = 0$$

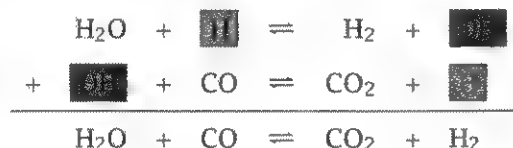
□

**2.3 Independent Reactions**

To motivate the discussion of independence of chemical reactions, let us again consider the water gas shift reaction



The issue of independence centers on the question of whether or not we can express any reaction in the network as a linear combination of the other reactions. If we can, then the set of reactions is not independent. It is not necessary to eliminate extra reactions and work with the smallest set, but it is sometimes preferable. In any case, the concept is important and is examined further. Before making any of these statements precise, we explore the question of whether or not the three reactions listed in Reactions 2.16 are independent. Can we express the first reaction as a linear combination of the second and third reactions? By *linear* combination we mean multiplying a reaction by a number and adding it to the other reactions. It is clear from inspection that the first reaction is the sum of the second and third reactions, so the set of three reactions is not independent.

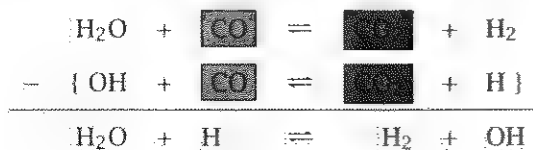


If we deleted the first reaction from the network, would the remaining two reactions be independent?



The answer is now yes, because no multiple of Reaction 2.17 can equal Reaction 2.18. There is no way to produce CO or CO<sub>2</sub> from only Reaction 2.17. Likewise there is no way to produce H<sub>2</sub> or H<sub>2</sub>O from only Reaction 2.18.

This discussion is *not* meant to imply that there is something *wrong* with the first reaction in Reactions 2.16. Indeed if we focus attention on the second reaction, we can again ask the question whether or not it can be written as a linear combination of the first and third reactions. The answer is yes because the second reaction is the first reaction minus the third reaction.



So the first and third reactions could be chosen as the independent set of two reactions. Finally, the third reaction in Reaction 2.16 is equal to the first reaction minus the second reaction, so the first and second reactions could be chosen as an independent set. For this example then, any two of the reactions comprise an independent set. The situation is not always this simple as we will see from the chemical vapor deposition chemistry.

Before making the problem more complicated, we explore how to automate the preceding analysis by exploiting the stoichiometric matrix. If you are familiar with linear algebra, the issue of independence of reactions is obviously related to the rank of the stoichiometric matrix. Familiarity with these concepts, although helpful, is not required to follow the subsequent development. We now consider the stoichiometric matrix for the water gas shift reaction presented in Equation 2.7

$$\mathbf{v} = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 1 \end{bmatrix} \quad (2.19)$$

We can make an important mathematical connection to the preceding physical arguments. The question of whether or not the *i*th reaction can be written as a linear combination of the other reactions is the same as the question of whether or not the *i*th row of the *v* matrix can be written as a linear combination of the other rows. The linear independence of the reactions in a reaction network is equivalent to the linear independence of the rows in the corresponding stoichiometric matrix.

The **rank** of a matrix is defined as the number of linearly independent rows (or equivalently, columns) in the matrix. Therefore, the number of linearly independent reactions in a network, *n<sub>i</sub>*, is equal to the rank of *v*. There are efficient numerical algorithms available for finding the rank of a matrix and a set of linearly independent rows. The focus of our attention is not on the algorithm, but on how we can exploit the results of the algorithm to analyze sets of chemical reactions. You should consult Strang [7] or another linear algebra text for a lucid explanation of the algorithm, Gaussian elimination with partial pivoting.

#### Example 2.6: More species than reactions

Show that mass conservation implies that any *independent* set of reactions has more species than reactions.



## Solution

From Example 2.5, we know that mass conservation is equivalent to

$$\mathbf{vM} = \mathbf{0}$$

Consider the columns of the  $\mathbf{v}$  matrix as column vectors. This matrix-vector multiplication can be expressed as a linear combination of the columns of  $\mathbf{v}$  with the elements of the  $\mathbf{M}$  vector as the coefficients in the linear combination

$$\begin{aligned} \mathbf{vM} &= \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n_s} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n_i1} & v_{n_i2} & \cdots & v_{n_in_s} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n_s} \end{bmatrix} \\ &= \begin{bmatrix} v_{11} \\ \vdots \\ v_{n_i1} \end{bmatrix} M_1 + \begin{bmatrix} v_{12} \\ \vdots \\ v_{n_i2} \end{bmatrix} M_2 + \cdots + \begin{bmatrix} v_{1n_s} \\ \vdots \\ v_{n_in_s} \end{bmatrix} M_{n_s} \\ &= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned} \quad (2.20)$$

The last equation implies the columns of  $\mathbf{v}$  are linearly *dependent* because the molecular weights are nonzero.<sup>4</sup> Because the rows are linearly independent, we conclude there are more columns (species) than rows (independent reactions),  $n_s > n_i$  and  $\mathbf{v}$  is a *wide* matrix (i.e., not a square or *tall* matrix).

Notice that one must consider linearly independent reactions for the statement in the example to be true. If we considered an arbitrary network for example,



<sup>4</sup>One could solve for column  $j$  of  $\mathbf{v}$  by moving the remaining terms to the other side of the equality and dividing by  $M_j$ , which is nonzero. That is possible as long as one of the  $M_j$  multipliers in Equation 2.20 is nonzero. In our case, all of the multipliers are nonzero.

These reactions are obviously not independent because they are the same reaction written four times. In this case

$$\mathbf{v} = \begin{bmatrix} -2 & -1 & 2 \\ -4 & -2 & 4 \\ -6 & -3 & 6 \\ -8 & -4 & 8 \end{bmatrix}$$

Since we have not used independent reactions,  $\mathbf{v}$  is *tall* and not *wide*. Recall  $\mathbf{v} = \begin{bmatrix} -2 & -1 & 2 \end{bmatrix}$  for the single independent reaction, which is in agreement with the example.  $\square$

**Maximal sets of linearly independent reactions.** Up to this point, we have started with a set of reactions and investigated constructing subsets of these reactions that are linearly independent. Now consider the reverse problem. We start with a set of species, and we would like to know the largest number of linearly independent valid chemical reactions among these species. In other words, a given set of reactions may be linearly independent, but we want to be sure we have not left out some valid reactions. By valid chemical reactions we mean *element conserving*, which is a sufficient condition implying the weaker condition of *mass conserving*.

The following describes a systematic approach to this problem. First, list formation reactions for every species in the list from its elements. We may use molecules of pure elements ( $\text{O}_2$ ) rather than atoms ( $\text{O}$ ) to save work as long as the atoms themselves do not appear in the species list. This set is guaranteed by construction to be a maximal linearly independent set for an *enlarged* species list, which includes the original species plus any new elements introduced in the formation reactions.

Then eliminate through linear combinations of reactions any new elements that were introduced in the formation reactions and that do not appear in the original species list. The remaining set is a maximal linearly independent set of reactions for the original species list.

This procedure is perhaps best illustrated by example.

### Example 2.7: Maximal set of reactions for methane oxidation

Consider the oxidation of methane in which the observed species are:

$$\mathbf{A} = \begin{bmatrix} \text{CO}_2 & \text{H}_2\text{O} & \text{CH}_4 & \text{CO} & \text{H}_2 & \text{O}_2 \end{bmatrix}^T$$

We first write formation reactions for all species from the elements. It is necessary to write these reactions in the order in which the species

appear in the species list. We can use  $O_2$  and  $H_2$  because O and H are not in the species list



Notice we have introduced the element carbon, which is not in the original species list, to express the formation reactions. We can add C to the list and the enlarged species list is

$$\tilde{A} = [CO_2 \ H_2O \ CH_4 \ CO \ H_2 \ O_2 \ C]^T$$

The stoichiometry of the formation reactions are then summarized by

$$\tilde{v}\tilde{A} = 0$$

in which

$$\tilde{v} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 & -1 & -2 \end{bmatrix} \quad (2.25)$$

Notice we have zeros below the diagonal of the first four rows and columns. We know *by inspection* that these reactions are linearly independent, which is why we wrote them in the first place.<sup>5</sup> So the rank of  $\tilde{v}$  is four. See Exercise 2.9 for a proof that there are no other valid linearly independent reactions among these species and elements.

We now wish to eliminate carbon from the species list. The approach is to replace formation reactions involving C with independent linear combinations of the four reactions that eliminate C from the set. For example we could replace Reaction 2.21 with the sum of Reaction 2.21 and the negative of Reaction 2.23. Equivalently we replace row 1 in Equation 2.25 with the sum of row 1 and negative of row 3. We always add linear combinations of rows *below* the row on which we are making the zero in order not to disturb the pattern of zeros below the diagonal in the first columns of the matrix. We leave the second row unchanged

<sup>5</sup>Examine the locations of the zeros in the first four rows and columns of  $\tilde{v}$ . Because these portions of the first four rows are independent, so are the entire rows, and, therefore, the reactions.

because the  $v_{2,7}$  is zero already; i.e., C does not take place in the second reaction. Proceeding down the rows we replace row 3 by twice row 3 minus row 4. When we reach the last row, either a zero already exists in the last column or we remove this last row because we have no rows below the last with which to zero that element. That reduces  $\tilde{v}$  to

$$\tilde{v}' = \begin{bmatrix} 1 & 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 2 & -2 & -4 & 1 & 0 \end{bmatrix}$$

Inspection of the zeros in the first three columns of  $\tilde{v}'$  tells us that the rows are independent, and, therefore, the matrix has full rank. If we now multiply out these equations, the last column of zeros removes the C from the species list and we have

$$vA = 0$$

in which

$$v = \begin{bmatrix} 1 & 0 & -1 & 0 & 2 & -1 \\ 0 & 2 & 0 & 0 & -2 & -1 \\ 0 & 0 & 2 & -2 & -4 & 1 \end{bmatrix}$$

Therefore the maximal linearly independent set for the original species contains *three* reactions. The stoichiometric matrix above corresponds to the following choice of reactions



□

If we were going to remove other elements besides C, we would repeat this procedure starting with  $v$ . See also Exercise 2.6.

## 2.4 Reaction Rates and Production Rates

In order to describe the change in composition in a reactor, one has to know the reaction rates. As an example, we consider the third reaction in the CVD chemistry, Reactions 2.14



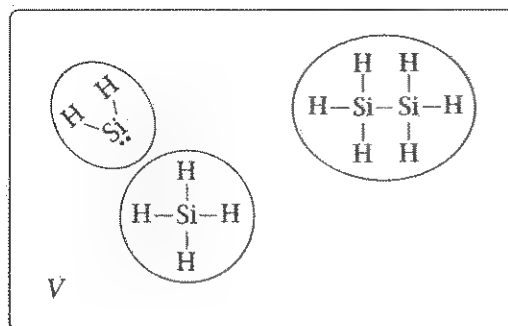


Figure 2.1: Defining the reaction rate,  $r$ , for the reaction  $\text{SiH}_2 + \text{SiH}_4 = \text{Si}_2\text{H}_6$ .

The **reaction rate**,  $r$ , is defined as the number of times this reaction event takes place per time per volume. One can imagine turning  $\text{SiH}_4$ ,  $\text{SiH}_2$  and  $\text{Si}_2\text{H}_6$  molecules loose in a box of some fixed volume  $V$  as depicted in Figure 2.1. We define the **reaction extent**,  $\varepsilon$ , to keep track of the number of times this reaction event occurs. Imagine that we could somehow count up the net number of times an  $\text{SiH}_4$  molecule hit an  $\text{SiH}_2$  molecule and turned into an  $\text{Si}_2\text{H}_6$  molecule during a short period of time. The change in the reaction extent,  $\Delta\varepsilon$ , is the net number of reaction events that occur in the time interval  $\Delta t$ . The reaction rate is then

$$r = \frac{\Delta\varepsilon}{\Delta t V} \quad (2.27)$$

If the forward event (an  $\text{SiH}_4$  molecule and an  $\text{SiH}_2$  molecule turning into an  $\text{Si}_2\text{H}_6$  molecule) occurs more often than the reverse event (an  $\text{Si}_2\text{H}_6$  molecule decomposing into an  $\text{SiH}_4$  molecule and an  $\text{SiH}_2$  molecule), then the change in  $\varepsilon$  is positive and the reaction rate is positive. If the reverse event occurs more often than the forward event, then the change in  $\varepsilon$  and reaction rate are negative. If the system is at equilibrium, then the change in  $\varepsilon$  is zero and the forward and reverse events occur in equal numbers. The extent  $\varepsilon$  is a number of molecular change events and therefore the units of  $r$  in Equation 2.27 are  $\#/(\text{time} \cdot \text{volume})$ . If one divides by Avogadro's number, the units of extent are moles and the units of reaction rate are moles/(time·volume), which are the usual units for extent and reaction rate in this text. Finally, we often deal with physical situations in which we assume the

material behaves as a continuum and we can ignore the discrete nature of the molecules. This means we can take the volume  $V$  large enough to average the random fluctuations of the molecules, but small enough that there is negligible spatial variation in the average concentrations of the components or the reaction rate within  $V$ . Under this continuum assumption, we can speak of the reaction rate as defined at a point in space within some larger reacting system or physical reactor equipment.

Notice in the definition of reaction rate, we are taking the reaction stoichiometry *literally*. We are postulating that these collision and transformation events are taking place at the molecular level. These literal reactions are known as **elementary reactions**. We delay a more complete discussion of elementary reactions and reaction mechanisms until Chapter 5. We will also see that for complex reacting systems, it may be difficult to know whether or not a reaction is an elementary reaction. But that is a separate issue, which we take up later, and that issue does not prevent us from defining the reaction rate.

It is difficult to measure reaction rates directly, because we do not directly sense molecular transformation events. We can measure concentrations, however. It is important to connect the reaction rate to the rate of change of the concentrations of the various species in the reactor, which are the quantities we usually care about in a commercial reactor. We define **production rate**,  $R_j$ , as the rate at which the  $j$ th species is produced (moles/(time·volume)) due to the chemical reactions taking place. It is clear looking at the stoichiometry in Equation 2.27 that each time the forward reaction event occurs, an  $\text{Si}_2\text{H}_6$  molecule is produced. Each time the reverse reaction occurs, an  $\text{Si}_2\text{H}_6$  molecule is consumed. The production rate of  $\text{Si}_2\text{H}_6$ ,  $R_{\text{Si}_2\text{H}_6}$ , is therefore directly related to the reaction rate,

$$R_{\text{Si}_2\text{H}_6} = r$$

Notice that if  $r$  is positive  $R_{\text{Si}_2\text{H}_6}$  is positive as we expect because  $\text{Si}_2\text{H}_6$  is being produced. Similar arguments lead to relating the other production rates to the reaction rate,

$$R_{\text{SiH}_4} = -r$$

$$R_{\text{SiH}_2} = -r$$

Notice that we have three production rates, one for each species, but only one reaction rate, because there is only a single reaction. If we

now introduce the production rate vector,  $R$ ,

$$R = \begin{bmatrix} R_{\text{SiH}_4} \\ R_{\text{SiH}_2} \\ R_{\text{Si}_2\text{H}_6} \end{bmatrix}$$

we can summarize the connection between the three production rates and the single reaction rate by

$$R = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} r \quad (2.28)$$

Notice that the column vector in Equation 2.28 is just the transpose of the row vector that comprises  $\nu = [-1 \ -1 \ 1]$ , which follows from Reaction 2.26

Consider what happens to the relationship between the production and reaction rates if there is more than one reaction. Recall the water gas shift reaction,



Three reaction rates are required to track all three reactions. Let  $r_i$  denote the reaction rate for the  $i$ th reaction. What production rate of atomic hydrogen, H, results from these three reactions? We notice that H does not take part in the first reaction, is consumed in the second reaction, and is produced in the third reaction. We therefore write

$$R_{\text{H}} = (0) r_1 + (-1) r_2 + (1) r_3 = -r_2 + r_3$$

Consider the second species,  $\text{H}_2$ . It is produced in the first and second reactions and does not take part in the third reaction. Its production

rate can therefore be expressed as

$$R_{\text{H}_2} = (1) r_1 + (1) r_2 + (0) r_3 = r_1 + r_2$$

You should examine the remaining four species and produce the following matrix equation,

$$\begin{bmatrix} R_{\text{H}} \\ R_{\text{H}_2} \\ R_{\text{OH}} \\ R_{\text{H}_2\text{O}} \\ R_{\text{CO}} \\ R_{\text{CO}_2} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad (2.29)$$

The fundamental relationship between the reaction rates and the production rates now emerges. Compare the matrices in Equations 2.7 and 2.29. Notice that the first *row* of the matrix in Equation 2.7 is the same as the first *column* of the matrix in Equation 2.29. Moreover, *each* row of the matrix in Equation 2.7 is the same as the corresponding column of the matrix in Equation 2.29. In other words, the two matrices are transposes of each other. We can therefore summarize Equation 2.29 as

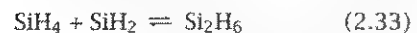
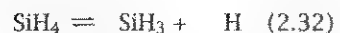
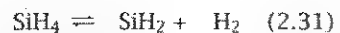
$$R = \nu^T r \quad (2.30)$$

in which  $\nu^T$  denotes the transpose of the stoichiometric matrix. Equation 2.30 implies that one can always compute the production rates from the reaction rates. That computation is a simple matter of matrix multiplication. The reverse problem, deducing the reaction rates from the production rates, is not so simple as it involves solving a set of equations. We will see in the next section under what conditions that solution can be found.

### Example 2.8: Production rate for $\text{SiH}_2$

What is the production rate of  $\text{SiH}_2$  in terms of the reaction rates for the CVD example?

## Solution



Looking at Reactions 2.31–2.50, we note that  $\text{SiH}_2$  takes part in Reactions 2.31, 2.33, 2.34, 2.37, and 2.39–2.43. Extracting the stoichiometric numbers of  $\text{SiH}_2$  for each of these reactions gives<sup>6</sup>

$$R_{\text{SiH}_2} = r_1 - r_3 + r_4 + r_7 + 2r_9 + r_{10} - r_{11} - r_{12} - r_{13}$$

or

$$R_{\text{SiH}_2} = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{r}$$

$\text{SiH}_2$  was chosen as the fifth species in Exercise 2.4 so the row vector above is indeed the transpose of the fifth column of the  $\mathbf{v}$  matrix in Equation 2.15. You may wish to choose another component such as  $\text{SiH}_4$  and check another column of the  $\mathbf{v}$  matrix.  $\square$

## 2.5 Computational Aspects of Stoichiometry

As we have seen in this chapter, problems of realistic and complex reaction stoichiometry involve matrices and linear algebra. After the fundamental concepts are in place, application of the fundamentals requires computational tools. Moreover, if the computing environment is organized properly, the experience of solving nontrivial problems reinforces the understanding of the fundamental concepts and further

<sup>6</sup>We are so sure you will not forget that products have positive stoichiometric numbers and reactants have negative ones that we will not repeat it again.

prepares one to apply the fundamentals in realistic and complex industrial situations. In Appendix A, we briefly summarize Octave and MATLAB as high-level programming languages for numerical solution of reactor analysis and design problems. Octave is freely available for a variety of hardware platforms and can be down-loaded from [www.octave.org](http://www.octave.org). MATLAB is commercially available from The MathWorks, Inc., and is becoming a commonly available tool of industrial engineering practice.

### 2.5.1 Computing Production Rates from Reaction Rates

As discussed previously, computing  $R$  from  $\mathbf{r}$  is a simple matter of matrix multiplication. Consider again the water gas shift reaction chemistry,

$$\begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \text{H} \\ \text{H}_2 \\ \text{OH} \\ \text{H}_2\text{O} \\ \text{CO} \\ \text{CO}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.51)$$

In Chapter 5 we discuss means for predicting reaction rates given species concentrations, but for now just assume we know the three reaction rates are, in some chosen units of moles/(time·volume),

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The production rates of the six species due to these reactions are then computed as

$$\begin{bmatrix} R_{\text{H}} \\ R_{\text{H}_2} \\ R_{\text{OH}} \\ R_{\text{H}_2\text{O}} \\ R_{\text{CO}} \\ R_{\text{CO}_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ -3 \\ -4 \\ 4 \end{bmatrix}$$

The effect of the three reactions is to produce H,  $\text{H}_2$  and  $\text{CO}_2$ , and to consume OH,  $\text{H}_2\text{O}$  and CO at the given rates. Please perform this calculation for yourself.



### 2.5.2 Computing Reaction Rates from Production Rates

Another common task arising in the analysis of reactors and reaction mechanisms is to measure production rates by monitoring changes in species concentrations, to help infer the corresponding reaction rates. To make the concept clear, consider the simple isomerization reactions between species A, B and C,



The stoichiometric matrix for these reactions is

$$\nu = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

and the production rates and reaction rates are related by

$$\begin{bmatrix} R_A \\ R_B \\ R_C \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

It might appear at first glance that we can compute  $r$  given  $R$  because we have three equations and three unknowns, but because the reactions are not linearly independent, such is not the case. It is clear from inspection of Reactions 2.52–2.54 that the third reaction is the sum of the first two. Computing the rank of  $\nu$  confirms that only two reactions are independent.

Before we continue with the full set of three reactions, we explore what happens if we use an independent set. If we omit the third reaction, for example,



the production rates and new reaction rates are related by

$$\begin{bmatrix} R_A \\ R_B \\ R_C \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{r}_1 \\ \hat{r}_2 \end{bmatrix}$$

By inspection, the first equation gives

$$\hat{r}_1 = -R_A$$

and the third equation gives

$$\hat{r}_2 = R_C$$

The second equation tells us  $R_B = \hat{r}_1 - \hat{r}_2 = -(R_A + R_C)$ . In other words, using the linearly independent reactions, we can compute both reaction rates and we find a restriction on the possible production rates. We explore subsequently what happens when this restriction is violated by the production-rate measurements.

If we now return to the original set of three isomerization reactions, we can deduce that  $r_3$  is not determined by the production rates, and that from the first equation

$$r_1 = -R_A + r_3 \quad (2.55)$$

and from the third equation

$$r_2 = R_C + r_3 \quad (2.56)$$

and  $r_3$  is arbitrary. The second equation places the restriction  $R_B = r_1 - r_2$ , which upon substitution of Equations 2.55 and 2.56 gives  $R_B = -(R_A + R_C)$  as before. Because  $r_3$  is arbitrary, we cannot deduce the reaction rates from production rates, which is characteristic of using sets of reactions that are not linearly independent.

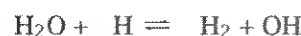
So we conclude that measuring production rates is not enough to tell us reaction rates. We require more information. As we discuss in Chapter 5, the extra information usually is provided by postulating a reaction mechanism and applying the laws of mass action to the elementary reactions. In that case we seek to determine the rate *constants* from the production rate measurements, not the reaction rates themselves. We take up that important problem in Chapter 9.

Finally, we emphasize that using linearly dependent sets of reactions presents no problem at all if one is interested only in computing the production rates from given reaction rates, which is the usual case when working with reaction mechanisms.

### 2.5.3 Measurement Errors and Least-Squares Estimation

As a final topic in this chapter, we explore what happens when the extra conditions on the production rates are violated by the data. Imagine

the following two linearly independent reactions of the water gas shift are taking place



so the production rates are given by

$$\begin{bmatrix} R_{\text{H}} \\ R_{\text{H}_2} \\ R_{\text{OH}} \\ R_{\text{H}_2\text{O}} \\ R_{\text{CO}} \\ R_{\text{CO}_2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ 0 & 1 \\ -1 & -1 \\ -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \quad (2.57)$$

We can compute the production rates when the two reaction rates are

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and we obtain

$$\mathbf{R} = \begin{bmatrix} -2 & 3 & 2 & -3 & -1 & 1 \end{bmatrix}^T \quad (2.58)$$

Now if  $\mathbf{R}$  is determined by measuring species concentrations, unmodeled effects undoubtedly cause discrepancy between predicted and measured values of  $\mathbf{R}$ . For example, let's assume that the production rate of the first species, H, is in error by a small amount so the measured  $\mathbf{R}$  is  $\mathbf{R} = \begin{bmatrix} -2.1 & 3 & 2 & -3 & -1 & 1 \end{bmatrix}^T$ . The effect of this error is to make the equations inconsistent, so there no longer is an exact solution for the reaction rates, which is not too surprising because we have six equations and only two unknowns in Equation 2.57. A system with more equations than unknowns is called over-determined. Normally we would not expect to find a solution for arbitrary  $\mathbf{R}$ , only those  $\mathbf{R}$  that are generated by multiplying an  $\mathbf{r}$  by  $\mathbf{v}^T$ . When the equations are inconsistent, one is usually interested in knowing the values of the two reaction rates that come *closest* to satisfying the six equations simultaneously. If we measure how close the equations are to zero by squaring the error in each equation and summing over all equations, then we are using the classic least-squares approach.

Computing least-squares solutions to over-determined equations is a useful computation in linear algebra. If one is given  $\mathbf{R}$  and trying to solve for  $\mathbf{r}$  from

$$\mathbf{R} = \mathbf{v}^T \mathbf{r} \quad (2.59)$$

then the least-squares solution is given by

$$\mathbf{r} = (\mathbf{v} \mathbf{v}^T)^{-1} \mathbf{v} \mathbf{R}$$

in which the superscript  $-1$  indicates a matrix inverse. If the reactions are linearly independent, then the matrix product  $\mathbf{v} \mathbf{v}^T$  has an inverse and the least-squares solution is unique. If the reactions are not linearly independent this inverse does not exist and the least-squares solution is not unique as before.

If we compute the least-squares solution to the inconsistent data given above, we find  $\mathbf{r} = \begin{bmatrix} 0.983 & 2.03 \end{bmatrix}^T$  instead of the *correct* value  $\mathbf{r} = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$ . Notice a small error in the H production rate has translated into small errors in both inferred reaction rates.

As a final example, let's consider the case in which we have repeated measurements of the production rates, all of which are subject to small random errors. If we add small amounts of random noise to the data given in Equation 2.58, we produce the following six measurements.

$$\mathbf{R}_{\text{meas}} = \begin{bmatrix} -2.05 & -2.06 & -1.93 & -1.97 & -2.04 & -1.92 \\ 2.94 & 3.02 & 3.04 & 2.93 & 3.06 & 3.04 \\ 2.01 & 1.94 & 2.01 & 1.92 & 2.01 & 2.04 \\ -2.98 & -2.98 & -2.98 & -2.99 & -2.96 & -2.96 \\ -1.03 & -1.03 & -0.98 & -1.07 & -0.95 & -1.08 \\ 0.97 & 1.05 & 1.06 & 1.09 & 1.00 & 1.07 \end{bmatrix}$$

If we take each column of  $\mathbf{R}_{\text{meas}}$ , that is each production-rate measurement, and compute the least-squares estimate of  $\mathbf{r}$  for that measurement, we obtain six estimates of the reaction rates, one for each measured production rate. We can perform these operations in one matrix equation via

$$\mathbf{r}_{\text{est}} = (\mathbf{v} \mathbf{v}^T)^{-1} \mathbf{v} \mathbf{R}_{\text{meas}}$$

and the result is

$$\mathbf{r}_{\text{est}} = \begin{bmatrix} 0.97 & 1.03 & 1.03 & 1.06 & 0.98 & 1.05 \\ 2.01 & 1.99 & 1.98 & 1.92 & 2.03 & 1.96 \end{bmatrix}$$

Figure 2.2 shows the estimated reaction rates for the six production-rate measurements. Notice that we obtain reasonable estimates of the true reaction rates,  $\mathbf{r} = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$ . Next consider what happens when we have many measurements available. Figure 2.3 displays the estimated reaction rates given 500 production-rate measurements. Notice that the estimated rates are again scattered about the true value, the

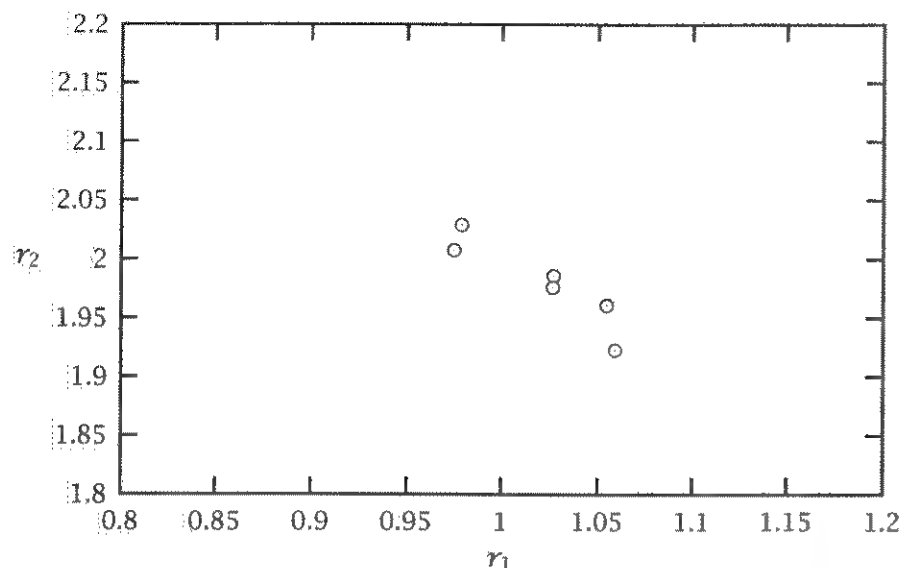


Figure 2.2: Estimated reaction rates from six production-rate measurements subject to measurement noise.

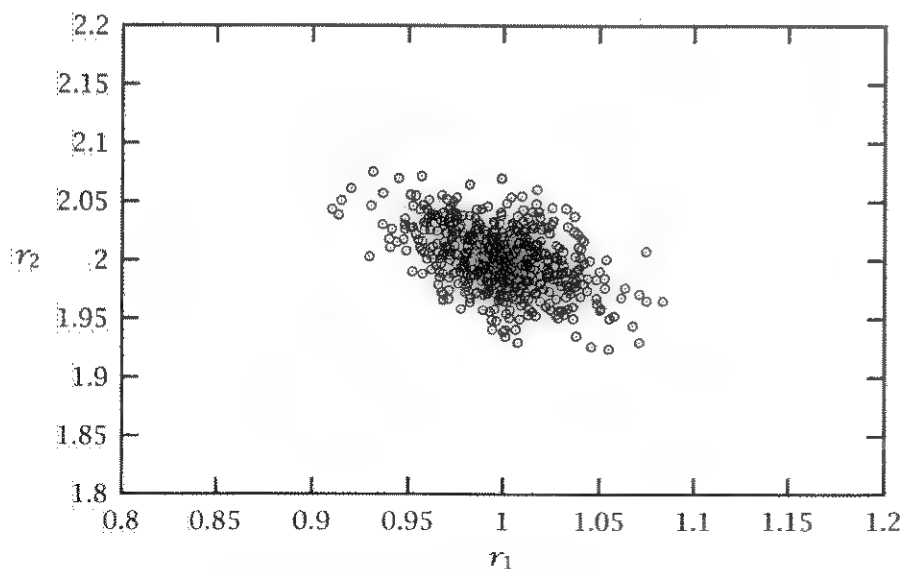


Figure 2.3: Estimated reaction rates from 500 production-rate measurements subject to measurement noise.

mean of these values is close to the true value, and the shape of the frequency distribution for the reaction rate estimates begins to emerge. You might want to generate the same plot with 5000 random measurements and see how much more detail is apparent in the probability distribution of estimates. Further discussion of parameter estimation from data is delayed until Chapter 9, but engineers benefit from thinking about extracting information from data and models at an early stage.

## 2.6 Summary

In this chapter we have introduced the compact notation for keeping track of the stoichiometry of chemical reactions,

$$\sum_{j=1}^{n_s} \nu_{ij} A_j = 0, \quad i = 1, 2, \dots, n_r$$

in which  $A_j$  represents chemical species  $j$ ,  $j = 1, \dots, n_s$  and  $n_s$  is the number of species in the reaction network. The stoichiometric coefficients are contained in the stoichiometric matrix  $\nu$ , in which  $\nu_{ij}$  is the stoichiometric coefficient for species  $j$  in reaction  $i$ ,  $i = 1, \dots, n_r$  and  $n_r$  is the number of reactions in the network. We can summarize the reaction stoichiometry with one vector equation

$$\nu A = 0$$

A set of reactions is linearly independent if no reaction in the set can be written as a linear combination of the other reactions in the set. Linear independence of reactions is equivalent to linear independence of the rows of  $\nu$ . The rank of a matrix is the number of linearly independent rows (equivalently columns) of the matrix, so the rank of  $\nu$  is the number of linearly independent reactions in the network.

The reaction rate is a fundamental concept that allows quantitative prediction of rates of conversions of reactants to products. We define the rate of reaction  $i$ ,  $r_i$ , to be the net number of times a reaction event occurs per time per volume. Given the rates of all reactions, we can calculate directly the production rates of all species,

$$R_j = \sum_{i=1}^{n_r} \nu_{ij} r_i, \quad j = 1, \dots, n_s$$

or as an equivalent vector equation,

$$R = \mathbf{v}^T \mathbf{r} \quad (2.60)$$

Given the rates of reactions, it is a simple matter to compute the species production rates with Equation 2.60. One cannot solve uniquely the reverse problem, in general. Given observed production rates, computing the corresponding reaction rates requires additional information, such as rate expressions for the elementary reactions in a reaction mechanism. If the set of chemical reactions is linearly independent, then one can uniquely solve the reverse problem. If the observed production rates contain experimental errors, there may not exist an exact solution of reaction rates,  $\mathbf{r}$ , that satisfy Equation 2.60. In this situation, one is normally interested in finding the reaction rates that *most closely* satisfy Equation 2.60. The closest solution in a least-squares sense is easily computed with standard linear algebra software.

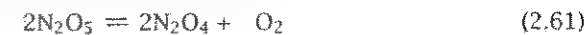
### Notation

$a_{jl}$	formula number for element $l$ in species $j$
$A_j$	$j$ th species in the reaction network
$E^l$	$l$ th element comprising the species
$i$	reaction index, $i = 1, 2, \dots, n_r$
$j$	species index, $j = 1, 2, \dots, n_s$
$M_j$	molecular weight of the $j$ th species
$n_i$	number of independent reactions in reaction network
$n_r$	total number of reactions in reaction network
$n_s$	total number of species in reaction network
$r_i$	reaction rate for $i$ th reaction
$R_j$	production rate for $j$ th species
$\varepsilon_i$	extent of reaction $i$
$v_{ij}$	stoichiometric number for the $j$ th species in the $i$ th reaction

## 2.7 Exercises

### Exercise 2.1: Finding independent sets of reactions

Consider the following set of chemical reactions,



- (a) Determine the stoichiometric matrix,  $\mathbf{v}$ , and the species list,  $A$ , for this reaction system so the reaction network is summarized by

$$\mathbf{v}A = 0$$

- (b) Use Octave, MATLAB, or your favorite software package to determine the rank of the stoichiometric matrix. How many of the reactions are linearly independent?
- (c) Now that you have found the number of independent reactions,  $n_i$ , which  $n_i$  of the original set of 6 reactions can be chosen as an independent set? Try guessing some set of  $n_i$  reactions and determine the rank of the new stoichiometric matrix. Stop when you have determined successfully one or more sets of  $n_i$  independent reactions.

Hint: you want to examine the rank of sub-matrices obtained by deleting rows (i.e., reactions) from the original stoichiometric matrix. In Octave, if you assign the original stoichiometric matrix to a name, `stoi`, then you can obtain the rank of the stoichiometric matrix associated with deleting the fifth reaction, for example, by

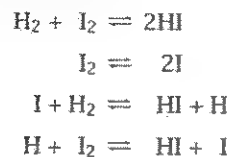
```
stoi2 = [stoi(1:4,:);stoi(6,:)]
rank(stoi2)
```

Do you see how the indices in forming `stoi2` work out? Notice we do not have to enter any more matrices after we build the original stoichiometric matrix to test the ranks of various reaction networks.

- (d) What do you think of a colleague's answer that contains Reactions 2.62 and 2.63 in the final set. Can this be correct? Why or why not?

**Exercise 2.2: The stoichiometric matrix**

- (a) What is the stoichiometric matrix for the following reaction network [2]? By inspection, how many of the reactions are linearly independent? How would you check your answer if you had access to a computer?



- (b) Given a stoichiometric matrix for a reaction network with  $n_s$  species and  $n_r$  reactions

$$\sum_{j=1}^{n_s} \nu_{ij} A_j = 0, \quad i = 1, 2, \dots, n_r$$

What is the production rate of the  $j$ th species,  $R_j$ , in terms of the reaction rates for the reactions,  $r_i$ ?

**Exercise 2.3: Finding reaction rates from production rates**

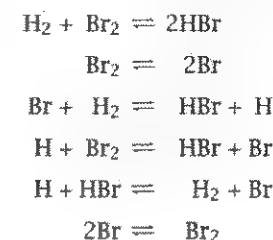
Consider again the water gas shift reaction presented in Equation 2.51. Assume the production rates have been measured and are, in some units of moles/(time·volume),

$$\begin{bmatrix} R_{\text{H}} \\ R_{\text{H}_2} \\ R_{\text{OH}} \\ R_{\text{H}_2\text{O}} \\ R_{\text{CO}} \\ R_{\text{CO}_2} \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -3 \\ -2 \\ 2 \end{bmatrix}$$

- (a) If you choose the first two reactions as a linearly independent set, what are the two reaction rates that are consistent with these data. Is this answer unique?
- (b) Repeat the calculation if you choose the second and third reactions as the linearly independent set of reactions. Is this answer unique?
- (c) How can these reaction rates differ, when the production rates are the same? Can we determine which set of reactions is really causing this measured production rate?

**Exercise 2.4: Independent reactions for bromine hydrogenation**

Consider the following set of chemical reactions [4, 5],



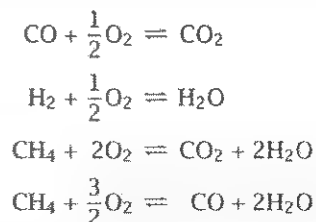
- (a) Determine the stoichiometric matrix,  $\nu$ , and the species list,  $A$ , for this reaction system so the reaction network is summarized by

$$\nu A = 0$$

- (b) Use Octave or MATLAB to determine the rank of the matrix using the rank function. How many reactions are linearly independent?
- (c) Now that you have found the number of independent reactions,  $n_i$ , which  $n_i$  of the original set of six reactions can be chosen as an independent set? Try guessing some set of  $n_i$  reactions and determine the rank of the new stoichiometric matrix. Stop when you have determined successfully one or more sets of  $n_i$  independent reactions.

**Exercise 2.5: Independent reactions for methane oxidation**

Consider a mixture of CO, H<sub>2</sub>, and CH<sub>4</sub> that is fed into a furnace with O<sub>2</sub> and produces CO, CO<sub>2</sub>, and H<sub>2</sub>O. The following chemical reactions have been suggested to account for the products that form.



- (a) Are these reactions linearly independent? How many reactions comprise a linearly independent set?
- (b) List all sets of linearly independent reactions. Which reaction is included in all of the linearly independent sets of reactions? Why?

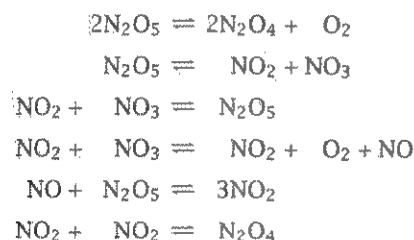


**Exercise 2.6: Methane oxidation and maximal independent sets**

- (a) List a maximal set of linearly independent reactions if  $O_2$  as well as C are not observed as species in the methane oxidation reactions in Example 2.7.
- (b) Repeat if  $H_2$ ,  $O_2$  and C are not observed as species.

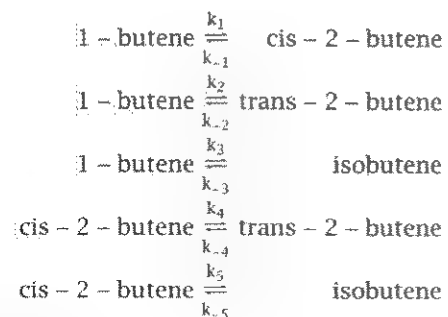
**Exercise 2.7: Production rates from reaction rates**

- (a) Consider the following set of chemical reactions,



Determine the rates of production of each component in terms of the rates of each reaction.

- (b) Butene isomerization reactions are shown below.



Determine the rates of production of each component in terms of the rates of each reaction.

**Exercise 2.8: Restrictions from element balancing**

Let the  $s$  species,  $A_j, j = 1, \dots, s$  be comprised of the  $e$  elements,  $E^l, l = 1, \dots, e$ . Writing the chemical formulas for the species in the usual way

$$\begin{aligned}
 A_1 &= E_{a_{11}}^1 E_{a_{12}}^2 \cdots E_{a_{1e}}^e \\
 A_2 &= E_{a_{21}}^1 E_{a_{22}}^2 \cdots E_{a_{2e}}^e \\
 &\vdots \\
 A_s &= E_{a_{s1}}^1 E_{a_{s2}}^2 \cdots E_{a_{se}}^e
 \end{aligned}$$

in which  $a_{jl}$  is the formula number for species  $j$  corresponding to element  $l$ . Show that any chemical reaction,  $\sum_j v_j A_j = 0$  satisfies the following  $e$  equations to balance the elements

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_s \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1e} \\ a_{21} & a_{22} & \cdots & a_{2e} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{se} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$$

If we define the  $s \times e$  matrix  $\mathcal{A}$  to hold the formula numbers, we can express the element balance by

$$v\mathcal{A} = 0 \quad (2.67)$$

Determine  $A$ ,  $E$  and  $\mathcal{A}$  for the species hydrogen (molecular), oxygen (molecular) and water.

**Exercise 2.9: Null space and fundamental theorem of linear algebra**

Equation 2.67 in Exercise 2.8 is begging to be analyzed by the fundamental theorem of linear algebra [7], so we explore that concept here. Consider an arbitrary  $m \times n$  matrix,  $B$ . The null space of matrix  $B$ , written  $\mathcal{N}(B)$ , is defined to be the set of all vectors  $x$  such that  $Bx = 0$ . The dimension of  $\mathcal{N}(B)$  is the number of linearly independent vectors  $x$  satisfying  $Bx = 0$ . One of the remarkable results of the fundamental theorem of linear algebra is the relation

$$\text{rank}(B) + \dim(\mathcal{N}(B)) = n$$

Armed with this result, consider Equation 2.67 and establish that the number of linearly independent reactions that satisfy the element balances is

$$i = s - \text{rank}(\mathcal{A})$$

The numerical support for computing null spaces is excellent. For example, the Octave command `null(B)` returns a matrix with columns consisting of linearly independent vectors in the null space of matrix  $B$ .

- (a) Determine  $A$ ,  $E$  and  $\mathcal{A}$  for the methane oxidation in Example 2.7.

- (b) Determine  $\text{rank}(\mathcal{A})$  and  $i$  for this example. Do you obtain the same size maximal set as in Example 2.7?

### Exercise 2.10: Limits on numbers of independent reactions

- (a) Species A, B and C are observed in a reacting system. What is the largest possible number of linearly independent reactions among these species,  $n_{\max}$ ?
- (b) What is the smallest possible number,  $n_{\min}$ ?
- (c) List a chemical example that has a maximal set with  $n$  reactions for each  $n$  in  $n_{\min} \leq n \leq n_{\max}$ .

### Exercise 2.11: Generating possible independent reaction sets

- (a) If one has  $s$  species comprised of  $e$  elements, what are the largest and smallest numbers of linearly independent reactions that can be written among the  $s$  species? Give an example with more than one species in which you cannot write any reactions.
- (b) If  $s > e$ , the usual case, how many reactions can be written? Prove you can write at least one valid reaction, or find a counterexample to this statement.

### Exercise 2.12: An alchemist's view of stoichiometry

In the early 1700s, the alchemists were unaware of the defining role that electrons played in bond formation and transformation of chemical species. The structure of the nucleus, elucidation of bond formation with electron sharing and the construction of the periodic table were major triumphs of the chemical and physical sciences in the nineteenth and early twentieth centuries [6, pp.113-166]. Before these developments, chemical transformation of the elements themselves, such as lead to gold, was on the table as an early goal of research.

Imagine a more flexible chemical world in which we demand conservation of *only* mass for a valid chemical reaction; any mass-conserving rearrangements of the protons and neutrons in the nuclei are also considered valid chemical reactions.

Assume we observe  $\text{H}_2$ ,  $\text{O}_2$  and  $\text{H}_2\text{O}$  as chemical species.

- (a) Write a few *mass-conserving* reactions among these species that are not valid chemical reactions.
- (b) How many linearly independent mass-conserving reactions can be constructed? You may want to use the result of Example 2.5 and the idea of

the null space introduced in Exercise 2.9 to be sure you have accounted for all linearly independent reactions.

- (c) If you enforce the stronger condition of element balancing, then, by inspection, how many linearly independent valid chemical reactions are possible?

### Exercise 2.13: Generalizing independent reactions

Consider  $s$  species comprised of  $e$  elements, in which  $c$  of the species are chemical compounds, i.e., not pure elements. Let  $i$  be the number of reactions in the maximal independent set. Show

$$\max(0, s - e) \leq i \leq \min(s - 1, c)$$

Hint: use the result  $i = s - \text{rank}(\mathcal{A})$  from Exercise 2.9 and determine upper and lower bounds on the rank of  $\mathcal{A}$  based on  $s$ ,  $e$  and  $c$ .

We should point out that these limits on numbers of reactions in Exercises 2.8-2.13 presume that conserving elements in the chemical reactions is the *only* restriction. If other quantities are conserved, the method of analysis remains valid, but the answers change. We will see one such conserved quantity, the surface site, when we study reactions with catalytic surfaces in Chapter 5.

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## 3

### Review of Chemical Equilibrium

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#### 3.1 Introduction

This chapter is intended to remind the reader of some important facts concerning the equilibrium state of a system that can undergo chemical reaction. It is by design a review, and assumes the reader is familiar with the fundamental concepts of classical thermodynamics as discussed in the many introductory textbooks on the subject [14, 5, 9, 16, 4].

Although essentially all reactors in operation are *not* at chemical equilibrium, it is still important to understand what equilibrium is and how to draw quantitative conclusions from considering the equilibrium state of a system. Chemical reactors are not at equilibrium because the net rates of chemical reactions are seldom allowed to become zero. However, the calculation of the equilibrium state can allow one to make conclusions about the limits of reactor performance that would be achievable if this state were reached. In addition, the consideration of the thermodynamic restrictions on reactor performance may motivate one to make operations or design changes that allow these restrictions to be changed and reactor performance improved. For these reasons we begin with a review of chemical equilibrium.

Consider the single-phase, multicomponent system depicted in Figure 3.1. It is convenient for our purposes to describe the state of this system by the variables temperature,  $T$ , pressure,  $P$ , and the number of moles of each component,  $n_j$ . Specifying the temperature, pressure,

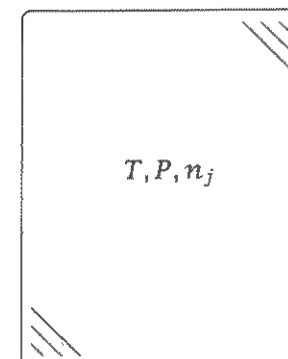


Figure 3.1: Thermodynamic system.

and number of moles of each component then completely specifies the equilibrium state of the system. The Gibbs energy of the system,  $G$ , is the energy function of these state variables that conveniently describes the condition of equilibrium. In particular the equilibrium state has a minimum Gibbs energy with respect to any changes in the number of moles at fixed  $T$  and  $P$ . The difference in Gibbs energy between two states at slightly different temperatures, pressures, and number of moles of each component is developed in standard textbooks to be

$$dG = -SdT + VdP + \sum_j \mu_j dn_j \quad (3.1)$$

in which  $S$  is the system entropy,  $V$  is the system volume and  $\mu_j$  is the chemical potential of component  $j$ .

### 3.2 Condition for Reaction Equilibrium

Consider a closed system in which the  $n_j$  can change only by the single chemical reaction,

$$\sum_j \nu_j A_j = 0$$

Given the definition of reaction extent in Chapter 2, for a small change in the extent of this single reaction, the numbers of moles of each component change by

$$dn_j = \nu_j d\varepsilon$$

Substituting this relation into Equation 3.1 gives an expression for the change in the Gibbs energy due to a small change in the reaction extent,

$$dG = -SdT + VdP + \sum_j (\nu_j \mu_j) d\varepsilon \quad (3.2)$$

For the closed system,  $G$  is only a function of  $T$ ,  $P$  and  $\varepsilon$ . The expansion of a total differential in terms of the partial derivatives allows one to deduce from Equation 3.2

$$S = -\left(\frac{\partial G}{\partial T}\right)_{P,\varepsilon} \quad (3.3)$$

$$V = \left(\frac{\partial G}{\partial P}\right)_{T,\varepsilon} \quad (3.4)$$

$$\sum_j \nu_j \mu_j = \left(\frac{\partial G}{\partial \varepsilon}\right)_{T,P} \quad (3.5)$$

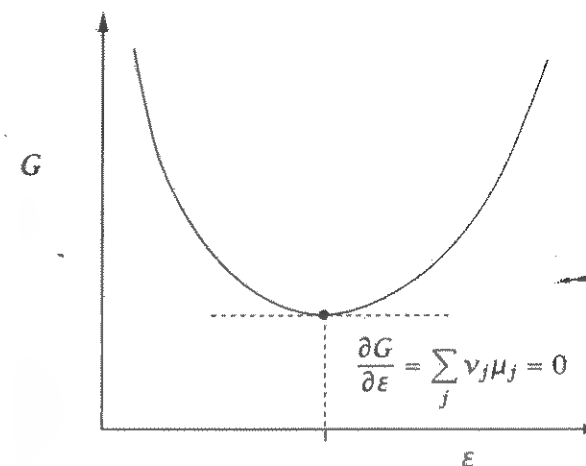


Figure 3.2: Gibbs energy versus reaction extent at constant  $T$  and  $P$ .

Consider a plot of  $G$  at some fixed  $T$  and  $P$  versus the reaction extent, sketched in Figure 3.2. A necessary condition for the Gibbs energy to be a minimum is that the derivative be zero, or in light of Equation 3.5,

$$\boxed{\sum_j \nu_j \mu_j = 0} \quad (3.6)$$

The direct use of Equation 3.6 to compute equilibrium composition is inconvenient because chemical potential is often expressed in terms of other quantities such as activity or fugacity. One can express  $\mu_j$  in terms of the activity of component  $j$ ,

$$\mu_j = G_j^\circ + RT \ln a_j \quad (3.7)$$

in which  $a_j$  is the activity of component  $j$  in the mixture referenced to some standard state and  $G_j^\circ$  is the Gibbs energy of component  $j$  in the same standard state. The activity and fugacity of component  $j$  are related by

$$a_j = \frac{f_j}{f_j^\circ}$$

in which  $f_j$  is the fugacity of component  $j$  and  $f_j^\circ$  is the fugacity of component  $j$  in the standard state.

**Standard state.** It is common convention in reaction equilibrium calculations to use the following standard state for gaseous, liquid, and solid mixtures: pure component  $j$  at 1.0 atm pressure and the system temperature. Sometimes the gaseous mixture standard state is defined to be the pressure corresponding to unit fugacity, but this pressure is close to 1.0 atm for most gases. It is obvious from the definition of the standard state that  $G_j^\circ$  and  $f_j^\circ$  are not functions of the system pressure or composition, because the standard state is 1.0 atm pressure and pure component  $j$ .  $G_j^\circ$  and  $f_j^\circ$  are, however, strong functions of the system temperature.

Multiplying both sides of Equation 3.7 by  $\nu_j$  and summing yields

$$\sum_j \nu_j \mu_j = \sum_j \nu_j G_j^\circ + RT \sum_j \nu_j \ln a_j \quad (3.8)$$

The term  $\sum_j \nu_j G_j^\circ$  is known as the standard Gibbs energy change for the reaction,  $\Delta G^\circ$ . Using the equilibrium condition, Equation 3.6, and simple logarithm addition and exponentiation identities yields

$$\Delta G^\circ + RT \ln \prod_j a_j^{\nu_j} = 0 \quad (3.9)$$

Equation 3.9 motivates the definition of the equilibrium constant,  $K$ ,

$$K = e^{-\Delta G^\circ/RT} \quad (3.10)$$

Rearrangement of Equation 3.9 and exponentiation leads to the following simple condition for chemical equilibrium

$$K = \prod_j a_j^{\nu_j} \quad (3.11)$$

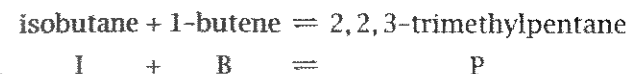
From the definitions of  $K$  (Equation 3.10) and the standard state, it is clear that  $K$  also is a function of the system temperature, but not a function of the system pressure or composition.

The following example illustrates these concepts and the calculation of the equilibrium composition in the simplest possible setting, an ideal-gas mixture.

### Example 3.1: Ideal-gas equilibrium

The reaction of isobutane and linear butenes to branched  $C_8$  hydrocarbons is used to synthesize high octane fuel additives. One such

reaction is



Determine the equilibrium composition for this system at a pressure of 2.5 atm and temperature of 400 K. You may assume the gases are ideal at this temperature and pressure and this reaction is the only one that occurs. The standard Gibbs energy change for this reaction at 400 K is  $-3.72$  kcal/mol [17].

### Solution

We recall that the fugacity of a component in an ideal-gas mixture is equal to its partial pressure,

$$f_j = P_j \quad (3.12)$$

The quantity  $f_j^\circ$  is the fugacity of component  $j$  in the standard state, which is pure component  $j$  at 1.0 atm pressure and the system temperature, 400 K. From Equation 3.12,  $f_j^\circ = 1.0$  atm because the partial pressure of a pure component  $j$  at 1.0 atm total pressure is 1.0 atm. The activity of component  $j$  is then simply

$$a_j = \frac{P_j}{1 \text{ atm}} \quad (3.13)$$

Notice that when we suppress the 1.0 atm in Equation 3.13 in subsequent discussion, the units of  $P_j$  have to be atm or  $a_j$  is not dimensionless as it should be.

We now attempt to calculate the equilibrium composition from Equation 3.11,

$$K = \frac{a_P}{a_I a_B} \quad (3.14)$$

in which  $P$  represents the trimethylpentane,  $I$  represents isobutane, and  $B$  represents the butene. Since  $\Delta G^\circ$  is given at the system temperature, we can simply calculate  $K$  at 400 K from Equation 3.10,  $K = 108$ . Substituting the activity of the components in Equation 3.14 then gives

$$K = \frac{P_P}{P_I P_B}$$

If we choose to use the mole fractions of the components,  $y_j$ , as the unknown variables rather than the partial pressures, we have

$$K = \frac{y_P}{y_I y_B P} \quad (3.15)$$



in which  $P$  is 2.5 atm. The problem is that we have three unknowns, the gas-phase mole fractions at equilibrium, and only one equation, Equation 3.15. We can of course add the equation that mole fractions sum to one,

$$\sum_j y_j = 1$$

but we still have three unknowns and only two equations. In other words, the equilibrium composition cannot be determined with the information given. All we know at this point is that any set of mole fractions satisfying Equation 3.15 (and summing to one) could be an equilibrium composition. It is important to realize that the mere statement that a system is at equilibrium at a certain temperature and pressure does not necessarily tell one everything about the system.  $\square$

The next example illustrates what kind of additional information is required to be able to find the equilibrium composition.

### Example 3.2: Ideal-gas equilibrium, revisited

Compute the equilibrium composition of Example 3.1 with the following additional information. The gas is contained in a closed vessel that is initially charged with an equimolar mixture of isobutane and butene.

#### Solution

We show how the initial composition of the gas in a closed vessel allows us to determine the equilibrium composition. Notice in particular that we do not know the volume of the vessel or the initial number of moles, just that the vessel initially contains an equimolar mixture of the two reactants. We proceed to describe the mole fractions of all components in terms of a single variable, the extent of reaction. Let  $n_{j0}$  represent the unknown number of moles of each component initially contained in the vessel. Since the vessel is closed, the number of moles at some other time can be related to the reaction extent from the stoichiometry using  $n_j = n_{j0} + \nu_j \varepsilon$ ,

$$n_I = n_{I0} - \varepsilon, \quad n_B = n_{B0} - \varepsilon, \quad n_P = n_{P0} + \varepsilon \quad (3.16)$$

Summing Equations 3.16 produces

$$n_T = n_{T0} - \varepsilon$$

in which  $n_T$  is the total number of moles in the vessel. The total number of moles decreases with reaction extent because more moles are

consumed than produced by the reaction. Dividing both sides of Equations 3.16 by  $n_T$  produces equations for the mole fractions in terms of the reaction extent,

$$y_I = \frac{n_{I0} - \varepsilon}{n_{T0} - \varepsilon} \quad y_B = \frac{n_{B0} - \varepsilon}{n_{T0} - \varepsilon} \quad y_P = \frac{n_{P0} + \varepsilon}{n_{T0} - \varepsilon} \quad (3.17)$$

Dividing top and bottom of the right-hand side of the previous equations by  $n_{T0}$  yields,

$$y_I = \frac{y_{I0} - \varepsilon'}{1 - \varepsilon'} \quad y_B = \frac{y_{B0} - \varepsilon'}{1 - \varepsilon'} \quad y_P = \frac{y_{P0} + \varepsilon'}{1 - \varepsilon'} \quad (3.18)$$

in which  $\varepsilon' = \varepsilon/n_{T0}$  is a dimensionless reaction extent that is scaled by the initial total number of moles. Notice that the mole fractions are now expressed in terms of the initial mole fractions, which are known, and a reaction extent, which is the single unknown. Notice also that the mole fractions in Equations 3.18 sum to one for all  $\varepsilon'$ . Substituting Equations 3.18 into Equation 3.15 then provides the single equation to determine  $\varepsilon'$ ,

$$K = \frac{(y_{P0} + \varepsilon')(1 - \varepsilon')}{(y_{B0} - \varepsilon')(y_{I0} - \varepsilon')P}$$

Multiplying through by the denominator and collecting terms gives

$$(y_{B0} - \varepsilon')(y_{I0} - \varepsilon')KP - (y_{P0} + \varepsilon')(1 - \varepsilon') = 0$$

The equilibrium equation is quadratic in  $\varepsilon'$ . Substituting in the given initial composition,  $y_{P0} = 0$ ,  $y_{B0} = y_{I0} = 1/2$  gives

$$\varepsilon'^2(1 + KP) - \varepsilon'(1 + KP) + (1/4)KP = 0$$

The two solutions are

$$\varepsilon' = \frac{1 \pm \sqrt{1 + KP}}{2} \quad (3.19)$$

The correct solution is chosen by considering the physical constraints that mole fractions must be positive. If the positive sign is chosen in Equation 3.19, the reaction extent is greater than 1/2, which is physically impossible because it would result in negative numbers of moles of the reactants. The negative sign is therefore chosen, and the solution is  $\varepsilon' = 0.469$ . The equilibrium mole fractions are then computed from Equation 3.18 giving

$$y_I = 5.73 \times 10^{-2} \quad y_B = 5.73 \times 10^{-2} \quad y_P = 0.885$$

The equilibrium at 400 K favors the product trimethylpentane. The reaction has moved from the initial condition of pure reactants to an equilibrium state of relatively pure product.  $\square$

**Second derivative of  $G$ .** Before leaving the equilibrium of ideal gases, it is instructive to show that the equilibrium state is in fact a minimum in the Gibbs energy and that this minimum is unique. The procedure in the preceding example determined the value of the reaction extent for which  $\partial G/\partial \varepsilon = 0$ . To show that this value corresponds to a minimum (and not a maximum or inflection), we would like to show  $\partial^2 G/\partial \varepsilon^2 > 0$  at this value of reaction extent as depicted in Figure 3.2. If we can show that  $\partial^2 G/\partial \varepsilon^2 > 0$  for every  $\varepsilon$ , then the minimum is unique.

We begin by performing the same steps we used to determine the mole fractions in Example 3.2, only for an arbitrary number of components. Recall that for the closed system undergoing a single reaction, we can describe the number of moles by

$$n_j = n_{j0} + \nu_j \varepsilon \geq 0 \quad (3.20)$$

Since the reaction extent only has physical significance when it corresponds to a nonnegative number of moles for all components, we have the inequality in Equation 3.20. Summing over all components yields

$$n_T = n_{T0} + \bar{\nu} \varepsilon > 0 \quad (3.21)$$

in which  $\bar{\nu} = \sum_j \nu_j$ . Notice  $\bar{\nu} = 0$  if moles are conserved in the reaction. The total number of moles is strictly positive to preclude the case of no material in the system, an uninteresting case. Dividing Equation 3.20 by Equation 3.21 gives

$$y_j = \frac{n_{j0} + \nu_j \varepsilon}{n_{T0} + \bar{\nu} \varepsilon} \quad (3.22)$$

Defining the dimensionless extent as before,

$$\varepsilon' = \varepsilon/n_{T0} \quad (3.23)$$

Equation 3.22 can be written as

$$y_j = \frac{y_{j0} + \nu_j \varepsilon'}{1 + \bar{\nu} \varepsilon'} \quad (3.24)$$

We remark in passing for later use that

$$\frac{n_T}{n_{T0}} = 1 + \bar{\nu} \varepsilon' > 0 \quad (3.25)$$

We now proceed to calculate  $\partial^2 G/\partial \varepsilon^2$ . We start with Equation 3.5, repeated here,

$$\left(\frac{\partial G}{\partial \varepsilon}\right)_{T,P} = \sum_j \nu_j \mu_j \quad (3.26)$$

From Equation 3.23, we can express the derivative with respect to the dimensionless reaction extent,

$$\frac{1}{n_{T0}} \left(\frac{\partial G}{\partial \varepsilon'}\right)_{T,P} = \sum_j \nu_j \mu_j \quad (3.27)$$

Differentiation with respect to  $\varepsilon'$  yields

$$\frac{1}{n_{T0}} \left(\frac{\partial^2 G}{\partial \varepsilon'^2}\right)_{T,P} = \sum_j \nu_j \left(\frac{\partial \mu_j}{\partial \varepsilon'}\right)_{T,P} \quad (3.28)$$

The chemical potential of a component in an ideal gas is

$$\mu_j = G_j^\circ + RT \ln(P_j/1 \text{ atm}) = G_j^\circ + RT [\ln y_j + \ln P] \quad (3.29)$$

Substituting Equation 3.24 into Equation 3.29 yields

$$\mu_j = G_j^\circ + RT [\ln(y_{j0} + \nu_j \varepsilon') - \ln(1 + \bar{\nu} \varepsilon') + \ln P]$$

Differentiating this equation with respect to  $\varepsilon'$  ( $T$  and  $P$  are constant) gives

$$\left(\frac{\partial \mu_j}{\partial \varepsilon'}\right)_{T,P} = RT \left[ \frac{\nu_j}{y_{j0} + \nu_j \varepsilon'} - \frac{\bar{\nu}}{1 + \bar{\nu} \varepsilon'} \right]$$

Substituting into Equation 3.28 gives

$$\frac{1}{n_{T0} RT} \left(\frac{\partial^2 G}{\partial \varepsilon'^2}\right)_{T,P} = \sum_j \nu_j \left[ \frac{\nu_j}{y_{j0} + \nu_j \varepsilon'} - \frac{\bar{\nu}}{1 + \bar{\nu} \varepsilon'} \right]$$

Performing the summation on the two terms yields

$$\frac{1}{n_{T0} RT} \left(\frac{\partial^2 G}{\partial \varepsilon'^2}\right)_{T,P} = \sum_j \frac{\nu_j^2}{y_{j0} + \nu_j \varepsilon'} - \frac{\bar{\nu}^2}{1 + \bar{\nu} \varepsilon'}$$

Multiplying through by  $1 + \bar{\nu} \varepsilon'$  gives

$$\frac{1 + \bar{\nu} \varepsilon'}{n_{T0} RT} \left(\frac{\partial^2 G}{\partial \varepsilon'^2}\right)_{T,P} = \sum_j \frac{\nu_j^2}{y_{j0} + \nu_j \varepsilon'} (1 + \bar{\nu} \varepsilon') - \bar{\nu}^2 \quad (3.30)$$

Although Equation 3.30 may not appear very useful, if we define two variables,

$$a_j = \frac{\nu_j}{\sqrt{y_{j0} + \nu_j \varepsilon'}} \quad b_j = \sqrt{y_{j0} + \nu_j \varepsilon'}$$

Equation 3.30 can be written as

$$\frac{1 + \bar{\nu}\epsilon'}{n_{T0}RT} \left( \frac{\partial^2 G}{\partial \epsilon'^2} \right)_{T,P} = \sum_j a_j^2 \sum_j b_j^2 - \left( \sum_j a_j b_j \right)^2 \quad (3.31)$$

The Cauchy inequality [2], also known as the Schwarz inequality, states that for any two vectors **a** and **b**

$$\sum_j a_j b_j \leq \left( \sum_j a_j^2 \right)^{1/2} \left( \sum_j b_j^2 \right)^{1/2} \quad (3.32)$$

The equality is achieved if and only if the two vectors are nonnegatively proportional ( $a_j = kb_j$  for every  $j$ , with  $k \geq 0$ ). Since  $a_j = \frac{v_j}{y_{j0} + v_j \epsilon'} b_j$ , it is clear that the vectors **a** and **b** are not proportional, and strict inequality holds in Equation 3.31. Since  $1 + \bar{\nu}\epsilon' > 0$  from Equation 3.25 and  $n_{T0}$ ,  $R$  and  $T$  are positive, we can conclude

$$\left( \frac{\partial^2 G}{\partial \epsilon'^2} \right)_{T,P} > 0 \quad (3.33)$$

Due to the definition of  $\epsilon'$ , Equation 3.33 is equivalent to

$$\left( \frac{\partial^2 G}{\partial \epsilon^2} \right)_{T,P} > 0$$

which is what we set out to show. Notice that this inequality is true for every  $\epsilon$  corresponding to physically meaningful compositions. Since the  $G$  function has positive curvature at every  $\epsilon$ , Figure 3.2 is indeed the correct picture for an ideal gas, and the equilibrium state is unique. Aris provides an alternative treatment of this same question [1].

**Evaluation of  $G$ .** As the last example on the equilibrium of an ideal gas, we calculate the actual function  $G(T, P, \epsilon')$ . As we shall see, the equilibrium extent then can be determined by plotting  $G$  versus  $\epsilon'$ .

The Gibbs energy of a single phase can be computed from the chemical potentials and mole numbers of the components from (see Exercise 3.6 for a derivation)

$$G = \sum_j \mu_j n_j \quad (3.34)$$

The chemical potential of a component in an ideal gas is given by Equation 3.29, repeated here,

$$\mu_j = G_j^\circ + RT [\ln y_j + \ln P]$$

Substituting the above into Equation 3.34 gives

$$G = \sum_j n_j G_j^\circ + RT \sum_j n_j [\ln y_j + \ln P] \quad (3.35)$$

Note that for this single reaction case,  $n_j = n_{j0} + v_j \epsilon$ , which gives

$$\sum_j n_j G_j^\circ = \sum_j n_{j0} G_j^\circ + \epsilon \Delta G^\circ \quad (3.36)$$

We now define a modified Gibbs function to measure changes from the initial standard Gibbs energy and to scale by the temperature and the total initial moles as follows,

$$\tilde{G}(T, P, \epsilon') = \frac{G - \sum_j n_{j0} G_j^\circ}{n_{T0} RT} \quad (3.37)$$

Substituting Equations 3.35 and 3.36 into Equation 3.37 gives

$$\tilde{G} = \epsilon' \frac{\Delta G^\circ}{RT} + \sum_j \frac{n_j}{n_{T0}} [\ln y_j + \ln P] \quad (3.38)$$

From Equation 3.20, the ratio  $n_j/n_{T0}$  is easily seen to be equal to  $y_{j0} + v_j \epsilon'$ . Substituting this relation, the definition of  $K$  and Equation 3.24 into Equation 3.38 yields

$$\tilde{G} = -\epsilon' \ln K + \sum_j (y_{j0} + v_j \epsilon') \left[ \ln \left( \frac{y_{j0} + v_j \epsilon'}{1 + \bar{\nu} \epsilon'} \right) + \ln P \right]$$

which can be rearranged to

$$\tilde{G} = -\epsilon' \ln K + (1 + \bar{\nu} \epsilon') \ln P + \sum_j (y_{j0} + v_j \epsilon') \ln \left( \frac{y_{j0} + v_j \epsilon'}{1 + \bar{\nu} \epsilon'} \right) \quad (3.39)$$

Notice that because  $T$  and  $P$  are known values,  $\tilde{G}$  is simply a shift of the  $G$  function up or down by a constant and then rescaling by the positive constant  $1/(n_{T0} RT)$ . In particular, the shape of the function  $\tilde{G}$  is the same as  $G$ , and the minimum with respect to  $\epsilon'$  is at the same value of  $\epsilon'$  for the two functions.

### Example 3.3: Minimum in $G$ for an ideal gas

Plot the  $\tilde{G}$  function for Example 3.2 and show that  $\epsilon'$  corresponding to the minimum of  $\tilde{G}$  agrees with the result of Example 3.2.

## Solution

We apply Equation 3.39 to the chemistry of Example 3.2,



For this stoichiometry,  $\sum_j \nu_j = \bar{\nu} = -1$ . The reaction started with an equimolar mixture of reactants, so  $y_{P0} = 0$ ,  $y_{I0} = y_{B0} = 0.5$ . Substituting these values into Equation 3.39 gives

$$\begin{aligned} \tilde{G}(T, P, \epsilon') = & -\epsilon' \ln K(T) + (1 - \epsilon') \ln P + \\ & \epsilon' \ln(\epsilon') + 2(0.5 - \epsilon') \ln(0.5 - \epsilon') - (1 - \epsilon') \ln(1 - \epsilon') \end{aligned} \quad (3.41)$$

Recall that the range of physically significant  $\epsilon'$  values is

$$0 \leq \epsilon' \leq 0.5$$

The equilibrium constant at 400 K has been computed to be  $K = 108$ . The pressure is 2.5 atm. Recall we must use atm for the pressure units because of the way we expressed chemical potential in Equation 3.29. Substituting these values into Equation 3.41 and plotting the results gives Figure 3.3. Because the minimum is close to  $\epsilon' = 0.5$  we see again that the equilibrium favors the product. Magnifying this region of the plot in Figure 3.3 shows that  $\epsilon' = 0.47$  at equilibrium, which is in good agreement with the value 0.469 calculated in Example 3.2. Although we prefer the simple calculation in Example 3.2 to obtain a numerical value for  $\epsilon'$ , the graphical representation of  $\tilde{G}$  is more informative on issues of whether or not the solution is indeed a minimum and whether or not the minimum is unique. It also gives us confidence that the analysis of the previous section is correct because  $\tilde{G}$  has positive curvature for all  $\epsilon'$  of physical interest.  $\square$

Notice from Equation 3.39 that for an ideal gas, the pressure enters directly in the Gibbs energy with the  $\ln P$  term. The effect of pressure on the equilibrium extent depends on the sign of  $\bar{\nu}$ . If positive, which means there are more product than reactant molecules in the stoichiometry of the reaction, then an increase in the pressure causes a decrease in the equilibrium reaction extent, in agreement with Le Chatelier's principle. This can be verified by examining the term  $(1 + \epsilon'\bar{\nu}) \ln P$  in Equation 3.39. If  $\bar{\nu}$  is positive, increasing the pressure increases this term more at larger values of  $\epsilon'$  than at smaller values, thus pushing the minimum in the  $\tilde{G}$  function to smaller values of  $\epsilon$ . Hence for  $\bar{\nu} > 0$  an

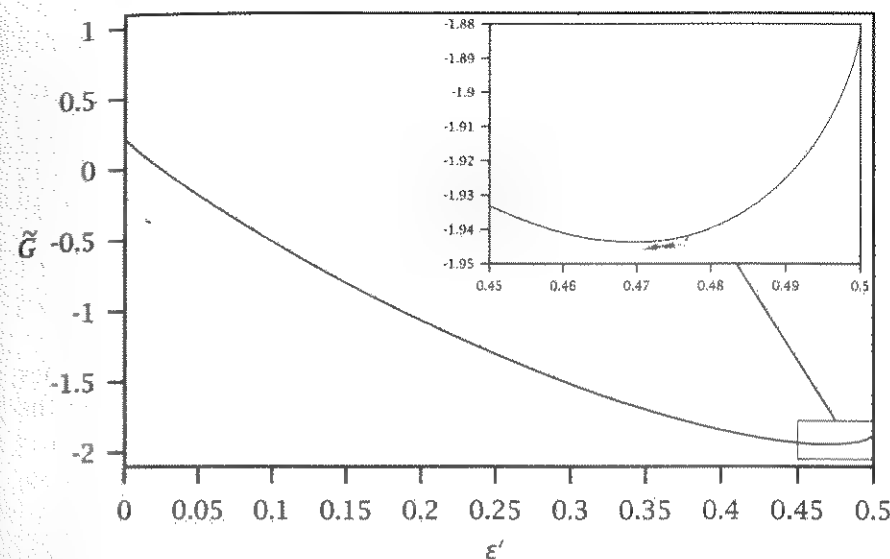


Figure 3.3: Gibbs energy  $\tilde{G}$  versus reaction extent  $\epsilon'$ .

equilibrium shift favoring reactants is the effect of a pressure increase. If  $\bar{\nu}$  is negative, the opposite situation arises. For  $\bar{\nu} = 0$ , the pressure does not affect the equilibrium, because the term  $(1 + \epsilon'\bar{\nu}) \ln P$  is then constant with  $\epsilon'$  and does not affect the location of the minimum. For single liquid-phase or solid-phase systems, the effect of pressure on equilibrium is usually small, because the chemical potential of a component in a liquid-phase or solid-phase solution is usually a weak function of pressure.

The temperature effect on the Gibbs energy is contained in the  $\ln K(T)$  term. This term often gives rise to a large effect of temperature on equilibrium. We turn our attention to the evaluation of this important temperature effect in the next section.

### 3.2.1 Evaluation of the Gibbs Energy Change of Reaction

We usually calculate the standard Gibbs energy change for the reaction,  $\Delta G^\circ$ , by using the Gibbs energy of formation of the species, which are commonly available thermochemical data. The standard Gibbs energy of formation of species  $j$  is defined as the difference in the Gibbs energy

of the species and its constituent elements in some standard state.

$$G_{jf}^\circ = G_j^\circ - \sum_{l=1}^{n_e} a_{jl} G_{El}^\circ \quad (3.42)$$

The standard state for the elements are usually the pure elements in their common form at 25°C and 1.0 atm. For example, the Gibbs energy of formation of H<sub>2</sub>O is

$$G_{\text{H}_2\text{O}f}^\circ = G_{\text{H}_2\text{O}}^\circ - G_{\text{H}_2}^\circ - \frac{1}{2} G_{\text{O}_2}^\circ \quad (3.43)$$

We then take the appropriate linear combinations of the species Gibbs energy of formation to compute the Gibbs energy change for the reaction at 25°C

$$\Delta G_i^\circ = \sum_j \nu_{ij} G_{jf}^\circ \quad (3.44)$$

As shown in Exercise 3.11, the Gibbs energy of the elements cancel, leading to Equation 3.44.

Finding appropriate thermochemical data remains a significant challenge for solving realistic, industrial problems. Vendors offer a variety of commercial thermochemical databases to address this need. Many companies also maintain their own private thermochemical databases for compounds of special commercial interest to them.<sup>1</sup>

The standard state temperature 25°C is often not the temperature of the system of interest, so we need to convert from 25°C to the system temperature. To accomplish this conversion, the temperature dependence of  $\Delta G^\circ$  is discussed in the next section.

### 3.2.2 Temperature Dependence of the Standard Gibbs Energy

Recall from Equation 3.3 that the change of the Gibbs energy with temperature is the negative of the entropy,

$$\left(\frac{\partial G}{\partial T}\right)_{P,n_j} = -S$$

<sup>1</sup>For educational purposes and to gain some experience with thermochemical databases, you may wish to try the Design Institute for Physical Property Data (DIPPR) database. A web-based student version of the database provides students with access to data for 100 common compounds at no charge: <http://dippr.byu.edu/students/chemsearch.asp>. The list of included compounds is available at: <http://dippr.byu.edu/Student100.txt>.

This expression can be used to evaluate the change in the Gibbs energy in the standard state, pure component  $j$  at a pressure of 1.0 atm,

$$\left(\frac{\partial G_j^\circ}{\partial T}\right)_{P,n_j} = -S_j^\circ \quad (3.45)$$

Notice in this context that the subscripts  $P$  and  $n_j$  denoting constant pressure and composition are redundant due to the definition of the standard state and are dropped to avoid confusion. Summing Equation 3.45 with the stoichiometric coefficients of a single reaction gives

$$\sum_j \frac{\partial(\nu_j G_j^\circ)}{\partial T} = \sum_j -\nu_j S_j^\circ$$

Defining the term on the right-hand side to be the standard entropy change of reaction,  $\Delta S^\circ$  gives

$$\frac{\partial \Delta G^\circ}{\partial T} = -\Delta S^\circ \quad (3.46)$$

Let  $H$  denote the enthalpy and recall its connection to the Gibbs energy,

$$G = H - TS \quad (3.47)$$

**Partial molar properties.** Recall the definition of a partial molar property is

$$X_j = \left(\frac{\partial X}{\partial n_j}\right)_{T,P,n_k}$$

in which  $X$  is any extensive mixture property ( $U, H, A, G, V, S$ , etc.). In other words, the partial molar property of component  $j$  measures the change in an extensive property as a small amount of component  $j$  is added to the mixture at constant  $T, P$  and other mole numbers.

Equation 3.47 is valid also for the corresponding partial molar quantities

$$G_j = H_j - TS_j \quad (3.48)$$

Writing Equation 3.48 for a component in its standard state gives  $G_j^\circ = H_j^\circ - TS_j^\circ$ , and summing with the stoichiometric coefficient yields

$$\Delta G^\circ = \Delta H^\circ - T\Delta S^\circ \quad (3.49)$$

Solving Equation 3.49 for the entropy change and substituting into Equation 3.46 gives

$$\frac{\partial \Delta G^\circ}{\partial T} = \frac{\Delta G^\circ - \Delta H^\circ}{T}$$

Rearranging this equation and division by  $RT$  gives

$$\frac{1}{RT} \frac{\partial \Delta G^\circ}{\partial T} - \frac{\Delta G^\circ}{RT^2} = -\frac{\Delta H^\circ}{RT^2}$$

Using differentiation formulas, the left-hand side can be rewritten as

$$\frac{\partial \left( \frac{\Delta G^\circ}{RT} \right)}{\partial T} = -\frac{\Delta H^\circ}{RT^2}$$

which finally can be expressed in terms of the equilibrium constant

$$\boxed{\frac{\partial \ln K}{\partial T} = \frac{\Delta H^\circ}{RT^2}} \quad (3.50)$$

Equation 3.50, known as the van't Hoff equation, expresses how the equilibrium constant varies with temperature. Equation 3.50 can be integrated between two temperatures to give

$$\int_{T_1}^{T_2} \frac{\partial \ln K}{\partial T} dT = \int_{T_1}^{T_2} \frac{\Delta H^\circ}{RT^2} dT$$

To evaluate the integral one needs to have the standard enthalpy change as a function of temperature. This information can be computed from the partial molar heat capacities. The partial molar heat capacity for component  $j$  is defined by

$$C_{P,j} = \left( \frac{\partial H_j}{\partial T} \right)_{P,n_k} \quad (3.51)$$

Evaluating this expression at the standard state, multiplying by the stoichiometric numbers and summing produces

$$\Delta C_P^\circ = \frac{\partial \Delta H^\circ}{\partial T}$$

in which  $\Delta C_P^\circ$  is the heat capacity change upon reaction

$$\Delta C_P^\circ = \sum_j \nu_j C_{P,j}^\circ$$

The standard enthalpy change between a reference temperature  $T_0$  and an arbitrary temperature  $T$  can be evaluated from

$$\Delta H^\circ(T) = \Delta H^\circ(T_0) + \int_{T_0}^T \Delta C_P dT$$

If the standard heat capacity change of reaction is zero over the temperature range, then  $\Delta H^\circ$  is constant and Equation 3.50 can be integrated directly to give

$$\ln \left( \frac{K_2}{K_1} \right) = -\frac{\Delta H^\circ}{R} \left( \frac{1}{T_2} - \frac{1}{T_1} \right) \quad (3.52)$$

Notice the  $H_j^\circ$  themselves may be strong functions of temperature, but if  $\Delta C_P^\circ = 0$ , then  $\Delta H^\circ$  is still independent of temperature and Equation 3.52 is correct. Equation 3.52 also may be a useful approximation, but the error in assuming constant standard change in enthalpy should be examined if heat capacity data are available.

### 3.3 Condition for Phase Equilibrium

Consider a multicomponent, multiphase system that is at equilibrium and denote two of the phases as  $\alpha$  and  $\beta$ . Let  $T^k$  and  $P^k$  represent the temperature and pressure of phase  $k$  and  $n_j^k$  represent the number of moles of component  $j$  in phase  $k$ . The chemical potential of component  $j$  in phase  $k$  is denoted by  $\hat{\mu}_j^k$ . The hat is to emphasize that this is the chemical potential of component  $j$  in the multicomponent mixture. The conditions for equilibrium when more than one phase are present are developed in standard texts to be

$$\begin{aligned} T^\alpha &= T^\beta \\ P^\alpha &= P^\beta \\ \hat{\mu}_j^\alpha &= \hat{\mu}_j^\beta, \quad j = 1, 2, \dots, n_s \end{aligned} \quad (3.53)$$

Since these equations apply to any two phases, the temperatures of all phases are equal and the distinction of temperatures for each phase is unnecessary and one can speak of the system temperature  $T$ . Similarly for the pressure. Most of our attention then is focused on the condition that the chemical potential of component  $j$  must be equal in all phases.

A loose but useful physical intuition is connected to the equilibrium conditions. If the temperatures of the phases of a two-phase mixture are not equal, one expects to see a transfer of heat from the hotter to the colder phase until the equilibrium condition is satisfied. Similarly if the pressures are not equal, one expects a momentum transfer from the system at higher pressure to the one at lower pressure. Finally, if the chemical potential of a component is higher in one phase than another phase, one expects a mass transfer of the component from the phase with higher chemical potential to the phase with lower chemical potential until the chemical potentials become equal. In reality, of course, the



gradients and fluxes do not separate so ideally and we observe the simultaneous transfer of heat, momentum and mass of all components while the system approaches equilibrium from some arbitrary initial condition [3, pp. 767-768].

Again, the chemical potential is often expressed in terms of the fugacity, which is defined by

$$\hat{\mu}_j = \mu_j^\circ + RT \ln \hat{f}_j \quad (3.54)$$

in which  $\mu_j^\circ$  denotes the chemical potential of component  $j$  in a standard state. The standard state is again pure component  $j$  at the system temperature and 1.0 atm pressure. If we express Equation 3.54 for two phases  $\alpha$  and  $\beta$  and equate their chemical potentials we deduce

$$\hat{f}_j^\alpha = \hat{f}_j^\beta \quad j = 1, 2, \dots, n_s \quad (3.55)$$

One can therefore use either the equality of chemical potentials or fugacities as the condition for equilibrium.

### 3.3.1 Ideal Mixtures

In this and the next section we briefly review the methods for expressing the fugacity of components in a mixture and calculating the phase equilibrium. The simplest rule for evaluating the fugacity of a component in a mixture is the ideal mixture assumption. We review this case first.

#### Gaseous Solutions

Consider a gas-phase mixture of interest at some temperature and pressure. Let  $f_j^G$  denote the fugacity of pure component  $j$  in the gas phase at the temperature and pressure of the mixture. The simplest mixing rule for calculating the fugacity of component  $j$  in the mixture containing component  $j$  at some mole fraction  $y_j$  is the following linear mixing rule

$$\hat{f}_j^G = f_j^G y_j \quad (\text{ideal mixture}) \quad (3.56)$$

An ideal gas, for example, obeys this mixing rule and furthermore the fugacity of pure  $j$  at the mixture  $T$  and  $P$  is simply the system pressure,  $f_j^G = P$ . The fugacity of component  $j$  in an ideal gas is therefore the partial pressure of component  $j$ .

$$\hat{f}_j^G = P_j = P y_j \quad (\text{ideal gas})$$

### Liquid (and Solid) Solutions

Again the simplest mixing rule for liquid (and solid) mixtures is that the fugacity of component  $j$  in the mixture is the fugacity of pure  $j$  at the temperature and pressure of the mixture times the mole fraction of  $j$  in the mixture. We let  $x_j$  denote the mole fraction to distinguish liquid-phase and solid-phase from gas-phase compositions.

$$\hat{f}_j^L = f_j^L x_j \quad (3.57)$$

This approximation is usually valid when the mole fraction of a component is near one. In a two-component mixture, one can establish from the Gibbs-Duhem relations that if the first component obeys the ideal mixture, then the second component follows Henry's law:

$$\hat{f}_2^L = k_2 x_2 \quad (3.58)$$

in which  $k_2$  is the Henry's law constant for the second component. Note that it is not generally equal to  $f_2^L$  (unless the mixture is ideal over the entire composition range).

**Fugacity pressure dependence.** Sometimes we know the fugacity of a component at one pressure and we need to evaluate it at another pressure. We start with the standard relation

$$\left( \frac{\partial \ln \hat{f}_j}{\partial P} \right)_{T, x_k} = \frac{V_j}{RT}$$

in which  $V_j$  is the partial molar volume. We integrate between the two pressures of interest to obtain

$$\hat{f}_j|_{P_2} = \hat{f}_j|_{P_1} \exp \left[ \frac{1}{RT} \int_{P_1}^{P_2} V_j dP \right] \quad (3.59)$$

For condensed phases, the partial molar volume is usually constant over reasonable pressure ranges giving

$$\hat{f}_j|_{P_2} = \hat{f}_j|_{P_1} \exp \left[ \frac{V_j(P_2 - P_1)}{RT} \right] \quad (3.60)$$

The exponential term in Equations 3.59 and 3.60 is called the Poynting correction factor. For liquids and solids, the partial molar volume is generally small, and the Poynting correction may be neglected if the pressure does not vary by a large amount.

### 3.3.2 Nonideal Mixtures

For mixtures that display significant deviations from ideality, correcting factors from the ideal mixture assumption are used. For gaseous mixtures, one defines the fugacity coefficient,  $\hat{\phi}_j$ , as follows

$$\hat{f}_j^G = P y_j \hat{\phi}_j$$

The analogous correcting factor for the liquid phase is the activity coefficient,  $\gamma_j$ .

$$\hat{f}_j^L = f_j^L x_j \gamma_j$$

The introduction of these coefficients obviously has shifted the burden from evaluating the fugacities to evaluating the activity and fugacity coefficients. These coefficients may be available in several forms. Correlations may exist for systems of interest or phase equilibrium data may be available from which the coefficients can be calculated. The following list is a representative sample of the numerous texts devoted to this important topic [11, 12, 18, 13, 5].

## 3.4 Equilibrium Composition for Heterogeneous Reactions

In this section we illustrate the calculation of chemical equilibrium when there are multiple phases as well as a chemical reaction taking place. The following example illustrates the important issues.

**Example 3.4: Chemical and phase equilibrium for a nonideal mixture**  
Consider the liquid-phase reaction



that occurs in the following three-phase system.

Phase I: nonideal liquid mixture of A and C only. For illustration purposes, assume the activity coefficients are given by the simple Margules equation,

$$\ln \gamma_A = x_C^2 [A_{AC} + 2(A_{CA} - A_{AC})x_A]$$

$$\ln \gamma_C = x_A^2 [A_{CA} + 2(A_{AC} - A_{CA})x_C]$$

Phase II: pure liquid B.

Phase III: ideal-gas mixture of A, B and C.

All three phases are in intimate contact and we have the following data:

$$A_{AC} = 1.4 \quad P_A^\circ = 0.65 \text{ atm}$$

$$A_{CA} = 2.0 \quad P_B^\circ = 0.50 \text{ atm}$$

$$P_C^\circ = 0.50 \text{ atm}$$

in which  $P_j^\circ$  is the vapor pressure of component  $j$  at the system temperature.

1. Plot the partial pressures of A and C versus  $x_A$  for a vapor phase that is in equilibrium with only the A-C liquid phase. Compute the Henry's law constants for A and C from the Margules equation. Sketch the meaning of Henry's law on the plot and verify your calculation from the plot.
2. Use Henry's law to calculate the composition of all three phases for  $K = 4.7$ . What is the equilibrium pressure?
3. Repeat for  $K = 0.23$ .
4. Assume  $K = 1$ . Use the Margules equation to calculate the composition of all three phases.
5. Repeat 4 with an ideal mixture assumption and compare the results.

### Solution

**Part 1.** The calculation of the partial pressures of A and C in a vapor phase in equilibrium with a liquid-phase mixture of A and C follows from equating the chemical potentials of the components in the different phases. Since the gas phase is assumed an ideal-gas mixture,

$$\hat{f}_A^G = P_A \quad (3.62)$$

In the liquid phase, the fugacity of A is computed from the activity coefficient

$$\hat{f}_A^L = f_A^L x_A \gamma_A \quad (3.63)$$

The fugacity of pure liquid A at the system  $T$  and  $P$  is not known. However, the fugacity of pure liquid A at the system temperature and the vapor pressure of A at the system temperature is known; it is simply the vapor pressure,  $P_A^\circ$ . If the system pressure is not greatly different

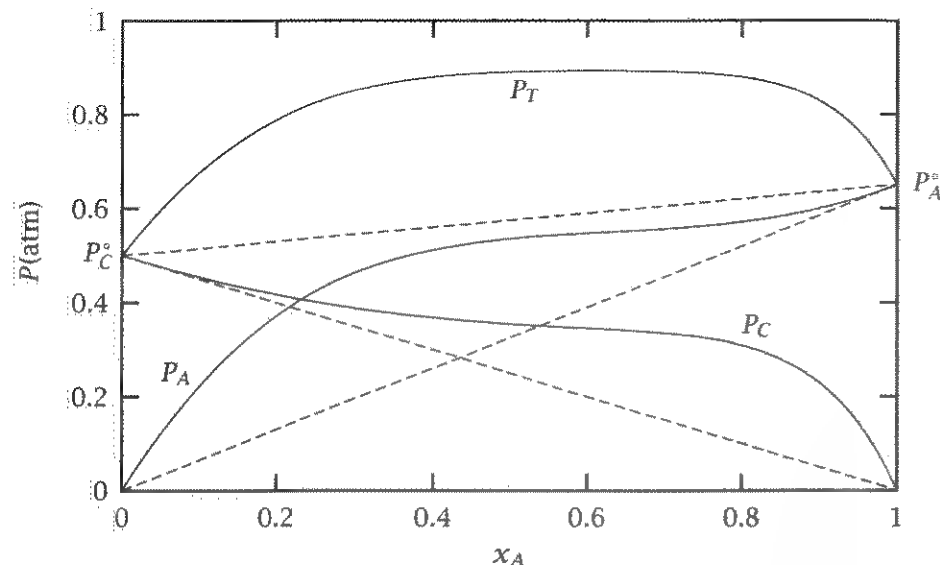


Figure 3.4: Partial pressures of components A and C versus liquid-phase composition in a nonideal solution; solid line: Margules equation; dashed line: ideal mixture.

from the vapor pressure of A at the system temperature, then the following approximation is valid because, as discussed in Section 3.3.1, fugacities of liquids are only weak functions of pressure

$$f_A^L = P_A^0 \quad (3.64)$$

Substituting Equation 3.64 into 3.63 and equating to 3.62 gives,

$$P_A = P_A^0 x_A \gamma_A \quad (3.65)$$

The analogous expression is valid for  $P_C$ . Figure 3.4 plots  $P_A$ ,  $P_C$  and  $P_T = P_A + P_C$  versus the composition in the liquid phase,  $x_A$ , using the Margules equation to compute the activity coefficient in Equation 3.65. Notice that the system exhibits positive deviations from an ideal mixture, which is also plotted in the figure.

Henry's law for component A is

$$\hat{f}_A^L = k_A x_A$$

which is valid for  $x_A$  small. Comparing to Equation 3.63 and using Equation 3.64 gives

$$k_A = P_A^0 \gamma_A$$

which is also valid for small  $x_A$ . Computing  $\gamma_A$  from the Margules equation for  $x_A = 0$  gives

$$\gamma_A(0) = e^{A_{AC}}$$

So the Henry's law constant for component A is

$$k_A = P_A^0 e^{A_{AC}}$$

The analogous expression holds for component C. Substituting in the values gives

$$k_A = 2.6, \quad k_C = 3.7$$

These values can be checked by examining Figure 3.4. The slope of the tangent line to the  $P_A$  curve at  $x_A = 0$  is equal to  $k_A$ . The negative of the slope of the tangent line to the  $P_C$  curve at  $x_A = 1$  is equal to  $k_C$ .

**Part 2.** For  $K = 4.7$ , one expects a large value of the equilibrium constant to favor the formation of the product, C. We therefore assume that  $x_A$  is small and Henry's law is valid for component A. The validity of this assumption can be checked after the equilibrium composition is computed. The unknowns in the problem are  $x_A$  and  $x_C$  in the A-C mixture,  $y_A$ ,  $y_B$  and  $y_C$  in the gas phase, and the system pressure. There are therefore six unknowns. We require six equations for a well-posed problem. We can equate the fugacities of each component in the gas and liquid phases, and use the fact that the mole fractions sum to one in the gas and A-C liquid phases. Finally, the chemical equilibrium provides the sixth equation and the problem can be solved. In general we would set up six equations in six unknowns and solve the problem numerically, but the structure of this problem allows an analytical solution. Consider first the condition for reaction equilibrium,

$$K = \frac{\hat{a}_C}{\hat{a}_A \hat{a}_B}$$

in which  $K$  is evaluated for a liquid-phase standard state. Evaluating the activities first of component A, the component for which we assume Henry's law,

$$\hat{a}_A^L = \frac{\hat{f}_A^L}{f_A^0} = \frac{k_A x_A}{f_A^0}$$

From the standard state for reactions,  $f_A^\circ$  is the fugacity of pure liquid A at the system temperature and 1.0 atm. Again, this value is unknown, but we do know that  $P_A^\circ$  is the fugacity of pure liquid A at the system temperature and the vapor pressure of A at this temperature. The difference between 0.65 and 1.0 atm is not large, so we assume  $f_A^\circ = P_A^\circ$ . Since  $x_A$  is assumed small,  $x_C$  is assumed near one. The fugacity is then evaluated from

$$\hat{a}_C^L = \frac{\hat{f}_C^L}{f_C^\circ} = \frac{f_C^L x_C}{f_C^\circ}$$

Now  $f_C^L$  and  $f_C^\circ$  are the fugacities of pure liquid C at the system temperature and the system pressure and 1.0 atm, respectively. If the system pressure turns out to be reasonably small, then it is a good assumption to assume these fugacities are equal giving,

$$\hat{a}_C^L = x_C$$

Since component B is in a pure liquid phase, the same reasoning leads to

$$\hat{a}_B^L = \frac{\hat{f}_B^L}{f_B^\circ} = \frac{f_B^L}{f_B^\circ} = 1$$

Substituting these activities into the reaction equilibrium condition gives

$$K = \frac{x_C}{x_A k_A / P_A^\circ \cdot 1} \quad (3.66)$$

Using  $x_C = 1 - x_A$  and solving Equation 3.66 for  $x_A$  yields

$$x_A = \left(1 + \frac{k_A K}{P_A^\circ}\right)^{-1} \quad x_C = \left(1 + \frac{P_A^\circ}{k_A K}\right)^{-1}$$

Substituting in the provided data gives

$$x_A = 0.05, \quad x_C = 0.95$$

Therefore, from Figure 3.4, the assumption of Henry's law for component A is reasonable.

The vapor compositions now are computed from the phase equilibrium conditions. The fugacities of all components in the ideal-gas mixture are equal to their partial pressures. Therefore, equating these to the fugacities of the liquid phase gives

$$P_A = k_A x_A, \quad P_B = P_B^\circ, \quad P_C = P_C^\circ x_C$$

Substituting in the provided data, Henry's law constant and known liquid-phase compositions gives

$$P_A = 0.13 \text{ atm}, \quad P_B = 0.50 \text{ atm}, \quad P_C = 0.48 \text{ atm}$$

The system pressure is therefore  $P = 1.11 \text{ atm}$ . This low pressure justifies the assumptions we have made about fugacities of pure liquids at the system pressure. Finally, the vapor-phase concentrations can be computed from the ratios of partial pressures to total pressure, which gives

$$y_A = 0.12, \quad y_B = 0.45, \quad y_C = 0.43$$

**Part 3.** For  $K = 0.23$  one expects the reactants to be favored so Henry's law is assumed for component C. You are encouraged to work through the preceding development again for this situation. The answers are

$$x_A = 0.97, \quad x_C = 0.03$$

$$y_A = 0.51, \quad y_B = 0.40, \quad y_C = 0.09$$

$$P = 1.24 \text{ atm}$$

Again the assumption of Henry's law is justified and the system pressure is low.

**Part 4.** For  $K = 1$ , the equilibrium composition may not allow Henry's law to be assumed for either A or C. In this case we must solve the reaction equilibrium condition using the Margules equation for the activity coefficients,

$$K = \frac{x_C y_C}{x_A y_A}$$

By expressing  $x_C = 1 - x_A$ , we have one equation in one unknown,

$$K = \frac{(1 - x_A) \exp \left[ x_A^2 (A_{CA} + 2(A_{AC} - A_{CA})(1 - x_A)) \right]}{x_A \exp \left[ (1 - x_A)^2 (A_{AC} + 2(A_{CA} - A_{AC})x_A) \right]} \quad (3.67)$$

Equation 3.67 can be solved numerically to give  $x_A = 0.35$ . From this value, the rest of the solution can be computed. Recall that the partial pressures of A and C now must be computed from the fugacity of the liquid phase using the activity coefficients

$$P_j = P_j^\circ x_j y_j, \quad j = A, C$$

The solution is

$$x_A = 0.35, \quad x_C = 0.65$$

$$y_A = 0.36, \quad y_B = 0.37, \quad y_C = 0.28$$

$$P = 1.37 \text{ atm}$$

Using the full Margules equation instead of Henry's law for the other  $K$  values gives  $x_A = 0.054$  for  $K = 4.7$  and  $x_A = 0.96$  for  $K = 0.23$ , in good agreement with the Henry's law solution presented above.

**Part 5.** Finally, if one assumes that the A-C mixture is ideal, the equilibrium condition becomes

$$K = \frac{x_C}{x_A}$$

which can be solved to give  $x_A = 1/(1 + K)$ . For  $K = 1$ , the solution is

$$x_A = 0.5, \quad x_C = 0.5$$

$$y_A = 0.30, \quad y_B = 0.47, \quad y_C = 0.23$$

$$P = 1.08 \text{ atm}$$

As can be seen from comparison with the solution to Part 4, the ideal mixture assumption leads to significant error as might be expected from the strong deviations from ideality shown in Figure 3.4.  $\square$

### 3.5 Multiple Reactions

Determining equilibria for reacting systems with multiple phases and reactions can require significant computational effort. The phase and reaction equilibrium conditions generally lead to mathematical problems of two types: solving nonlinear algebraic equations and minimizing a nonlinear function subject to constraints. In this section it is assumed that the required thermochemical data are available, but finding or measuring these data is often another significant challenge in computing equilibria for systems of industrial interest.

#### 3.5.1 Algebraic Approach

We again consider a single-phase system but allow  $n_r$  reactions

$$\sum_j \nu_{ij} A_j = 0, \quad i = 1, 2, \dots, n_r$$

Letting  $\varepsilon_i$  be the reaction extent for the  $i$ th reaction, the change in the number of moles of component  $j$  for a closed system due to the  $n_r$  chemical reactions is

$$n_j = n_{j0} + \sum_i \nu_{ij} \varepsilon_i \quad (3.68)$$

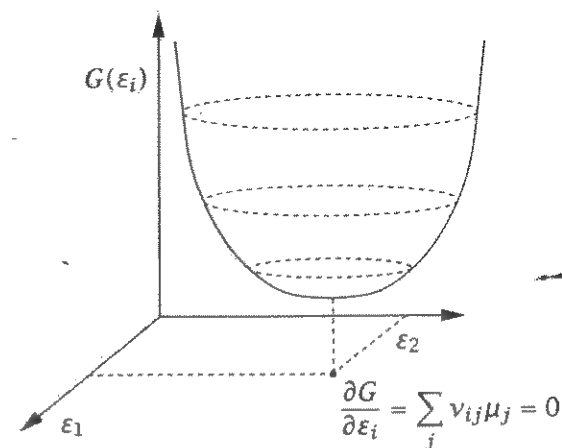


Figure 3.5: Gibbs energy versus two reaction extents at constant  $T$  and  $P$ .

We can compute the change in Gibbs energy as before

$$dG = -SdT + VdP + \sum_j \mu_j dn_j$$

Substituting in the change in the number of moles with reaction extents,  $dn_j = \sum_i \nu_{ij} d\varepsilon_i$ , gives

$$\begin{aligned} dG &= -SdT + VdP + \sum_j \mu_j \sum_i \nu_{ij} d\varepsilon_i \\ &= -SdT + VdP + \sum_i \left( \sum_j \nu_{ij} \mu_j \right) d\varepsilon_i \end{aligned} \quad (3.69)$$

At constant  $T$  and  $P$ ,  $G$  is a minimum as a function of the  $n_r$  reaction extents. Necessary conditions are therefore

$$\left( \frac{\partial G}{\partial \varepsilon_i} \right)_{T,P,\varepsilon_{i \neq i}} = 0, \quad i = 1, 2, \dots, n_r$$

as illustrated in Figure 3.5 for the case of two reactions. Evaluating the partial derivatives in Equation 3.69 gives

$$\sum_j \nu_{ij} \mu_j = 0, \quad i = 1, 2, \dots, n_r \quad (3.70)$$

In other words, the necessary condition for a single reaction applies for each reaction in the network for the multiple reaction case.

Substituting Equation 3.7 for the chemical potential in terms of the activity into Equation 3.70 gives

$$\sum_j \nu_{ij} \mu_j = \sum_j \nu_{ij} G_j^\circ + RT \sum_j \nu_{ij} \ln a_j$$

Defining the standard Gibbs energy change for reaction  $i$ ,  $\Delta G_i^\circ = \sum_j \nu_{ij} G_j^\circ$  gives

$$\sum_j \nu_{ij} \mu_j = \Delta G_i^\circ + RT \sum_j \nu_{ij} \ln a_j$$

Finally, defining the equilibrium constant for reaction  $i$  as

$$K_i = e^{-\Delta G_i^\circ / RT} \quad (3.71)$$

allows one to express the reaction equilibrium condition as

$$K_i = \prod_j a_j^{\nu_{ij}}, \quad i = 1, 2, \dots, n_r \quad (3.72)$$

### Example 3.5: Equilibrium composition for multiple reactions

In addition to the formation of 2,2,3-trimethylpentane in Example 3.2, it is known that 2,2,4-trimethylpentane may also form with the same stoichiometry,



Recalculate the equilibrium composition for this example given that  $\Delta G^\circ = -4.49$  kcal/mol for this reaction at 400 K.

### Solution

Let reaction 1 be the formation of 2,2,3 trimethylpentane denoted as  $P_1$  and reaction 2 be the formation of 2,2,4-trimethylpentane denoted as  $P_2$ . From the Gibbs energy changes, we have

$$K_1 = 108, \quad K_2 = 284$$

We now set up a table to calculate the four compositions from the extents of the two reactions,

$$\begin{aligned} n_I &= n_{I0} - \varepsilon_1 - \varepsilon_2 & n_{P_1} &= n_{P_{10}} + \varepsilon_1 \\ n_B &= n_{B0} - \varepsilon_1 - \varepsilon_2 & n_{P_2} &= n_{P_{20}} + \varepsilon_2 \end{aligned}$$

The total number of moles is then  $n_T = n_{T0} - \varepsilon_1 - \varepsilon_2$ . Forming the mole fractions yields

$$\begin{aligned} y_I &= \frac{y_{I0} - \varepsilon'_1 - \varepsilon'_2}{1 - \varepsilon'_1 - \varepsilon'_2} & y_{P_1} &= \frac{y_{P_{10}} + \varepsilon'_1}{1 - \varepsilon'_1 - \varepsilon'_2} \\ y_B &= \frac{y_{B0} - \varepsilon'_1 - \varepsilon'_2}{1 - \varepsilon'_1 - \varepsilon'_2} & y_{P_2} &= \frac{y_{P_{20}} + \varepsilon'_2}{1 - \varepsilon'_1 - \varepsilon'_2} \end{aligned} \quad (3.74)$$

Applying Equation 3.72 to the two reactions gives

$$K_1 = \frac{y_{P_1}}{y_I y_B P} \quad K_2 = \frac{y_{P_2}}{y_I y_B P}$$

Substituting in the mole fractions gives two equations for the two unknown reaction extents,

$$\begin{aligned} PK_1(y_{I0} - \varepsilon'_1 - \varepsilon'_2)(y_{B0} - \varepsilon'_1 - \varepsilon'_2) - (y_{P_{10}} + \varepsilon'_1)(1 - \varepsilon'_1 - \varepsilon'_2) &= 0 \\ PK_2(y_{I0} - \varepsilon'_1 - \varepsilon'_2)(y_{B0} - \varepsilon'_1 - \varepsilon'_2) - (y_{P_{20}} + \varepsilon'_2)(1 - \varepsilon'_1 - \varepsilon'_2) &= 0 \end{aligned}$$

These two equations are solved with the initial condition of an equimolar mixture of reactants,  $y_I = y_B = 0.5$ ,  $y_{P_1} = y_{P_2} = 0$ .<sup>2</sup> Using the initial guess from the solution of Example 3.2,  $\varepsilon_1 = 0.469$ ,  $\varepsilon_2 = 0$ , gives the solution

$$\varepsilon_1 = 0.133, \quad \varepsilon_2 = 0.351$$

Other initial guesses result in the same solution, leading us to believe that this solution is the unique, physically meaningful solution.<sup>3</sup> The mole fractions are computed from Equations 3.74 giving

$$y_I = 0.031, \quad y_B = 0.031, \quad y_{P_1} = 0.258, \quad y_{P_2} = 0.680$$

Notice that the existence of the second reaction causes us to produce considerably less 2,2,3-trimethylpentane in favor of the 2,2,4 isomer. This is bad news in this example because the production of 2,2,4 is preferable due to its high octane content. The news only gets worse when we consider that a host of other products are also possible, such as the 2,3,3 and 2,3,4 isomers as well as 2,2,3,3-tetramethylbutane and still others. The Gibbs energy of formation of many of these possible

<sup>2</sup>It is best to use a numerical tool to solve these equations when there is more than one reaction.

<sup>3</sup>The mole fractions are real and positive.



products show that they are present in significant amounts at equilibrium. It is clear that one cannot allow the system to reach equilibrium and still hope to obtain a high yield of the desired product. The solution to problems like these is often to develop a catalyst that selectively increases the reaction rate for the desired reaction in comparison to the undesired side reactions. If such a catalyst is available, the reaction is halted after a high yield of the desired product is formed and before the equilibration to the other products can reduce the yield. This use of catalysts to promote the rates of desirable reactions is discussed further in Chapter 5.  $\square$

### 3.5.2 Optimization Approach

The other main approach to finding the reaction equilibrium is to minimize the appropriate energy function, in this case the Gibbs energy [19]. This optimization-based formulation of the problem, as shown in Example 3.3, can be more informative than the algebraic approach discussed above.

We first extend our evaluation of the Gibbs energy for the single reaction case to allow multiple reactions. We start with

$$G = \sum_j \mu_j n_j \quad (3.75)$$

and express the chemical potential in terms of activity

$$\mu_j = G_j^\circ + RT \ln a_j$$

We again use Equation 3.68 to track the change in mole numbers due to multiple reactions,

$$n_j = n_{j0} + \sum_i \nu_{ij} \varepsilon_i$$

Using the two previous equations we have

$$\mu_j n_j = n_{j0} G_j^\circ + G_j^\circ \sum_i \nu_{ij} \varepsilon_i + \left[ n_{j0} + \sum_i \nu_{ij} \varepsilon_i \right] RT \ln a_j \quad (3.76)$$

It is convenient to define the same modified Gibbs energy function that we used in Equation 3.37

$$\tilde{G}(T, P, \varepsilon'_i) = \frac{G - \sum_j n_{j0} G_j^\circ}{n_{T0} RT} \quad (3.77)$$

in which  $\varepsilon'_i = \varepsilon_i / n_{T0}$ . If we sum on  $j$  in Equation 3.76 and introduce this expression into Equations 3.75 and 3.77, we obtain

$$\tilde{G} = \sum_i \varepsilon'_i \frac{\Delta G_i^\circ}{RT} + \sum_j \left[ \gamma_{j0} + \sum_i \nu_{ij} \varepsilon'_i \right] \ln a_j$$

Using the definition of the equilibrium constant, Equation 3.71, we can write the energy function as

$$\tilde{G} = - \sum_i \varepsilon'_i \ln K_i + \sum_j \left[ \gamma_{j0} + \sum_i \nu_{ij} \varepsilon'_i \right] \ln a_j \quad (3.78)$$

To find the equilibrium composition, we minimize this modified Gibbs energy over the physically meaningful values of the  $n_r$  extents. The main restriction on these extents is, again, that they produce non-negative mole numbers, or, if we wish to use intensive variables, non-negative mole fractions. We can express these constraints as

$$-\gamma_{j0} - \sum_i \nu_{ij} \varepsilon'_i \leq 0, \quad j = 1, \dots, n_s \quad (3.79)$$

Our final statement, therefore, for finding the equilibrium composition for multiple reactions is to solve the optimization problem

$$\min_{\varepsilon'_i} \tilde{G}(\varepsilon'_i) \quad (3.80)$$

subject to Equation 3.79. The min notation means to minimize the function  $\tilde{G}(\varepsilon'_i)$  with respect to the argument  $\varepsilon'_i$ .

#### Example 3.6: Multiple reactions with optimization

Revisit the two-reaction trimethylpentane example, and find the equilibrium composition by minimizing the Gibbs energy.

#### Solution

We start with the modified energy function, Equation 3.78. We recall for an ideal-gas mixture, the activity of component  $j$ , is simply its partial pressure divided by 1.0 atm,

$$a_j = \frac{P}{1 \text{ atm}} \gamma_j \quad (\text{ideal-gas mixture})$$

Substituting this relation into Equation 3.78 and rearranging gives

$$\begin{aligned} \tilde{G} = & - \sum_i \epsilon'_i \ln K_i + \left( 1 + \sum_i \bar{v}_i \epsilon'_i \right) \ln P \\ & + \sum_j \left( \gamma_{j0} + \sum_i \nu_{ij} \epsilon'_i \right) \ln \left[ \frac{\gamma_{j0} + \sum_i \nu_{ij} \epsilon'_i}{1 + \sum_i \bar{v}_i \epsilon'_i} \right] \end{aligned} \quad (3.81)$$

in which

$$\bar{v}_i = \sum_j \nu_{ij}$$

The minimization of this function of  $\epsilon'_i$  then determines the two equilibrium extents. The constraints on the extents are found from Equation 3.79. For this problem they are

$$\begin{aligned} -\gamma_{I0} + \epsilon'_1 + \epsilon'_2 &\leq 0 & -\gamma_{P10} - \epsilon'_1 &\leq 0 \\ -\gamma_{B0} + \epsilon'_1 + \epsilon'_2 &\leq 0 & -\gamma_{P20} - \epsilon'_2 &\leq 0 \end{aligned}$$

Substituting in the initial conditions gives the constraints

$$\epsilon'_1 + \epsilon'_2 \leq 0.5, \quad 0 \leq \epsilon'_1, \quad 0 \leq \epsilon'_2$$

Figure 3.6 shows the lines of constant Gibbs energy determined by Equation 3.81 as a function of the two reaction extents. We see immediately that the minimum is unique. Notice in Figure 3.6 that  $\tilde{G}$  is defined only in the region specified by the constraints. The numerical solution of the optimization problem is

$$\epsilon'_1 = 0.133, \quad \epsilon'_2 = 0.351, \quad \tilde{G} = -2.569$$

The solution is in good agreement with the extents computed using the algebraic approach, and the Gibbs energy contours depicted in Figure 3.6.  $\square$

Optimization is a powerful tool for solving many types of engineering modeling and design problems. We also rely heavily on optimization tools in Chapter 9 on parameter estimation.

### 3.6 Summary

In this review of reaction equilibrium, the Gibbs energy was chosen as the convenient function for solving reaction equilibrium problems when the temperature and pressure are specified. The fundamental

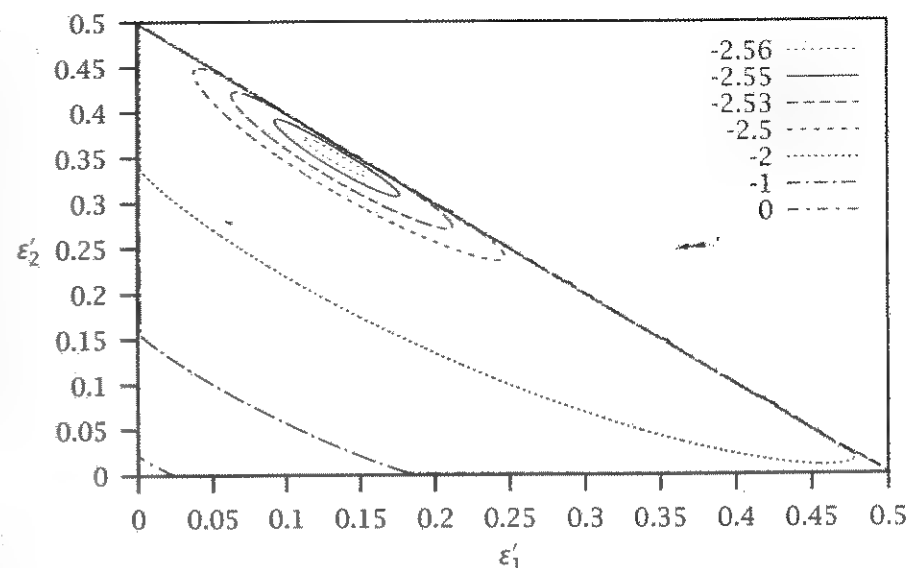


Figure 3.6: Gibbs energy contours for the pentane reactions as a function of the two reaction extents.

equilibrium condition is that the Gibbs energy is minimized. This fundamental condition leads to several conditions for equilibrium such as

$$\sum_j \nu_j \mu_j = 0$$

$$K = \prod_j a_j^{\nu_j}$$

for a single reaction or

$$\sum_j \nu_{ij} \mu_j = 0, \quad i = 1, \dots, n_r$$

$$K_i = \prod_j a_j^{\nu_{ij}}, \quad i = 1, \dots, n_r$$

for multiple reactions, in which the equilibrium constant is defined to be

$$K_i = e^{-\Delta G_i^\circ / RT}$$

You should feel free to use whichever formulation is most convenient for the problem. The equilibrium “constant” is not so constant, because it depends on temperature via

$$\frac{\partial \ln K}{\partial T} = \frac{\Delta H^\circ}{RT^2}$$

or, if the enthalpy change does not vary with temperature,

$$\ln \left( \frac{K_2}{K_1} \right) = -\frac{\Delta H^\circ}{R} \left( \frac{1}{T_2} - \frac{1}{T_1} \right)$$

The conditions for phase equilibrium were presented: equalities of temperature, pressure and chemical potential of each species in all phases. The evaluation of chemical potentials of mixtures was discussed, and the following methods and approximations were presented: ideal mixture, Henry's law, and simple correlations for activity coefficients.

When more than one reaction is considered, which is the usual situation faced in applications, we require numerical methods to find the equilibrium composition. Two approaches to this problem were presented. We either solve a set of nonlinear algebraic equations or solve a nonlinear optimization problem subject to constraints. If optimization software is available, the optimization approach is more powerful and provides more insight.

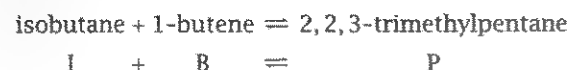
## Notation

$a_j$	activity of component $j$
$a_{jl}$	formula number for element $l$ in species $j$
$A_j$	$j$ th species in the reaction network
$C_{p,j}$	partial molar heat capacity of component $j$
$E_l$	$l$ th element constituting the species in the reaction network
$f_j$	fugacity of component $j$
$G$	Gibbs energy
$G_j$	Partial molar Gibbs energy of component $j$
$\Delta G_i^\circ$	standard Gibbs energy change for reaction $i$
$H$	enthalpy
$H_j$	partial molar enthalpy of component $j$
$\Delta H_i^\circ$	standard enthalpy change for reaction $i$
$i$	reaction index, $i = 1, 2, \dots, n_r$
$j$	species index, $j = 1, 2, \dots, n_s$
$k$	phase index, $k = 1, 2, \dots, n_p$
$K$	equilibrium constant
$K_i$	equilibrium constant for reaction $i$
$l$	element index, $l = 1, 2, \dots, n_e$
$n_r$	total number of reactions in reaction network
$n_s$	total number of species in reaction network
$P$	pressure
$P_j$	partial pressure of component $j$
$S$	entropy
$T$	temperature
$V$	volume
$\epsilon$	reaction extent
$\epsilon_i$	reaction extent for reaction $i$
$\mu_j$	chemical potential of component $j$
$\nu_{ij}$	stoichiometric number for the $j$ th species in the $i$ th reaction
$\nu_j$	stoichiometric number for the $j$ th species in a single reaction
$\bar{\nu}$	$\sum_j \nu_j$
$\bar{\nu}_i$	$\sum_j \nu_{ij}$

### 3.7 Exercises

#### Exercise 3.1: Calculating an equilibrium composition

Consider the following reaction in a closed vessel at a pressure of 1.0 atm and temperature of 500°K

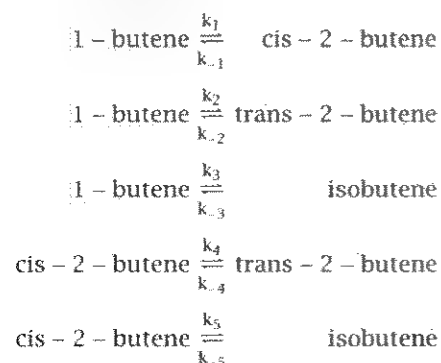


The standard Gibbs energy and enthalpy changes for this reaction at 500 K are  $\Delta G^\circ = -4.10$  kcal/mol,  $\Delta H^\circ = -20.11$  kcal/mol.

- Determine the equilibrium composition for this system for an initial equimolar mixture of isobutane and butene. What assumptions did you make?
- What is the equilibrium *conversion* of the reactants in part 3.1a? Assume for economic reasons that you must increase the equilibrium conversion to 97%. At what pressure must you run this reaction at a temperature of 500 K to achieve this conversion?
- At what temperature must you run this reaction at a pressure of 1.0 atm to achieve 97% conversion?
- What do you know about this system that prevents you from taking these answers too seriously?

#### Exercise 3.2: Equilibrium and linearly independent reactions

Butene isomerization reactions are shown below.



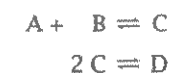
The Gibbs energy of formation of the components at 400 K are [17]

Components	$\Delta G_f^\circ$ (kcal/mol)
1-butene	23.10
cis-2-butene	21.94
trans-2-butene	21.33
isobutene	20.23

- Determine the minimum number of reactions that are needed to calculate the equilibrium composition of the butenes.
- Set up the linear algebra problem to determine the equilibrium extents of the reactions. What assumptions have you made?
- Compute the equilibrium composition at 400 K and 1.0 atm pressure starting from pure 1-butene.
- Why is isobutene the predominate butene formed?

#### Exercise 3.3: More than one reaction

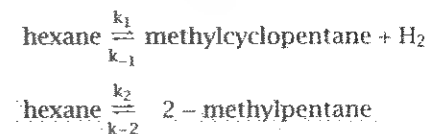
Consider the following two gas-phase reactions



- What are the two equilibrium equations relating the two unknown extents of reaction and the initial gas-phase compositions? You can assume the pressure is low enough that the gases are ideal.
- What is the equilibrium gas-phase composition at 2.0 atm total pressure given initial gas-phase composition  $y_A = y_B = 0.5$ ,  $y_C = y_D = 0$ , and  $K_1 = 1.0$ ,  $K_2 = 2.5$ ? Solve the two nonlinear equations numerically. If a computer is not available, you can use trial and error (partial answer:  $y_A = 0.31$ ).
- What is the equilibrium gas-phase composition at 2.0 atm total pressure given initial gas-phase composition  $y_A = y_C = 0.5$ ,  $y_B = y_D = 0$  (partial answer:  $y_A = 0.62$ )?

#### Exercise 3.4: Temperature effect on reaction equilibria

Hexane can equilibrate to methyl-substituted pentanes and methylcyclopentane according to

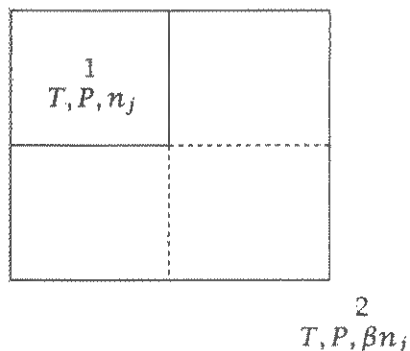


Component	$\Delta H_f^{600\text{ K}}$ (kcal/mol)	$\Delta G_f^{600\text{ K}}$ (kcal/mol)
n-hexane	-46.10	43.02
methylcyclopentane	-31.69	45.76
2-methylpentane	-47.63	42.39
hydrogen	0	0

in which  $\Delta H_f^{600\text{ K}}$  and  $\Delta G_f^{600\text{ K}}$  are the heats of formation and the Gibbs free energy of formation of the compounds at 600 K from the elements, respectively [17]. Find the temperature where equal amounts of methylcyclopentane and 2-methylpentane form if pure hexane is allowed to equilibrate at 1.0 atm total pressure. Also find the composition of the mixture at this temperature. At this condition the hydrocarbons are gases.

### Exercise 3.5: Other equilibrium conditions

In the text, we focused our attention on minimizing Gibbs energy to determine the chemical equilibrium at fixed temperature and pressure. To be fair to the other energy functions, what system function would be at an extremum (maximum or minimum) if the following were specified. Provide a short derivation verifying your result starting with the second law,  $dS_{\text{tot}} \geq 0$ .



**Figure 3.7:** Two systems with identical intensive properties and different extensive properties.

- $T$  and  $P$  are fixed.
- $T$  and  $V$  are fixed.
- $S$  and  $P$  are fixed.
- $S$  and  $V$  are fixed.
- The system is insulated and  $V$  is fixed. In addition to  $V$ , what other thermodynamic property is fixed?
- The system is insulated and  $P$  is fixed. In addition to  $P$ , what other thermodynamic property is fixed?

### Exercise 3.6: Partial molar Gibbs energy and chemical potential

- Derive Equation 3.34 in the text. Consider first a system at equilibrium depicted in Figure 3.7 at given  $T, P, n_j, j = 1, \dots, n_s$ . Now consider system 2 to be simply  $\beta$  identical copies of system 1. The equilibrium state

of system 2 therefore has the same temperature and pressure (intensive properties) as system 1 but the number of moles of each component (extensive property) changes by constant factor  $\beta$ . System 2 is specified by the variables  $T, P, \beta n_j, j = 1, \dots, n_s$ . Consider a reversible change from system 1 to system 2, imagine slowly removing the partition depicted in Figure 3.7, for example, and integrate Equation 3.1

$$\int_1^2 dG = \int_1^2 \left[ -SdT + VdP + \sum_j \mu_j dn_j \right]$$

Because  $\mu_j$  is an intensive variable, it remains constant during the process, along with  $T$  and  $P$ . Perform the integration and express  $G_2$  and  $n_{j2}$  in terms of  $G_1$  and  $n_{j1}$  and  $\beta$  to complete the argument.

- Show this same argument produces the following result for any partial molar property

$$X = \sum_j X_j n_j$$

in which  $X$  is any extensive thermodynamic property.

### Exercise 3.7: Creating a second phase with reaction

Consider the gas-phase reaction



Product C has a fairly low vapor pressure, so we are concerned about the formation of a liquid phase in the reactor. The Clausius-Clapeyron equation well represents the vapor pressure of component C as a function of temperature

$$\ln P_C^c = c - \frac{\Delta H_{\text{vap}}}{RT} \quad (3.83)$$

The reactor is initially filled with an equimolar mixture of A and B. The equilibrium constant at  $T = 298\text{ K}$  is  $K = 8$ , the reaction is exothermic with  $\Delta H^\circ = -10\text{ kcal/mol}$ , and the system pressure is  $P = 1.0\text{ atm}$ . Components A and B are not very soluble in liquid C. The heat of vaporization of component C is  $\Delta H_{\text{vap}} = 5\text{ kcal/mol}$ , and value of the Clausius-Clapeyron constant is  $c = 7.53$ .

- Over what temperature range does the reactor contain a liquid phase?
- If the reaction is endothermic with  $\Delta H^\circ = 10\text{ kcal/mol}$ , over what temperature range does the reactor contain a liquid phase?

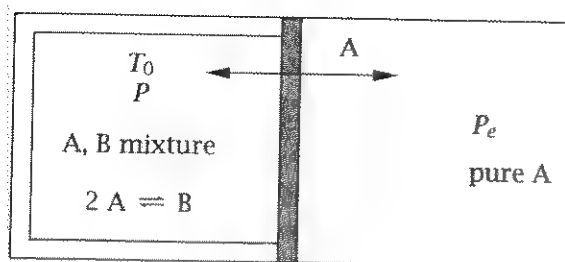


Figure 3.8: Vessel with semipermeable membrane and chemical reaction  $2A \rightleftharpoons B$ .

### Exercise 3.8: The van't Hoff relation

Methanol can be manufactured by the gas-phase reaction



A batch reactor is charged with a  $\text{H}_2$  and  $\text{CO}$  mixture at 1.0 atm and the pressure is maintained at 1.0 atm. The initial mole fraction of  $\text{H}_2$  is 0.6 and  $\text{CO}$  is 0.4. At 400 K,  $\Delta G^\circ = -333$  cal/mol and  $\Delta H^\circ = -22,580$  cal/mol.

Determine the temperature at which the equilibrium mole fraction of  $\text{H}_2$  is 0.05.

### Exercise 3.9: Semipermeable membrane and chemical reaction

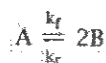
Consider the membrane separation shown in Figure 3.8. An isothermal rigid vessel initially contains pure component B at pressure  $P_0$  and temperature  $T_0$ . It is separated by a semipermeable membrane from a source of pure component A at pressure  $P_e$ . The membrane is permeable to component A but not component B. The system undergoes chemical reaction



inside the vessel due to the presence of a catalyst. No reaction occurs outside the vessel. The equilibrium constant at temperature  $T_0$  is  $K = 0.5$ . What is the equilibrium composition and pressure in the vessel if  $P_e = 5.0$  atm,  $P_0 = 1.0$  atm? An ideal-gas mixture may be assumed.

### Exercise 3.10: Reconciling kinetics and thermodynamics

Consider the gas-phase reaction



Anticipating the discussion of kinetics in Chapter 5, we suppose everyone is familiar with the classical rate expressions for the forward and reverse reactions

$$r_f = k_f c_A, \quad r_r = k_r c_B^2$$

The forward rate expression follows from assuming that the unimolecular forward reaction is proportional to the concentration of A molecules. The reverse rates expression follows from assuming the bimolecular reverse reaction is proportional to the probability of collisions of two B molecules, which is proportional to  $c_B^2$  given the typically large numbers of molecules. The net reaction rate is given by the difference of forward and reverse rates

$$r = k_f c_A - k_r c_B^2$$

The forward and reverse rate constants,  $k_f, k_r$ , quantify the fraction of collisions that result in successful reactions. We expect these rate constants to depend on temperature because the frequency and energy of collisions increases with the mean velocity of the molecules which increases with temperature. A kinetic view of the equilibrium condition is obtained by setting the net reaction rate to zero.

- (a) In order for this kinetic view to be in agreement with the results of thermodynamics, show that the rate constants also satisfy the restriction

$$\frac{k_r}{k_f} = K \frac{\phi_A}{\phi_B^2}$$

What can you conclude about the dependences of rate constants  $k_f$  and  $k_r$  on the composition?

- (b) If A and B form an ideal mixture, what can you conclude about the dependence of  $k_f, k_r$  on the composition?

### Exercise 3.11: Gibbs energy of formation

Consider the Gibbs energy of formation of species  $j$

$$G_{jff}^\circ = G_j^\circ - \sum_{l=1}^{n_e} a_{jl} G_{El}^\circ$$

in which  $G_{El}^\circ$  is the Gibbs energy of element  $l$  and  $a_{jl}$  is the formula number for element  $l$  in species  $j$ . Multiply this equation by the stoichiometric coefficient  $v_{ij}$  and sum on  $j$  to produce

$$\sum_{j=1}^{n_s} v_{ij} G_{jff}^\circ = \Delta G_i^\circ - \sum_{j=1}^{n_s} v_{ij} \sum_{l=1}^{n_e} a_{jl} G_{El}^\circ \quad (3.84)$$

Using the result of Exercise 2.8, verify Equation 3.44 in the text for the Gibbs energy of reaction.



**Exercise 3.12: Reactions going to completion**

In a single-phase system, show the equilibrium condition

$$\sum_j \nu_{ij} \mu_j = 0, \quad i = 1, \dots, n_r$$

cannot be satisfied by any reaction going to completion and completely eliminating one of the components. Hint: consider Equations 3.71 and 3.72.

Can reactions go to completion in a multiple-phase system? If no, prove it. If yes, provide an example.

**Exercise 3.13: Phase rule**

Consider a system at equilibrium with  $n$  components and  $\pi$  phases. Let  $f$  be the number of independently variable intensive properties of the composite system not allowing chemical reactions. Gibbs's phase rule states

$$f = n + 2 - \pi \quad (3.85)$$

- (a) In many textbooks, the following style of argument is used to derive the phase rule. Specify the following *intensive* variables to define the state of the system: temperature, pressure, and  $n - 1$  mole fractions in each phase

$$T^k, p^k, x_j^k \quad j = 1, \dots, n - 1 \quad k = 1, \dots, \pi \quad (3.86)$$

Count these unknowns and show there are  $\pi(n - 1)$  unknowns.

Apply the set of equilibrium relations given in Equation 3.53. Count these equations and show there are  $(n + 2)(\pi - 1)$  equations.

Call  $f$  the difference between the number of equations and the number of unknowns, and show that this difference produces Equation 3.85.

In particular, if  $f = 0$ , we have the same number of equations and unknowns, and are supposed to conclude that the intensive variables of the equilibrium state are therefore uniquely determined. What is *missing* from this argument? Provide sets of two equations with two unknowns that have (i) no solution, (ii) exactly one solution, and (iii) infinitely many solutions.

- (b) Rather than reject the phase rule, for which we have abundant experimental evidence, let's work a little harder to derive it, following the discussion by Feinberg [6]. Gibbs himself used the following style of argument [8, pp.87-96]. Start with the usual set of variables

$$dG = -SdT + VdP + \sum_j \mu_j n_j$$

To obtain a relation among only intensive variables, differentiate Equation 3.75,  $G = \sum_j \mu_j n_j$ , and subtract to obtain the Gibbs-Duhem relations

$$0 = -SdT + VdP - \sum_j n_j d\mu_j$$

which applies to every phase, so we can write the set of equations

$$\begin{bmatrix} S^1 & -V^1 & n_1^1 & \dots & n_n^1 \\ S^2 & -V^2 & n_1^2 & \dots & n_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S^\pi & -V^\pi & n_1^\pi & \dots & n_n^\pi \end{bmatrix} \begin{bmatrix} dT \\ dP \\ d\mu_1 \\ \vdots \\ d\mu_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3.87)$$

The phase equilibrium relations tell us the variables  $T, P, \mu_j, j = 1, \dots, n$  are equal in all phases. Therefore, the number of independently variable intensive properties is simply the number of linearly independent solutions to the set of *linear* equations listed in Equation 3.87.

So, how many linearly independent solutions does Equation 3.87 admit? Use the result of Exercise 2.9 to show

$$f = n + 2 - \text{rank}(M) \quad (3.88)$$

in which  $M$  is the  $\pi \times (n + 2)$  matrix appearing in Equation 3.87. Because the rank can exceed neither the number of rows ( $\pi$ ) nor the number of columns ( $n + 2$ ), we know that  $f$  is nonnegative, and that

$$f \geq n + 2 - \pi \quad (3.89)$$

- (c) What assumption about equilibrium states is required before we can conclude  $\text{rank}(M) = \pi$  and therefore

$$f = n + 2 - \pi \quad (3.90)$$

See Noll [10] and Feinberg [6] for further discussion of this issue.

- (d) Consider  $r$  *linearly independent* chemical reactions and add the reaction equilibrium conditions to Equation 3.87. What are the dimensions of the modified  $M$  matrix? Show the modified phase rule with reaction is

$$f = n + 2 - \pi - r$$

Notice we cannot state a phase rule with chemical reaction if we have not defined linearly independent reactions, which is another motivation for defining linearly independent reactions in Chapter 2.

### Exercise 3.14: Multiple reactions and ideal-gas equilibrium

We would like to show that, for multiple reactions, the equilibrium state of an ideal gas is unique. The following argument extends the single-reaction discussion starting on page 70 to multiple reactions [19, 15, 7]. In the single reaction case, we showed that the Gibbs energy  $\tilde{G}(T, P, \epsilon)$  has a positive second derivative in the single reaction extent, or

$$\frac{\partial^2 \tilde{G}}{\partial \epsilon^2} > 0$$

corresponding to Figure 3.2. To reduce notational complexity we consider  $T$  and  $P$  constant in all partial derivatives of this problem. Assume we have  $n_r$  linearly independent reactions. Then we must show that the  $n_r \times n_r$  matrix of second derivatives,  $\tilde{H}$

$$\tilde{H}_{ik} = \frac{\partial^2 \tilde{G}}{\partial \epsilon_i \partial \epsilon_k}$$

is a positive definite matrix, corresponding to Figure 3.5 for  $n_r = 2$ . A matrix  $H$  is positive definite if

$$\mathbf{x}^T \mathbf{H} \mathbf{x} > 0, \quad \text{for every } \mathbf{x} \neq 0$$

Consider the function  $G(T, P, n_j) = \sum_l \mu_l n_l$  and its first derivative  $\partial G / \partial n_j = \mu_j$ . Taking the second derivative gives a matrix  $H$  with elements

$$H_{jl} = \frac{\partial^2 G}{\partial n_j \partial n_l} = \frac{\partial \mu_j}{\partial n_l}$$

(a) Use the ideal-gas chemical potential  $\mu_j = G_j^\circ + RT \ln(P y_j)$  and show

$$\frac{\partial \mu_j}{\partial y_l} = \frac{RT}{y_j} \delta_{jl} \quad \text{in which} \quad \delta_{jl} = \begin{cases} 0, & j \neq l \\ 1, & j = l \end{cases}$$

Using the definition of mole fraction  $y_j = n_j / n_T$ , show

$$\frac{\partial y_j}{\partial n_l} = \frac{\delta_{jl}}{n_T} - \frac{y_j}{n_T} \quad (3.91)$$

(b) Using the chain rule

$$\frac{\partial \mu_j}{\partial n_l} = \sum_p \frac{\partial \mu_j}{\partial y_p} \frac{\partial y_p}{\partial n_l}$$

show

$$H_{jl} = \frac{RT}{n_T} \left[ \frac{\delta_{jl}}{y_j} - 1 \right] \quad (3.92)$$

(c) To show  $H$  is positive definite, we must show that the following quadratic form is positive

$$\mathbf{x}^T \mathbf{H} \mathbf{x} = \sum_j \sum_l x_j H_{jl} x_l > 0, \quad \text{for every } \mathbf{x} \neq 0$$

Substitute Equation 3.92 into the above equation and show

$$\mathbf{x}^T \mathbf{H} \mathbf{x} = \frac{RT}{n_T} \left[ \sum_l \frac{x_l^2}{y_l} - \left( \sum_l x_l \right)^2 \right]$$

Next use the Cauchy inequality, Equation 3.31, to show the right-hand side is nonnegative. When is equality achieved in the Cauchy inequality? We have therefore shown  $H$  is a positive definite matrix.

(d) We now find the second derivatives of  $\tilde{G}$  with respect to reaction extent. Given the relation  $n_j = n_{j0} + \sum_i \nu_{ij} \epsilon_i$  use the chain rule to show

$$\frac{\partial \tilde{G}}{\partial \epsilon_i} = \sum_j \frac{\partial G}{\partial n_j} \nu_{ij} \quad \frac{\partial^2 \tilde{G}}{\partial \epsilon_i \partial \epsilon_k} = \tilde{H}_{ik} = \sum_j \sum_l \nu_{ij} H_{jl} \nu_{kl}$$

Let  $\tilde{H}$  be the matrix of second derivatives in extents. In matrix notation we have shown

$$\tilde{H} = \mathbf{v} \mathbf{H} \mathbf{v}^T$$

(e) Finally, since  $H$  is positive definite and the reactions are linearly independent, show that  $\tilde{H}$  is positive definite also.

Notice the same argument applies if we make the weaker assumption of an ideal mixture rather than an ideal gas.

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## 4

### The Material Balance for Chemical Reactors

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#### 4.1 General Mole Balance

Consider an arbitrary reactor volume element depicted in Figure 4.1, which has inlet and outlet streams with volumetric flowrates  $Q_0$  and  $Q_1$ , respectively. The molar concentrations of component  $j$  in the two streams are given by  $c_{j0}$  and  $c_{j1}$  and the production rate of component  $j$  due to chemical reactions is  $R_j$ . The statement of conservation of mass for this system takes the form,

$$\left\{ \begin{array}{c} \text{rate of} \\ \text{accumulation} \\ \text{of component } j \end{array} \right\} = \left\{ \begin{array}{c} \text{rate of inflow} \\ \text{of component } j \end{array} \right\} - \left\{ \begin{array}{c} \text{rate of outflow} \\ \text{of component } j \end{array} \right\} + \left\{ \begin{array}{c} \text{rate of generation} \\ \text{of component } j \text{ by} \\ \text{chemical reactions} \end{array} \right\} \quad (4.1)$$

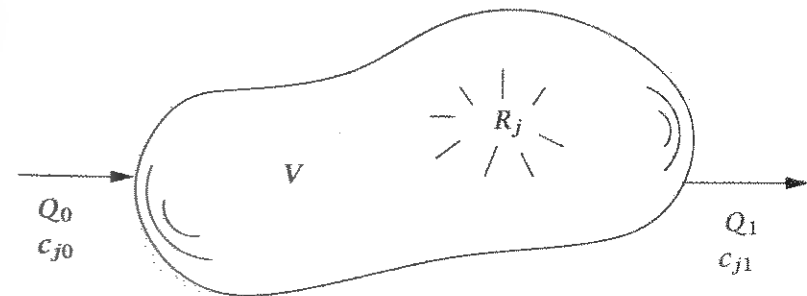


Figure 4.1: Reactor volume element.

In terms of the defined variables, we can write Equation 4.1 as,

$$\frac{d}{dt} \int_V c_j dV = Q_0 c_{j0} - Q_1 c_{j1} + \int_V R_j dV \quad (4.2)$$

Equation 4.2 applies to every chemical component in the system,  $j = 1, 2, \dots, n_s$ , including inerts, which do not take place in any reactions. One can, of course, include volume elements with more than two flow streams by summing with the appropriate sign over all streams entering and leaving the reactor. For the balances in this chapter, there will be two or fewer flow streams. Notice also that we are assuming that component  $j$  enters and leaves the reactor volume element only by convection with the inflow and outflow streams. In particular, we are neglecting diffusional flux through the boundary of the volume element due to a concentration gradient. The diffusional flux will be considered during the development of the material balance for the packed-bed reactor.

**Rate expressions.** To solve the reactor material balance, we require an expression for the production rate,  $R_j$ , for each component. As shown in Chapter 2, the production rate can be computed directly from the stoichiometry and the reaction rates for all reactions,  $r_i$ . Therefore we require an expression for the reaction rates in terms of the concentrations of the species. This topic occupies the majority of Chapter 5. For the purposes of illustrating the material balances in this chapter, we simply use some common reaction-rate expressions without derivation. These rate expressions may be regarded as empirical facts until the next chapter when the theoretical development of the rate expressions is provided.

## 4.2 The Batch Reactor

The batch reactor is assumed to be well stirred, so there are no concentration gradients anywhere in the reactor volume. In this case it is natural to consider the entire reactor contents to be the reactor volume element as in Figure 4.2, and  $V = V_R$ . Because the reactor is well stirred, the integrals in Equation 4.2 are simple to evaluate,

$$\int_{V_R} c_j dV = c_j V_R \quad (4.3)$$

$$\int_{V_R} R_j dV = R_j V_R \quad (4.4)$$

## 4.2 The Batch Reactor

Because the reactor is charged with reactants at  $t = 0$ , and nothing is added or removed from the reactor until the stopping time, the inflow and outflow stream flowrates are zero,  $Q_0 = Q_1 = 0$ .

Substituting these results into Equation 4.2 gives the general batch reactor design equation,

$$\frac{d(c_j V_R)}{dt} = R_j V_R \quad (4.5)$$

Equation 4.5 applies whether the reactor volume is constant or changes during the course of the reaction. If the reactor volume is constant, which is sometimes a good approximation for liquid-phase reactions,  $V_R$  can be divided out of both sides of Equation 4.5 to give

$$\frac{dc_j}{dt} = R_j \quad (4.6)$$

Be sure to use Equation 4.5 rather than Equation 4.6 if the reactor volume changes significantly during the course of the reaction.

### 4.2.1 Analytical Solutions for Simple Rate Laws

In complex and realistic situations, the material balance for the batch reactor must be solved numerically. However, if the reactor is isothermal, and the rate laws are assumed to be quite simple, then analytical solutions of the material balance are possible. Analytical solutions are valuable for at least two reasons. First, due to the closed form of the solution, analytical solutions provide insight that is difficult to achieve with numerical solutions. The effect of parameter values on the solution is usually more transparent, and the careful study of analytical solutions can often provide insight that is hard to extract from numerical computations. Secondly, even if one must compute a numerical solution for a problem of interest, the solution procedure should be checked for errors by comparison to known solutions. Comparing a numerical solution procedure to an analytical solution for a simplified problem provides some assurance that the numerical procedure has been constructed correctly. Then the verified numerical procedure can

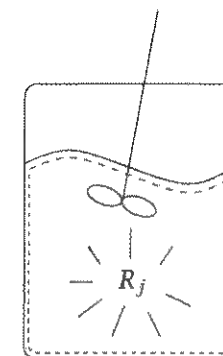


Figure 4.2: Batch reactor volume element.

be used with more assurance on the full problem for which no other solution is available.

The next several sections derive analytical solutions for some simple rate laws. Of course, the batch reactor is assumed to be operating at constant temperature in this discussion.

**First-order, irreversible.** Consider the first-order, irreversible reaction



in which the reaction rate is given by  $r = kc_A$ . Application of the material balance for a constant-volume reactor gives the following differential equation

$$\frac{dc_A}{dt} = -kc_A \quad (4.8)$$

in which the negative sign arises because the production rate of A is  $R_A = -r$  due to the stoichiometry of the reaction. Equation 4.8 requires an initial condition to have a unique solution. We denote the initial concentration of A in the reactor as  $c_{A0}$ ,

$$c_A(t) = c_{A0}, \quad t = 0$$

The solution to the differential equation with this boundary condition is

$$c_A = c_{A0}e^{-kt} \quad (4.9)$$

which is plotted in Figure 4.3 for several values of the rate constant  $k$ . Because the reaction is irreversible, the A concentration decreases exponentially from its initial value to zero with increasing time. The rate constant determines the shape of this exponential decrease. Rearranging Equation 4.9 and taking logarithms gives

$$\ln(c_A/c_{A0}) = -kt$$

which is plotted by using a log scale in Figure 4.4. Notice one can get an approximate value of the rate constant by calculating the slope of the straight line given by  $\ln(c_A/c_{A0})$  versus  $t$ . This procedure is sometimes recommended as a way to determine rate constants for first-order reactions by plotting experimental concentration data and determining this slope. As will be discussed in more detail in Chapter 9, this procedure is a poor way to determine a rate constant and should be viewed only as a rough approximation.

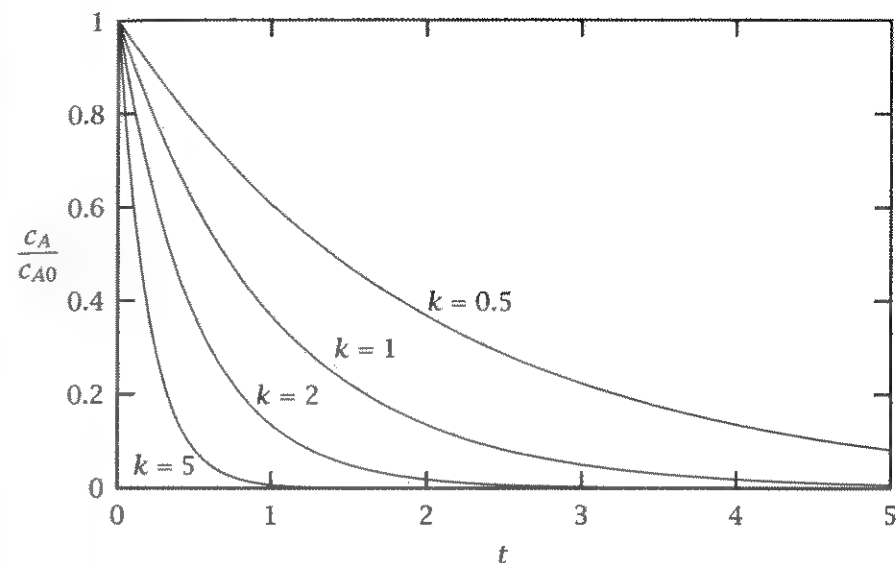


Figure 4.3: First-order, irreversible kinetics in a batch reactor.

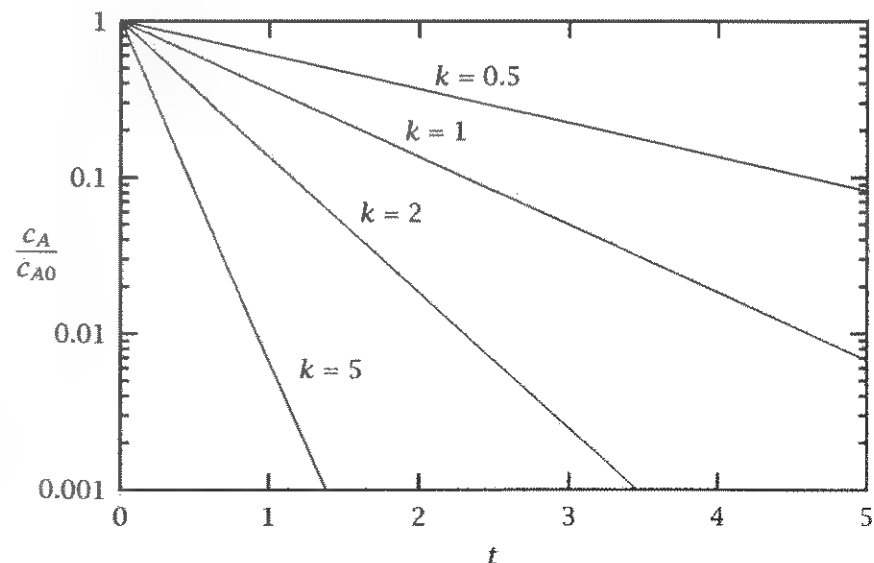


Figure 4.4: First-order, irreversible kinetics in a batch reactor, log scale.



The B concentration is easily determined from the A concentration. One could write down the material balance for component B,

$$\frac{dc_B}{dt} = R_B = kc_A \quad (4.10)$$

and solve this differential equation with the initial condition for B,  $c_B(0) = c_{B0}$ , after substituting the known solution for  $c_A(t)$ . It is simpler to note, however, that the sum of concentrations A and B is a constant. Adding Equations 4.8 and 4.10 gives

$$\frac{d(c_A + c_B)}{dt} = R_A + R_B = 0$$

Therefore,  $c_A + c_B$  is a constant and independent of time. The value of this constant is known at  $t = 0$ ,

$$c_A + c_B = c_{A0} + c_{B0}$$

which can be rearranged for the B concentration,

$$c_B = c_{A0} + c_{B0} - c_A \quad (4.11)$$

**First-order, reversible.** Consider now the same first-order reaction, but assume it is reversible



and the reaction rate is  $r = k_1c_A - k_{-1}c_B$ . The material balances for A and B are now

$$\begin{aligned} \frac{dc_A}{dt} &= -r = -k_1c_A + k_{-1}c_B \\ \frac{dc_B}{dt} &= r = k_1c_A - k_{-1}c_B \end{aligned}$$

with the same initial condition  $c_A(0) = c_{A0}$ ,  $c_B(0) = c_{B0}$ . Notice that  $c_A + c_B$  remains constant, so  $c_B$  can be computed from Equation 4.11. Substituting Equation 4.11 into the material balance for A gives

$$\frac{dc_A}{dt} = -k_1c_A + k_{-1}(c_{A0} + c_{B0} - c_A)$$

which can be rearranged into

$$\frac{dc_A}{dt} + (k_1 + k_{-1})c_A = k_{-1}(c_{A0} + c_{B0}) \quad (4.13)$$

Equation 4.13 is a nonhomogeneous, linear differential equation. The solution can be written as the sum of what is called the particular solution and the solution to the homogeneous equation [2]. One particular solution to the equation is the constant solution

$$c_{Ap} = \frac{k_{-1}}{k_1 + k_{-1}}(c_{A0} + c_{B0})$$

You should substitute this back into Equation 4.13 to check that it is indeed a solution. The homogeneous equation refers to the differential equation with a zero forcing term on the right-hand side,

$$\frac{dc_{Ah}}{dt} + (k_1 + k_{-1})c_{Ah} = 0$$

The solution to this equation already has appeared in the previous section,  $c_{Ah} = a \exp(-(k_1 + k_{-1})t)$ , in which  $a$  is an arbitrary constant to be determined from the initial condition. The full solution to Equation 4.13 is then  $c_A = c_{Ah} + c_{Ap}$ ,

$$c_A = a e^{-(k_1 + k_{-1})t} + \frac{k_{-1}}{k_1 + k_{-1}}(c_{A0} + c_{B0}) \quad (4.14)$$

The constant  $a$  is now determined from the initial condition. Writing Equation 4.14 for  $t = 0$  gives

$$c_{A0} = a + \frac{k_{-1}}{k_1 + k_{-1}}(c_{A0} + c_{B0})$$

Solving this equation for  $a$  yields

$$a = c_{A0} - \frac{k_{-1}}{k_1 + k_{-1}}(c_{A0} + c_{B0})$$

Substituting in this value of  $a$  into Equation 4.14 and rearranging terms gives the final solution

$$c_A = c_{A0} e^{-(k_1 + k_{-1})t} + \frac{k_{-1}}{k_1 + k_{-1}}(c_{A0} + c_{B0}) [1 - e^{-(k_1 + k_{-1})t}] \quad (4.15)$$

The B concentration can be determined by substituting Equation 4.11 into 4.15 and rearranging, or more simply, by switching the roles of A and B and  $k_1$  and  $k_{-1}$  in Reaction 4.12, yielding

$$c_B = c_{B0} e^{-(k_1 + k_{-1})t} + \frac{k_1}{k_1 + k_{-1}}(c_{A0} + c_{B0}) [1 - e^{-(k_1 + k_{-1})t}] \quad (4.16)$$

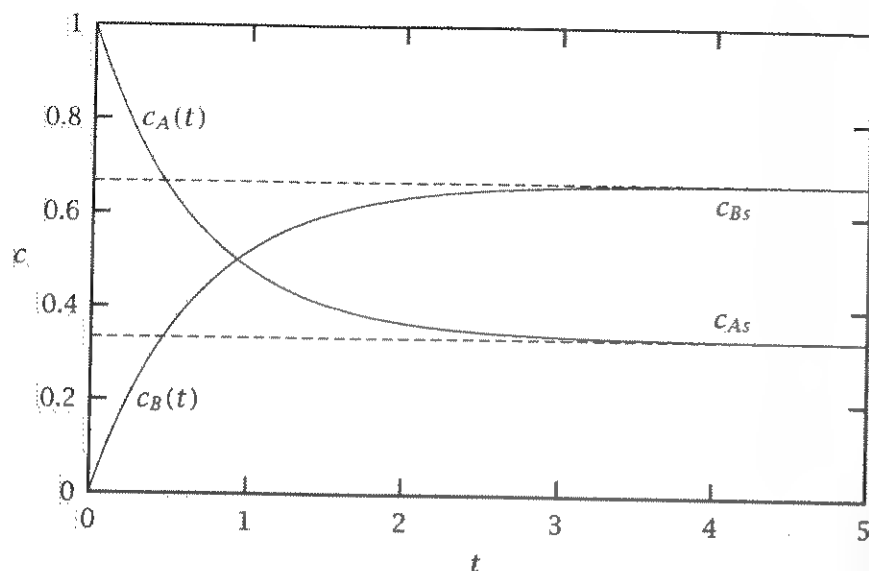


Figure 4.5: First-order, reversible kinetics in a batch reactor,  $k_1 = 1$ ,  $k_{-1} = 0.5$ ,  $c_{A0} = 1$ ,  $c_{B0} = 0$ .

Equations 4.15 and 4.16 are plotted in Figure 4.5. Notice that with the reversible reaction, the concentration of A does not go to zero as in the irreversible case, but goes to a nonzero steady-state value. We next calculate the values of the steady-state concentrations. Taking the limit  $t \rightarrow \infty$  in Equation 4.15 gives

$$c_{As} = \frac{k_{-1}}{k_1 + k_{-1}} (c_{A0} + c_{B0})$$

in which  $c_{As}$  is the steady-state concentration of A. Defining  $K_1 = k_1/k_{-1}$  allows us to rewrite this as

$$c_{As} = \frac{1}{1 + K_1} (c_{A0} + c_{B0})$$

Performing the same calculation for  $c_B$  gives

$$c_{Bs} = \frac{K_1}{1 + K_1} (c_{A0} + c_{B0})$$

These results are shown in Figure 4.5 for  $K_1 = 1/0.5 = 2$  and  $c_{A0} = 1$ ,  $c_{B0} = 0$ . Notice that because  $K_1$  is larger than 1, the forward reaction is

favored and the steady state favors the product B,  $c_{Bs} = 2/3$ ,  $c_{As} = 1/3$ . For small  $K_1$  values, the steady state would favor the reactant A.

**Second-order, irreversible.** Consider the irreversible reaction



in which the rate expression is second order,  $r = kc_A^2$ . The material balance and initial condition are

$$\frac{dc_A}{dt} = -kc_A^2, \quad c_A(0) = c_{A0} \quad (4.18)$$

This is our first nonlinear differential equation. Nonlinear differential equations do not have analytical solutions *in general*, but certain ones do. Equation 4.18 can be solved analytically because the equation is separable. Dividing both sides by  $c_A^2$  and putting the time differential on the right-hand side gives

$$\frac{dc_A}{c_A^2} = -k dt$$

in which only a function of  $c_A$  appears on the left-hand side and only a function of time appears on the right-hand side. Integrating both sides between the initial condition and a final condition of interest gives

$$\int_{c_{A0}}^{c_A} \frac{dc_A}{c_A^2} = -k \int_0^t dt$$

Performing the integrals gives

$$\frac{1}{c_{A0}} - \frac{1}{c_A} = -kt$$

Finally solving for  $c_A$  gives

$$c_A = \left( \frac{1}{c_{A0}} + kt \right)^{-1} \quad (4.19)$$

You should check that this solution does indeed satisfy the differential equation and initial condition.

Figure 4.6 displays this solution and the first-order solution with a rate constant chosen such that the initial rates are equal. Notice that although both solutions have the same qualitative features, the second-order reaction decays more slowly to zero than the first-order reaction.

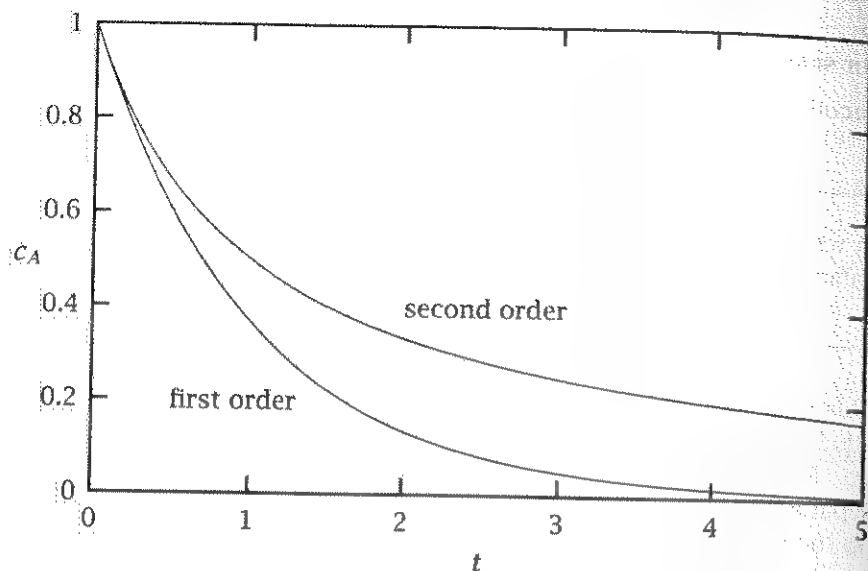


Figure 4.6: Second-order and first-order kinetics in a batch reactor,  $c_{A0} = 1, k = 1$ .

Consider another second-order, irreversible reaction,



in which the rate law is  $r = kc_A c_B$ . The material balance for components A and B are

$$\frac{dc_A}{dt} = -r = -kc_A c_B \quad (4.21)$$

$$\frac{dc_B}{dt} = -r = -kc_A c_B \quad (4.22)$$

In general we need to solve sets of nonlinear equations numerically, but again the special structure of this problem allows an analytical solution. Notice that we can subtract the material balance for B from the material balance for A to obtain

$$\frac{d(c_A - c_B)}{dt} = 0$$

which implies that in this reaction,  $c_A - c_B$  is independent of time. Because this value is known initially, one can solve for the concentration

## 4.2 The Batch Reactor

of B in terms of A,

$$c_B = c_A - c_{A0} + c_{B0} \quad (4.23)$$

Substituting this expression into the material balance for A yields

$$\frac{dc_A}{dt} = -kc_A(c_A - c_{A0} + c_{B0})$$

This equation also is separable and can be integrated to give (you should work through these steps),

$$c_A = (c_{A0} - c_{B0}) \left[ 1 - \frac{c_{B0}}{c_{A0}} e^{(c_{B0} - c_{A0})kt} \right]^{-1}, \quad c_{A0} \neq c_{B0} \quad (4.24)$$

Notice that if  $c_{A0} = c_{B0}$ , then from Equation 4.23  $c_A(t) = c_B(t)$  for all  $t$  and this case reduces to the solution previously presented in Equation 4.19. One also can obtain that result by taking the limit in Equation 4.24 as discussed in Exercise 4.11. Component B can be computed from Equation 4.23, or by switching the roles of A and B in Reaction 4.20, giving

$$c_B = (c_{B0} - c_{A0}) \left[ 1 - \frac{c_{A0}}{c_{B0}} e^{(c_{A0} - c_{B0})kt} \right]^{-1} \quad (4.25)$$

The concentration of component C also can be computed from  $c_A$  by noticing that the material balance for C is

$$\frac{dc_C}{dt} = kc_A c_B$$

and therefore,  $d(c_A + c_C)/dt = 0$ . The concentration of C is given by

$$c_C = c_{A0} - c_A + c_{C0}$$

Notice that if  $c_{A0} > c_{B0}$ , the steady-state solution is

$$c_{As} = c_{A0} - c_{B0}, \quad c_{Bs} = 0, \quad c_{Cs} = c_{B0} + c_{C0}$$

In this case, A starts out in excess and all of the B is depleted. Component A remains at steady state even though the reaction is irreversible. The final amount of C is equal to its starting value plus all the B that is present initially. For  $c_{B0} > c_{A0}$ , the steady-state solution is

$$c_{As} = 0, \quad c_{Bs} = c_{B0} - c_{A0}, \quad c_{Cs} = c_{A0} + c_{C0}$$

In this case, B starts out in excess and all of the A is depleted. Component B is present at steady state and the final amount of C is equal to its starting value plus all the A that is present initially.

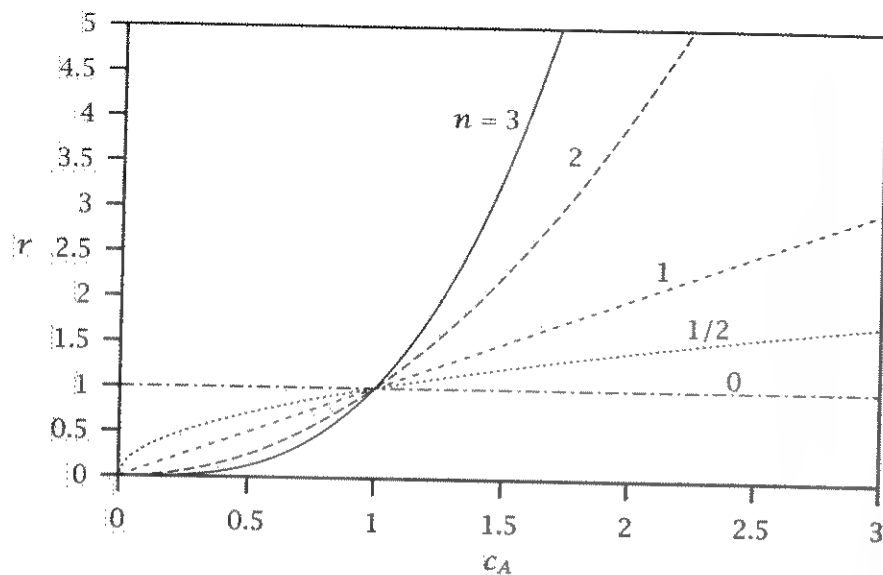


Figure 4.7: Reaction rate versus concentration for  $n$ th-order kinetics,  $r = kc_A^n$ ,  $n \geq 0$ .

**$n$ th-order, irreversible.** The  $n$ th-order rate expression  $r = kc_A^n$  is displayed in Figure 4.7 for a variety of positive  $n$  values. The material balance for Reaction 4.17 with  $n$ th-order reaction rate expression is

$$\frac{dc_A}{dt} = -r = -kc_A^n$$

This equation also is separable and can be rearranged to

$$\frac{dc_A}{c_A^n} = -k dt$$

Performing the integration and solving for  $c_A$  gives

$$c_A = [c_{A0}^{-n+1} + (n-1)kt]^{-\frac{1}{n-1}}, \quad n \neq 1$$

We can divide both sides by  $c_{A0}$  to obtain

$$\frac{c_A}{c_{A0}} = [1 + (n-1)k_0 t]^{-\frac{1}{n-1}}, \quad n \neq 1 \quad (4.26)$$

in which

$$k_0 = kc_{A0}^{n-1}$$

has units of inverse time. The  $n = 1$  case already is given in Equation 4.9, which also can be recovered by taking the appropriate limit in Equation 4.26 as discussed in Exercise 4.11. Figure 4.8 shows the behavior of  $c_A$  versus  $t$  for values of  $n$  greater than one. Notice the larger the value of  $n$ , the more slowly the A concentration approaches zero at large time. Some care should be exercised when using Equation 4.26 for  $n < 1$ . First notice in Figure 4.9 that  $c_A$  reaches zero in *finite time* for  $n < 1$ . This time can be obtained by setting the term in brackets to zero in Equation 4.26 yielding

$$t_{\text{zero}} = \begin{cases} 1/k_0(1-n), & n < 1 \\ \infty, & n \geq 1 \end{cases} \quad (4.27)$$

Finally notice in Figure 4.10 that for  $n < 0$ , the rate decreases with increasing reactant concentration; the reactant *inhibits the reaction*. Inhibition reactions are not uncommon, but care must be exercised in using this kinetic model when the concentrations are small. Notice the rate becomes unbounded as  $c_A$  approaches zero, which is not physically realistic. When using an ODE solver to compute solutions that can reach zero in finite time, it is often necessary to modify the right-hand sides of the material balance as follows

$$\frac{dc_A}{dt} = \begin{cases} -kc_A^n, & c_A > 0 \\ 0, & c_A = 0 \end{cases} \quad (4.28)$$

Note that this amendment of the material balance is equivalent to putting a discontinuity in  $r(c_A)$  at  $c_A = 0$  in Figure 4.10, so, again, models with order less than one, and especially order less than zero, should be examined carefully if the concentration reaches zero during model solution.

**Two reactions in series.** Consider the following two irreversible reactions,



Reactant A decomposes to form an intermediate B that can further react to form a final product C. Let the reaction rates be given by simple first-order rate expressions in the corresponding reactants,

$$r_1 = k_1 c_A, \quad r_2 = k_2 c_B$$

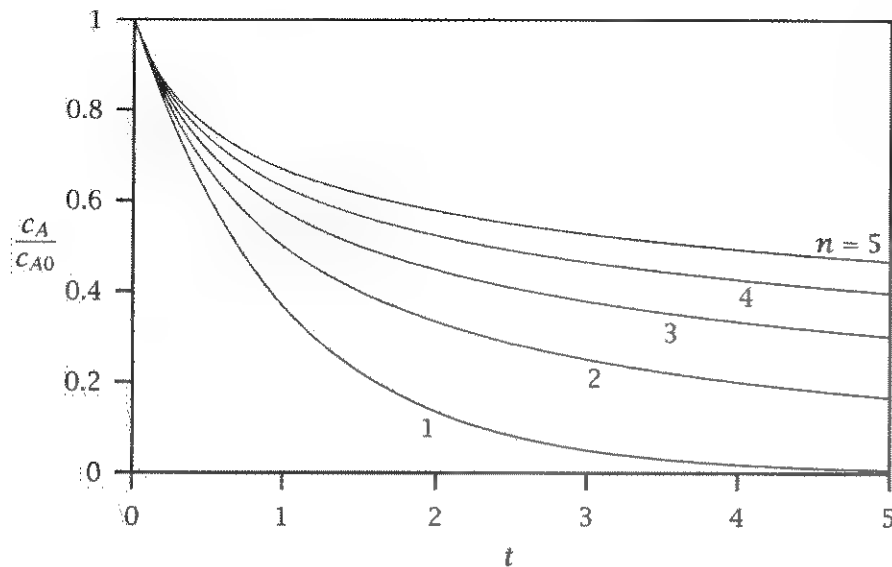


Figure 4.8: Batch reactor with  $n$ th-order kinetics,  $r = kc_A^n$ ,  $k_0 = kc_{A0}^{n-1} = 1$ ,  $n \geq 1$ .

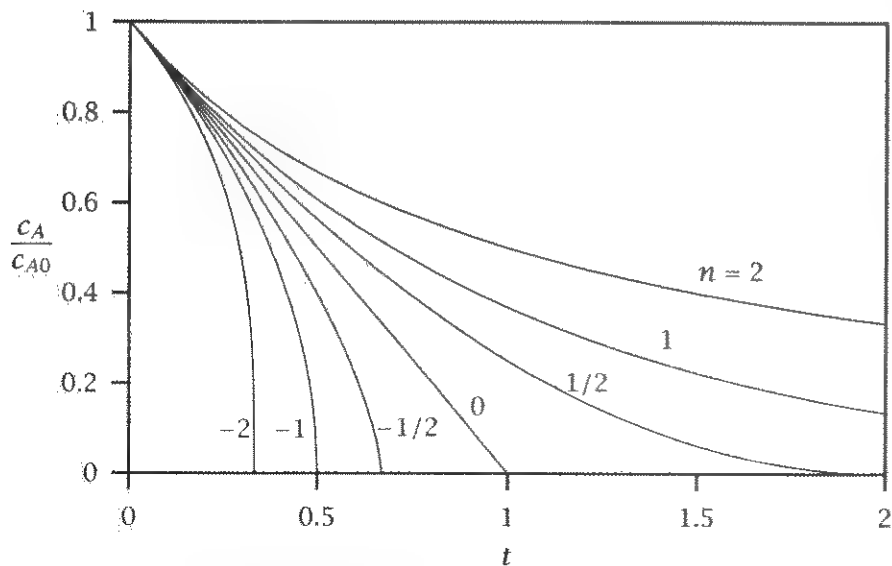


Figure 4.9: Batch reactor with  $n$ th-order kinetics,  $r = kc_A^n$ ,  $k_0 = kc_{A0}^{n-1} = 1$ ; note the concentration reaches zero in finite time  $t = 1/(k_0(1-n))$  for  $n < 1$ .

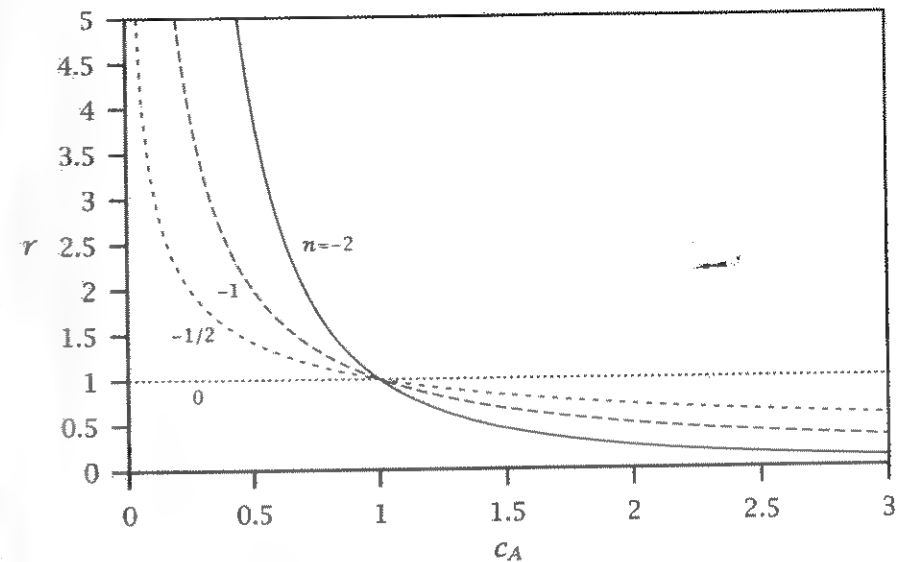


Figure 4.10: Reaction rate versus concentration for  $n$ th-order kinetics,  $r = kc_A^n$ ,  $n \leq 0$ .

The material balances for the three components are

$$\begin{aligned}\frac{dc_A}{dt} &= R_A = -r_1 = -k_1 c_A \\ \frac{dc_B}{dt} &= R_B = r_1 - r_2 = k_1 c_A - k_2 c_B \\ \frac{dc_C}{dt} &= R_C = r_2 = k_2 c_B\end{aligned}$$

The material balance for component A can be solved immediately to give  $c_A = c_{A0}e^{-k_1 t}$  as before. The material balance for B becomes

$$\frac{dc_B}{dt} + k_2 c_B = k_1 c_{A0} e^{-k_1 t}$$

The homogeneous solution is again  $c_{Bh} = a_1 e^{-k_2 t}$ . The particular solution can be guessed to be of the form  $c_{Bp} = a_2 e^{-k_1 t}$  due to the form of the forcing term on the right-hand side. Substituting the particular solution into the differential equation and solving for  $a_2$  gives

$$a_2 = c_{A0} \frac{k_1}{k_2 - k_1}, \quad k_1 \neq k_2$$

so the particular solution is known. The initial condition then determines  $a_1$ ,

$$c_B(0) = c_{B0} = a_1 + a_2$$

Solving for  $a_1$  yields  $a_1 = c_{B0} - k_1 c_{A0} / (k_2 - k_1)$ . Substituting  $a_1$  and  $a_2$  into the  $c_B$  solution gives

$$c_B = c_{B0} e^{-k_2 t} + c_{A0} \frac{k_1}{k_2 - k_1} [e^{-k_1 t} - e^{-k_2 t}], \quad k_1 \neq k_2 \quad (4.31)$$

Notice that the case of  $k_1 = k_2$  can be handled by taking limits of Equation 4.31. To determine the C concentration, notice from the material balances that  $d(c_A + c_B + c_C)/dt = 0$ , which implies  $c_A + c_B + c_C$  is constant. Therefore  $c_C$  can be computed from

$$c_C = c_{A0} + c_{B0} + c_{C0} - c_A - c_B$$

The concentrations of A, B and C are displayed in Figure 4.11 for an initial condition of pure reactant A. Notice that A decays exponentially as before; B forms at intermediate times and then disappears leaving only C as the final product. Exercise 4.10 asks for the optimal time to quench the reaction to achieve the maximum amount of intermediate B.

**Two reactions in parallel.** Consider next two parallel reactions of A to two different products, B and C,



Assume the rates of the two irreversible reactions are given by  $r_1 = k_1 c_A$  and  $r_2 = k_2 c_A$ . The material balances for the components are

$$\frac{dc_A}{dt} = R_A = -r_1 - r_2 = -k_1 c_A - k_2 c_A$$

$$\frac{dc_B}{dt} = R_B = r_1 = k_1 c_A$$

$$\frac{dc_C}{dt} = R_C = r_2 = k_2 c_A$$

The material balance for A can be solved directly to give

$$c_A = c_{A0} e^{-(k_1 + k_2)t} \quad (4.34)$$

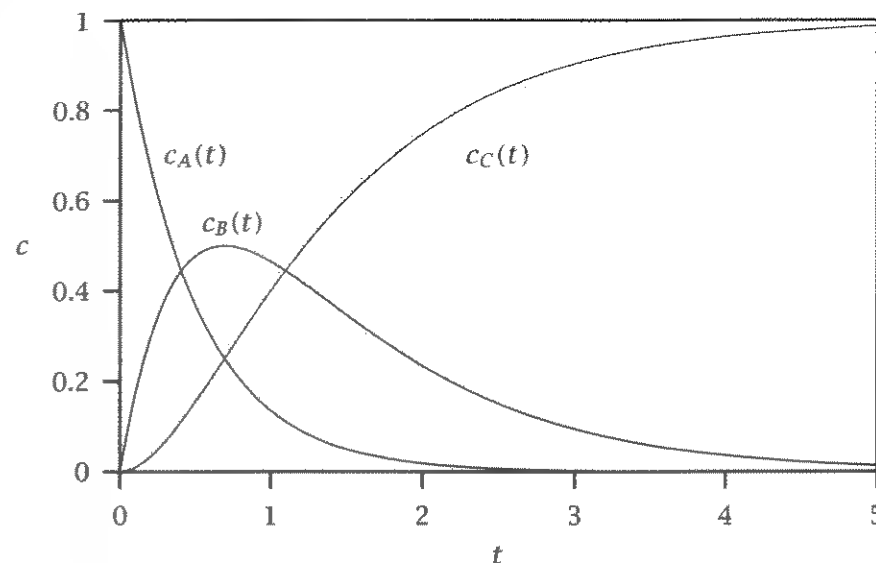


Figure 4.11: Two first-order reactions in series in a batch reactor,  $c_{A0} = 1$ ,  $c_{B0} = c_{C0} = 0$ ,  $k_1 = 2$ ,  $k_2 = 1$ .

Substituting  $c_A(t)$  into the material balance for B gives

$$\frac{dc_B}{dt} = k_1 c_{A0} e^{-(k_1 + k_2)t}$$

This equation is now separable and can be integrated directly to give

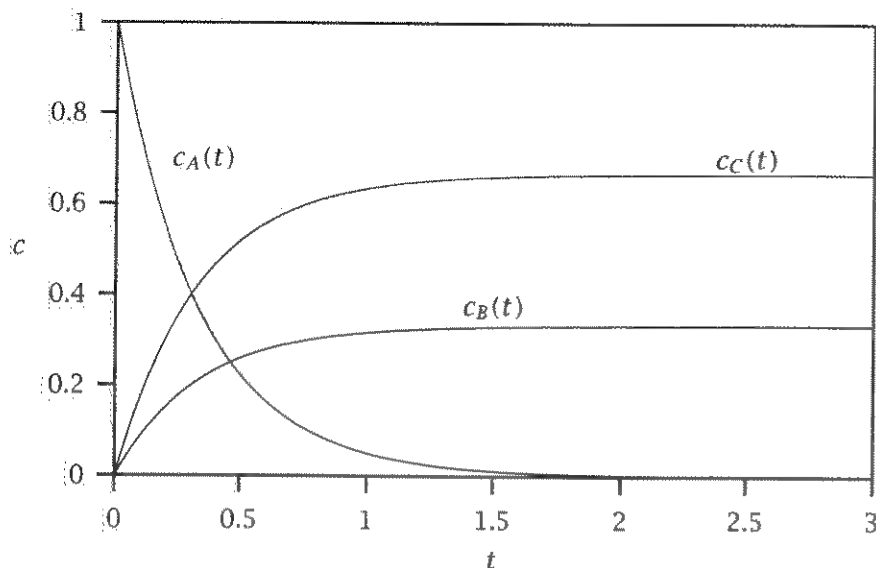
$$c_B = c_{B0} + c_{A0} \frac{k_1}{k_1 + k_2} (1 - e^{-(k_1 + k_2)t}) \quad (4.35)$$

Finally, component C can be determined from the condition that  $c_A + c_B + c_C$  is constant or by switching the roles of B and C, and  $k_1$  and  $k_2$  in Equation 4.35,

$$c_C = c_{C0} + c_{A0} \frac{k_2}{k_1 + k_2} (1 - e^{-(k_1 + k_2)t}) \quad (4.36)$$

These results are plotted in Figure 4.12. Notice that because the two parallel reactions compete for the same reactant, A, the rate constants determine which product is favored. Large values of  $k_1/k_2$  favor the formation of component B compared to C and vice versa.





**Figure 4.12:** Two first-order reactions in parallel in a batch reactor,  $c_{A0} = 1$ ,  $c_{B0} = c_{C0} = 0$ ,  $k_1 = 1$ ,  $k_2 = 2$ .

When two or more reactions are in this kind of competition, it is convenient to define **selectivity**, **yield**, and **conversion** to quantify the efficiency of the reaction in forming a desired final product. There are several ways to define selectivity, yield and conversion. However, these terms are often invoked without careful definition in the hope that the meaning is clear from context. This practice often confuses the issue and should be avoided.

**Point selectivity:** The point (or instantaneous) selectivity is the ratio of the production rate of one component to the production rate of another component.

**Overall selectivity:** The overall selectivity is the ratio of the amount of one component produced to the amount of another component produced.

**Yield:** The yield of component  $j$  is the fraction of a reactant that is converted into component  $j$ .

**Conversion:** Conversion is normally defined to be the fraction of a component that has been converted to products by the reaction network. Conversion has several definitions and conventions. It is best to state the definition in the context of the problem being solved.

### 4.3 The Continuous-Stirred-Tank Reactor (CSTR)

The continuous-stirred-tank reactor (CSTR) is also a well-stirred reactor so there are no concentration gradients anywhere in the reactor volume. We again consider the entire reactor contents to be the reactor volume element as in Figure 4.13, and  $V = V_R$ . Since the reactor is well stirred, the relations in Equations 4.3 and 4.4 apply to the CSTR also. The difference between the CSTR and batch reactor is the flow streams shown in Figure 4.13. We denote the feed stream with flowrate  $Q_f$  and component  $j$  concentration  $c_{jff}$ . The outflow stream is flowing out of a well-mixed reactor and is therefore assumed to be at the same concentration as the reactor. Its flowrate is denoted  $Q$ . Writing Equation 4.2 for this reactor gives,

$$\frac{d(c_j V_R)}{dt} = Q_f c_{jff} - Q c_j + R_j V_R, \quad j = 1, \dots, n_s \quad (4.37)$$

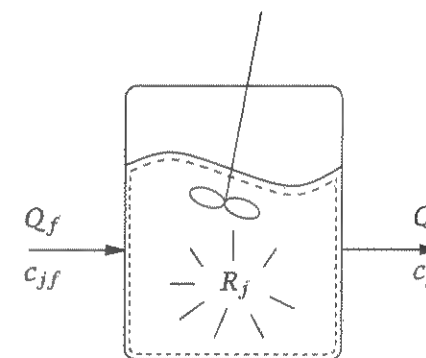
If the reactor volume is constant and the volumetric flowrates of the inflow and outflow streams are the same, Equation 4.37 reduces to

$$\frac{dc_j}{dt} = \frac{1}{\theta} (c_{jff} - c_j) + R_j \quad (4.38)$$

in which

$$\theta = V_R / Q_f$$

is called the **mean residence time** of the CSTR. The residence time distribution will be discussed further in Chapter 8. We refer to this



**Figure 4.13:** CSTR volume element.

balance as the constant-density case. It is often a good approximation for liquid-phase reactions.

The steady state of the CSTR is described by setting the time derivative in Equation 4.37 to zero,

$$0 = Q_f c_{jf} - Q c_j + R_j V_R \quad (4.39)$$

Conversion of reactant  $j$  is defined for a steady-state CSTR as follows

$$x_j = \frac{Q_f c_{jf} - Q c_j}{Q_f c_{jf}} \quad (\text{steady state}) \quad (4.40)$$

One can divide Equation 4.39 through by  $V_R$  to obtain for the constant-density case

$$c_j = c_{jf} + R_j \theta \quad (\text{steady state, constant density}) \quad (4.41)$$

#### Example 4.1: Reaching steady state in a CSTR

Consider a first-order, liquid-phase reaction in an isothermal CSTR shown in Figure 4.13



the feed concentration of A is  $c_{Af} = 2 \text{ mol/L}$ , the residence time of the reactor is  $\theta = 100 \text{ min}$ , and the rate constant is  $k = 0.1 \text{ min}^{-1}$ .

1. Find the steady-state concentration of A in the effluent for the given feed.
2. Plot the concentration of A versus time for constant feed concentration  $c_{Af} = 2 \text{ mol/L}$  if the reactor is initially filled with an inert so  $c_{A0} = 0 \text{ mol/L}$ .
3. Plot the concentration of A versus time for constant feed concentration  $c_{Af} = 2 \text{ mol/L}$  if the reactor is initially filled with feed so  $c_{A0} = 2 \text{ mol/L}$ .

#### Solution

**Part 1.** Because the reaction is isothermal and takes place in the liquid phase, we assume the fluid density is constant. Starting with Equation 4.41 for component A

$$c_A = c_{Af} + R_A \theta$$

Substituting the production rate  $R_A = -k c_A$  and solving for  $c_A$  gives the steady-state concentration

$$c_{As} = \frac{c_{Af}}{1 + k\theta}$$

Substituting in the numerical values gives

$$c_{As} = \frac{2 \text{ mol/L}}{1 + (0.1 \text{ min}^{-1})(100 \text{ min})} = 0.182 \text{ mol/L}$$

**Parts 2 and 3.** The constant-density mass balance for component A is given by Equation 4.38

$$\begin{aligned} \frac{dc_A}{dt} &= \frac{1}{\theta} (c_{Af} - c_A) - k c_A \\ c_A(0) &= c_{A0} \end{aligned} \quad (4.42)$$

Equation 4.42 can be arranged in the form

$$\frac{dc_A}{dt} + \left( \frac{1}{\theta} + k \right) c_A = \frac{1}{\theta} c_{Af}$$

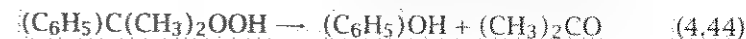
a linear differential equation of the same form as Equation 4.13, which was solved earlier. Exercise 4.6 provides another convenient method for solving linear differential equations analytically. Using either of these methods, we find the analytical solution

$$c_A(t) = c_{A0} e^{-((1/\theta)+k)t} + \frac{c_{Af}}{1+k\theta} [1 - e^{-((1/\theta)+k)t}] \quad (4.43)$$

Figure 4.14 shows the transient solution for the two initial conditions  $c_{A0} = 0$  and  $c_{A0} = 2 \text{ mol/L}$ . Notice both solutions converge to the same steady-state solution as time increases even though the starting conditions are quite different. We will see in Chapter 6 that the nonisothermal reactor behavior can be much more complex than that shown in Figure 4.14. □

#### Example 4.2: Phenol production in a CSTR

Next we consider a simple reactor-sizing problem. Consider the reaction of cumene hydroperoxide (CHP) to phenol and acetone



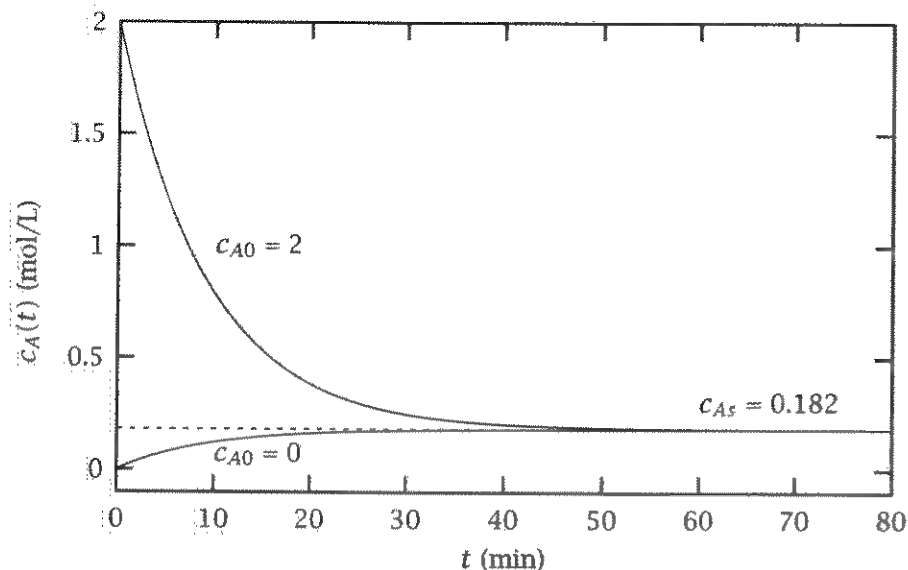


Figure 4.14: Reaching steady state in a CSTR.

This liquid-phase reaction is conducted using a small amount of acid as a catalyst [12], and the reaction rate is first order in the concentration of CHP and acid [5]. For the reactor temperature (85°C) and feed conditions, the reaction rate is given by  $r = kc_{\text{CHP}}$ , which means this is a pseudo-first-order reaction. Find the reactor volume to achieve 85% conversion of CHP at steady state. The flowrate into the reactor is  $Q_f = 26.9 \text{ m}^3/\text{hr}$  and  $k = 4.12 \text{ hr}^{-1}$ .

#### Solution

The reactants and products are liquids at 85°C and the volume change is neglected. Therefore we assume  $Q = Q_f$  and Equation 4.41 applies,

$$c_A = c_{Af} + R_A \theta$$

where A designates CHP. Substituting in for the production rate of CHP,  $R_A = -kc_A$ , and solving for the CHP concentration gives

$$c_A = \frac{c_{Af}}{1 + k\theta} \quad (4.45)$$

The conversion of CHP (for  $Q = Q_f$ ) is

$$x_A = \frac{c_{Af} - c_A}{c_{Af}} = 1 - \frac{c_A}{c_{Af}}$$

Substituting Equation 4.45 into the above equation gives

$$x_A = \frac{k\theta}{1 + k\theta}$$

Solving for  $\theta$  in terms of conversion gives

$$\theta = \frac{x_A}{k(1 - x_A)}$$

Substituting the relation  $V_R = Q_f \theta$  and solving for  $V_R$  gives

$$V_R = \frac{Q_f x_A}{k(1 - x_A)}$$

Substituting in the known values gives the required CSTR volume

$$V_R = \frac{(26.9 \text{ m}^3/\text{hr})(0.85)}{(4.12 \text{ hr}^{-1})(0.15)} = 37 \text{ m}^3$$

□

## 4.4 The Semi-Batch Reactor

The semi-batch reactor is a cross between the batch reactor and CSTR. The semi-batch reactor is initially charged with reactant, like the batch reactor, but allows a feed addition policy while the reaction takes place, like the CSTR. Normally there is no outflow stream. We then set  $Q = 0$  in Equation 4.37 to obtain for the semi-batch reactor

$$\frac{d(c_j V_R)}{dt} = Q_f c_{jf} + R_j V_R, \quad j = 1, \dots, n_s \quad (4.46)$$

One may choose to operate a semi-batch reactor to control the reaction rate or heat release during reaction by slowly adding one of the reactants in the feed stream. Compared to the batch reactor, the semi-batch reactor provides more complete use of the reactor volume in reactions such as polymerizations that convert from lower density to higher density during the course of the reaction.

## 4.5 Volume Change Upon Reaction

The preceding examples in the batch reactor and CSTR sections were solved under the simplifying assumption that the composition of the

reaction mixture has no effect on the density of the mixture. This assumption is useful for ideal-liquid mixtures in which the pure components have similar densities, and is a correct statement for ideal-gas mixtures. In this section we establish the appropriate modeling equations for the general case. We start with the material balance for a well-mixed reactor, Equation 4.37,

$$\frac{d(c_j V_R)}{dt} = Q_f c_{jf} - Q c_j + R_j V_R, \quad j = 1, 2, \dots, k, \dots, n_s \quad (4.47)$$

Equation 4.47 covers the batch, CSTR and semi-batch reactors, depending on how we specify  $Q_f$  and  $Q$ .

If we multiply Equation 4.47 by the molecular weight of species  $j$  and sum over all species we obtain,

$$\frac{d(\sum_j c_j M_j V_R)}{dt} = Q_f \sum_j c_{jf} M_j - Q \sum_j c_j M_j + \sum_j R_j M_j V_R \quad (4.48)$$

The term  $\sum_j c_j M_j$  is the mass density of the reactor contents, which we denote  $\rho$

$$\rho = \sum_{j=1}^{n_s} c_j M_j \quad (4.49)$$

The term  $\sum_j c_{jf} M_j$  is the mass density of the feedstream,  $\rho_f$ . We know that conservation of mass in chemical reactions implies  $\sum_j R_j M_j = 0$  (see Example 2.5). Substitution into Equation 4.48 leads to

$$\frac{d(\rho V_R)}{dt} = Q_f \rho_f - Q \rho \quad (4.50)$$

Equation 4.50 is clearly a total mass balance, in which the total mass in the reactor changes in time due to the inflow and outflow of mass. Notice that chemical reactions play no role in the total mass balance. The total mass balance is the starting point for the derivation of governing equations for numerous special situations in which the density and the volume of the reactor change, depending on the physical properties of the reaction mixture and the manner in which the reactor is operated.

Notice that because the total mass balance is a linear combination of the mole balances, it is not independent information. We may sometimes prefer to include the mass balance *in place* of one of the mole balances,  $j = k$ , to ensure that numerical roundoff does not cause small violations of the mass balance. The concentration of the  $k$ th species is

then obtained by subtraction

$$c_k = \frac{\rho - \sum_{j \neq k} c_j M_j}{M_k} \quad (4.51)$$

Notice that one should always replace a high-concentration species with the mass balance so that catastrophic loss of precision does not occur in the subtraction to obtain  $c_k$ .

If we have a single-phase system at equilibrium, the molar concentrations of the components, and the temperature and the pressure completely specify all intensive variables of the system. In this chapter we consider the temperature and the pressure to be known, fixed quantities. Therefore, the density of the reaction mixture, which is an intensive variable, is known if the  $c_j$  are known. This relationship is one form of the equation of state for the mixture

$$\rho = \tilde{f}(T, P, c_1, c_2, \dots, c_{n_s}) \quad (4.52)$$

Substituting the definition of density, Equation 4.49, into Equation 4.52 shows that the equation of state is a single constraining relationship on the set of possible mixture concentrations. Because  $T$  and  $P$  are fixed constants, it is convenient to express the equation of state as

$$f(c_1, c_2, \dots, c_{n_s}) = 0 \quad (4.53)$$

For example, we could express the equation of state in terms of the partial molar volumes as

$$\sum_j c_j V_j = 1$$

in which  $V_j$  is the partial molar volume of component  $j$  in the mixture. The partial molar volumes are functions of  $T$ ,  $P$  and  $c_j$ . If we assume an ideal mixture, this reduces to

$$\sum_j c_j V_j^\circ = 1, \quad \text{ideal mixture}$$

in which  $V_j^\circ$  is the specific volume of pure component  $j$ , which is a function of only  $T$  and  $P$ . We shall assume that a thermodynamic equation of state is valid even when the reactor is not at equilibrium. We briefly review the situation when the density does not vary with composition and then treat the general case.

**Constant density.** Because the mixture density,  $\rho$ , is independent of composition, it does not vary with time either and we can set it to the feed value,

$$\rho = \rho_f \quad (4.54)$$

The total mass balance then reduces to

$$\frac{dV_R}{dt} = Q_f - Q \quad (4.55)$$

which is sometimes referred to as a "volume balance." This terminology should be avoided because one cannot write a conservation statement for volume. The differential equation for the reactor volume is a consequence of the mass balance and the simplifying assumption of constant-mixture density. The three reactor types we have considered thus far are summarized below.

- **Batch reactor.** For the batch reactor,  $Q = Q_f = 0$ . We can therefore conclude from Equation 4.55 that a batch reactor with constant density has constant volume.
- **CSTR (dynamic and steady state).** If the outflow of the CSTR is regulated so that the CSTR has constant volume, then we can conclude from Equation 4.55 that  $Q = Q_f$ .
- **Semi-batch reactor.** In the semi-batch reactor, the reactor is filled during operation so  $Q_f$  is specified and positive for some time and  $Q = 0$ . The solution to Equation 4.55 then determines the change in volume of the reactor during the filling operation.

**Nonconstant density.** In the general case, we can consider the following variables to fully determine the state of the reactor:  $T, P, n_j, V_R$ . We assume throughout that the feed conditions,  $Q_f, c_{jf}$ , are specified. In the nonconstant density case, we switch to  $n_j$  from  $c_j$  because the material balance is fundamentally a balance on total moles in the reactor, not molar concentrations. If we know the  $n_j$  and  $V_R$ , then the values of the  $c_j$  are known, as required in the reaction rate expressions and the equation of state. To specify the right-hand sides of the material balances, Equation 4.47, we also require the value of  $Q$ . Because  $T, P$  are regarded as known, fixed constants in this chapter, the set of unknowns is  $n_j, V_R, Q$ . We therefore have  $n_s + 2$  unknowns to determine and require  $n_s + 2$  equations. We have the  $n_s$  equations from the component mole balances, Equations 4.47. The equation of state, Equation 4.53, provides one additional equation. The final equation is provided by

a statement of reactor operation. We consider here three important cases of reactor operations.

1. **Constant-volume reactor.** The constant-volume reactor can be achieved by allowing overflow of the reactor to determine flow-rate out of the reactor. In this situation,  $V_R$  is specified as the additional equation.
2. **Constant-mass reactor.** The constant-mass reactor can be achieved if a differential pressure measurement is used to control the flow-rate out of the reactor and the reactor has constant cross-sectional area. The difference in pressure between the top and bottom of the reactor is determined by  $\rho h$  in which  $h$  is the height difference. If the cross-sectional area is independent of height, then the pressure difference is also proportional to  $\rho V_R$ , the total mass in the reactor. So if the flow controller maintains constant pressure difference, it also maintains constant mass. In this situation  $\rho V_R$  is specified as the additional equation.
3. **Flowrate out of the reactor is specified.** This type of operation may be achieved if the flowrate out of the reactor is controlled by a flow controller. In this case  $Q(t)$  is specified. A semi-batch reactor is operated in this way with  $Q = 0$  until the reactor is filled with the reactants.

The calculations for the nonconstant-density case may be greatly simplified by using a differential-algebraic equation (DAE) solver. All three cases enumerated above can be handled by modifying the residual equations provided to the DAE solver. We do not have to differentiate the equation of state or perform other algebraic manipulations that are required if one uses an ordinary differential equation (ODE) solver.

In the general case, to produce a differential equation for the reactor volume, we begin by taking the differential of the equation of state  $f(c_j) = 0$  to obtain

$$\frac{df}{dt} = \sum_j \frac{\partial f}{\partial c_j} \frac{dc_j}{dt} = \sum_j f_j \frac{dc_j}{dt} = 0 \quad (4.56)$$

in which

$$f_j = \frac{\partial f}{\partial c_j}$$

Now differentiating the product term  $(c_j V_R)$  gives the relation

$$V_R \frac{dc_j}{dt} = \frac{d(c_j V_R)}{dt} - c_j \frac{dV_R}{dt}$$

Multiplying by  $f_j$  and summing gives

$$V_R \sum_j f_j \frac{dc_j}{dt} = \sum_j f_j \frac{d(c_j V_R)}{dt} - \sum_j f_j c_j \frac{dV_R}{dt} = 0$$

in which we have used Equation 4.56. Substituting the material balance into this equation and solving for the rate of change of reactor volume gives

$$\frac{dV_R}{dt} = \frac{\sum_j f_j (Q_f c_{jf} - Q c_j + R_j V_R)}{\sum_j f_j c_j}$$

Expressing the production rate in terms of the reaction rate yields

$$\frac{dV_R}{dt} = Q_f \frac{\sum_j f_j c_{jf}}{\sum_j f_j c_j} - Q + \frac{\sum_i \Delta f_i r_i V_R}{\sum_j f_j c_j} \quad (4.57)$$

in which  $\Delta f_i$  is defined to be

$$\Delta f_i = \sum_j v_{ij} f_j = \sum_j v_{ij} \frac{\partial f}{\partial c_j}$$

which is the change in the derivative property upon reaction. For example, if an ideal mixture is assumed for the equation of state,  $f_j = V_j^\circ$ , the pure component specific volumes, and  $\Delta f_i = \Delta V_i^\circ$ , the change in specific volume upon reaction  $i$ .

The constant-density and ideal-mixture cases for the three reactor operations are summarized in Table 4.1. We have  $n_s + 2$  unknowns and the component balances at the top of Table 4.1 provide  $n_s$  equations. The remaining two equations for the constant-mass, constant-volume and specified-outflow operations are provided lower in the table. In these cases, these two extra equations are either differential equations or merely specify the values of constants.

The general equation of state case for the three reactor operations is summarized in Table 4.2. The  $n_s$  component balances are the same as in Table 4.1. But for the remaining two equations, we have a choice. We can either specify algebraic equations to obtain DAEs for the model, or differentiate the algebraic relations to obtain ODEs. DAEs may be preferred because the ODEs require the partial derivatives of the equation of state,  $f_j = \partial f / \partial c_j$ , as shown in Equation 4.57. Depending on the form of the equation of state, these derivatives may be complex expressions. In the following example, we set up and solve a nonconstant-density case, in which the products are denser than the reactants.

Unknowns ( $n_s + 2$ ):		$V_R, Q, n_j, \quad j = 1, \dots, n_s$	
Component balances:		$\frac{dn_j}{dt} = Q_f c_{jf} - Q c_j + R_j V_R, \quad j = 1, \dots, n_s$	
Defined quantities:		$n_j = c_j V_R \quad \rho = \sum_j c_j M_j \quad \Delta V_i^\circ = \sum_j v_{ij} V_j^\circ$	
	(i) constant density: $\rho = \rho_0$	(ii) ideal mixture: $\sum_j c_j V_j^\circ = 1$	
1. vol	$V_R = V_{R0}$	$V_R = V_{R0}$	$Q = Q_f + \sum_i \Delta V_i^\circ r_i V_R$
2. mass	$V_R = V_{R0}$	$\frac{dV_R}{dt} = Q_f (1 - \rho_f / \rho) + \sum_i \Delta V_i^\circ r_i V_R$	$Q = Q_f \rho_f / \rho$
3. $Q$	$\frac{dV_R}{dt} = Q_f - Q$	$Q = Q_f - Q + \sum_i \Delta V_i^\circ r_i V_R$	$Q_{\text{specified}}$

Table 4.1: Reactor balances for constant-density and ideal-mixture assumptions.



Unknowns ( $n_s + 2$ ):	$V_R, Q, n_j, \quad j = 1, \dots, n_s$		
Component balances:	$\frac{dn_j}{dt} = Q_f c_{jf} - Q c_j + R_j V_R, \quad j = 1, \dots, n_s$		
Defined quantities:	$n_j = c_j V_R \quad \rho = \sum_j c_j M_j \quad \Delta f_i = \sum_j v_{ij} \frac{\partial f}{\partial c_j}$		
Equation of state:	$f(c_1, c_2, \dots, c_{n_s}) = 0$		
	DAEs	ODEs	
1. vol	$V_R = V_{R0}$	$f(c_j) = 0$	$Q = Q_f \frac{\sum_j f_{jcj}}{\sum_j f_{jcj}} + \frac{\sum_i \Delta f_i V_R}{\sum_j f_{jcj}}$
2. mass	$\rho V_R = \rho_0 V_{R0}$	$f(c_j) = 0$	$Q = Q_f \rho_f / \rho$
3. Q	Q specified	$f(c_j) = 0$	Q specified

Table 4.2: Reactor balances for general equation of state.

**Example 4.3: Semi-batch polymerization**

Consider a solution polymerization reaction, which can be modeled as a first-order, irreversible reaction



A 20 m<sup>3</sup> semi-batch reactor is initially charged with solvent and initiator to half its total volume. A pure monomer feed is slowly added at flowrate  $Q_{f0} = 1$  m<sup>3</sup>/min to fill the reactor in semi-batch operation to control the heat release.

Consider two cases for the subsequent reactor operation.

1. The monomer feed is shut off and the reaction goes to completion.
2. The monomer feed is adjusted to keep the reactor filled while the reaction goes to completion.

Calculate the total polymer mass production, and the percentage increase in polymer production achieved in the second operation. You may assume an ideal mixture; the densities of monomer and polymer are  $\rho_M = 800$  kg/m<sup>3</sup>,  $\rho_P = 1100$  kg/m<sup>3</sup>. The monomer molecular weight is  $M_M = 100$  kg/kmol, and the rate constant is  $k = 0.1$  min<sup>-1</sup>.

**Solution**

While the reactor is filling, the monomer mole balance is

$$\frac{d(c_M V_R)}{dt} = Q_{f0} c_{Mf} - k c_M V_R$$

in which  $c_{Mf} = \rho_M / M_M$  is given, and  $Q_f = Q_{f0}$  is constant during the filling operation. We denote the total number of moles of monomer by  $M = c_M V_R$ , and can write the monomer balance as

$$\frac{dM}{dt} = Q_{f0} c_{Mf} - k M \quad (4.58)$$

$$M(0) = 0$$

For an ideal mixture, the volume is given by

$$\frac{dV_R}{dt} = Q_{f0} + \Delta V k M \quad (4.59)$$

$$V_R(0) = 10 \text{ m}^3$$

in which

$$\Delta V = (1/\rho_P - 1/\rho_M) M_M$$

To compute the polymer mass, we note from the stoichiometry that the mass production rate of polymer  $\tilde{R}_p$  is

$$\tilde{R}_p = -R_M M_M$$

so the mass balance for total polymer  $\tilde{P}$  is given by

$$\frac{d\tilde{P}}{dt} = \tilde{R}_p V_R = k c_M M_M V_R = (k M_M) M \quad (4.60)$$

Let  $t_1$  be the time that the reactor fills. Equation 4.58 is of the same form as Equation 4.13 and can be solved in the same manner, yielding

$$M(t) = \frac{Q_{f0} c_{Mf}}{k} (1 - e^{-kt}), \quad t \leq t_1 \quad (4.61)$$

Substituting this result into Equations 4.59 and 4.60 and integrating gives

$$V_R(t) = V_R(0) + (1 + \Delta V c_{Mf}) Q_{f0} t - \frac{\Delta V Q_{f0} c_{Mf}}{k} (1 - e^{-kt}), \quad t \leq t_1 \quad (4.62)$$

$$\tilde{P}(t) = Q_{f0} c_{Mf} M_M \left( t - \frac{1}{k} (1 - e^{-kt}) \right), \quad t \leq t_1 \quad (4.63)$$

We can solve Equation 4.62 for the time  $t_1$  when the reactor fills,  $V_R(t_1) = 20 \text{ m}^3$ . Note we have a single nonlinear equation that must be solved numerically. Substituting in the numerical values and solving for  $t_1$  yields

$$t_1 = 11.2 \text{ min}$$

Note the reactor would have filled in 10 min if the density were constant. The extra time reflects the available volume created by converting some of the monomer to polymer during filling. After  $t_1$  we consider the two operations.

**Operation 1.** In the first operation,  $Q_f = 0$  after  $t_1$ . Substituting this value of  $Q_f$  into Equations 4.58 and 4.59 gives

$$\frac{dM}{dt} = -kM \quad (4.64)$$

$$\frac{dV_R}{dt} = \Delta V k M \quad (4.65)$$

in which initial condition  $M(t_1)$  is given by substituting  $t_1$  in Equation 4.61, and  $V_R(t_1) = 20 \text{ m}^3$ . We solve Equation 4.64 to obtain

$$M(t) = M(t_1) e^{-k(t-t_1)}, \quad t_1 \leq t$$

and substitution into Equations 4.65 and 4.60 and integration gives

$$V_R(t) = V_R(t_1) + \Delta V M(t_1) (1 - e^{-k(t-t_1)}), \quad t_1 \leq t \quad (4.66)$$

$$\tilde{P}(t) = \tilde{P}(t_1) + M_M M(t_1) (1 - e^{-k(t-t_1)}), \quad t_1 \leq t \quad (4.67)$$

Notice the reactor volume decreases after  $t_1$  because  $\Delta V$  is negative. These results are plotted in Figures 4.15–4.18.

**Operation 2.** Because the reactor volume is constant, we can solve Equation 4.59 for the feed flowrate during the secondary monomer addition

$$Q_f = -\Delta V k M$$

Substitution of this flowrate into Equation 4.58 gives

$$\frac{dM}{dt} = -(1 + \Delta V c_{Mf}) k M$$

which can be solved to give

$$M(t) = M(t_1) \exp \left( -(1 + \Delta V c_{Mf}) k (t - t_1) \right), \quad t_1 \leq t$$

The polymer mass can be calculated by substituting this result into Equation 4.60 and integrating

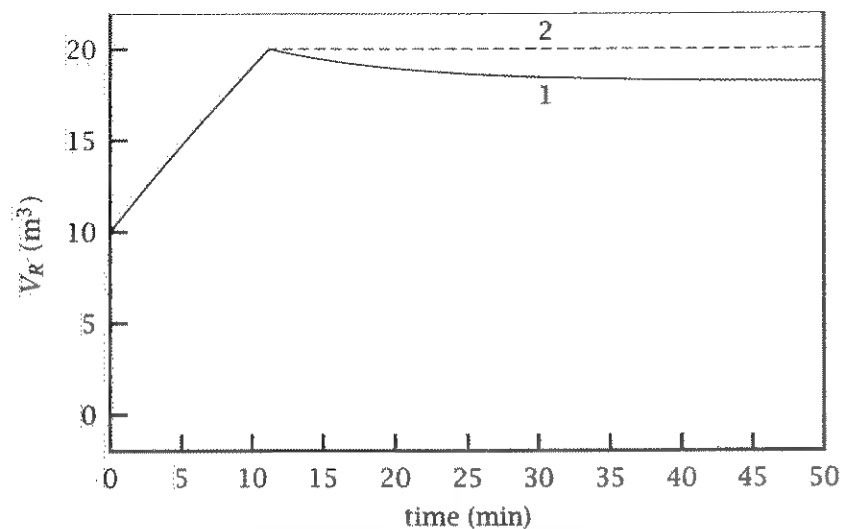
$$\tilde{P}(t) = \tilde{P}(t_1) + \frac{M_M M(t_1)}{(1 + \Delta V c_{Mf})} \left[ 1 - \exp \left( -(1 + \Delta V c_{Mf}) k (t - t_1) \right) \right], \quad t_1 \leq t$$

These results also are plotted in Figures 4.15–4.18. Notice the final polymer production is larger in Operation 2 because of the extra monomer addition. We can perform an independent, simple calculation of the total polymer in Operation 2. In Operation 2,  $10 \text{ m}^3$  of polymer are produced because in an ideal mixture, the volumes are additive. Therefore

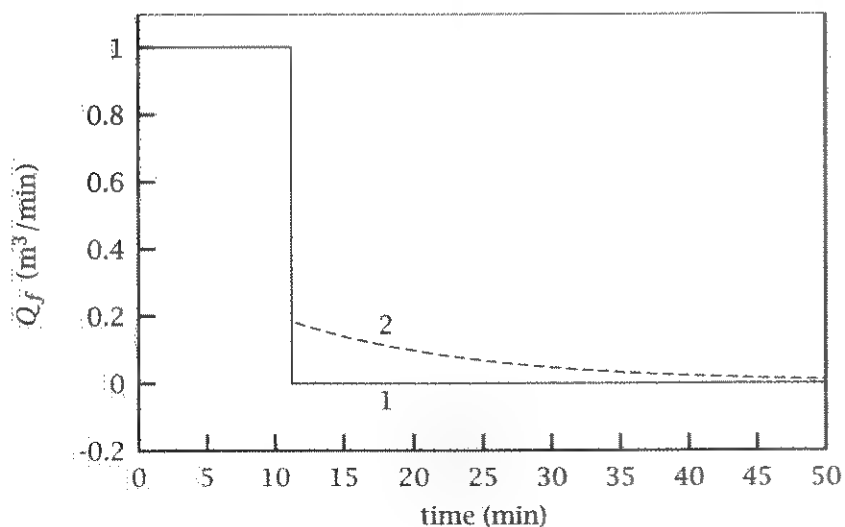
$$\tilde{P}_2 = (V_R - V_{R0}) \rho_P = 10 \text{ m}^3 \times 1100 \text{ kg/m}^3 = 11000 \text{ kg}$$

in good agreement with the long-time solution for Operation 2 in Figure 4.18. For Operation 1, we have no simple calculation, so we take the limit as  $t \rightarrow \infty$  in Equation 4.67 to obtain

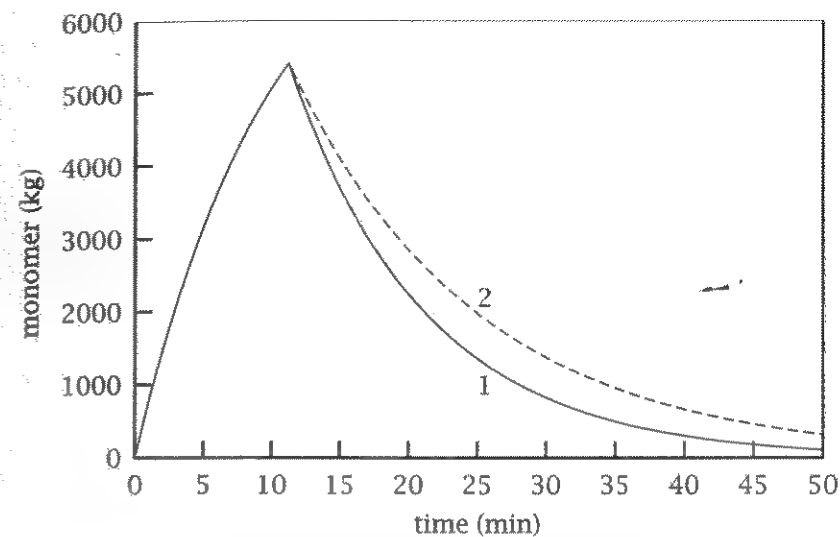
$$\tilde{P}_1 = \tilde{P}(t_1) + M_M M(t_1)$$



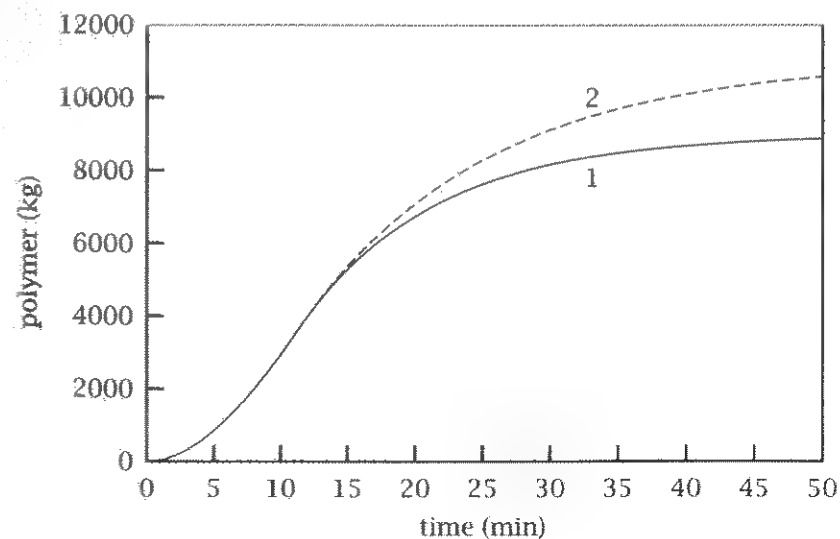
**Figure 4.15:** Semi-batch reactor volume for primary monomer addition (Operation 1) and primary plus secondary monomer additions (Operation 2).



**Figure 4.16:** Semi-batch reactor feed flowrate for primary monomer addition (Operation 1) and primary plus secondary monomer additions (Operation 2).



**Figure 4.17:** Semi-batch reactor monomer content for primary monomer addition (Operation 1) and primary plus secondary monomer additions (Operation 2).



**Figure 4.18:** Semi-batch reactor polymer content for primary monomer addition (Operation 1) and primary plus secondary monomer additions (Operation 2).

Both  $\tilde{P}(t_1)$  and  $M(t_1)$  can be evaluated by substituting  $t_1 = 11.2$  min in Equations 4.61 and 4.63, respectively. The results are

$$\tilde{P}(t_1) = 3582 \text{ kg}, \quad M(t_1) = 53.95 \text{ kmol}, \quad \tilde{P}_1 = 8977 \text{ kg}$$

Note the value  $\tilde{P}_1 = 8977$  kg is in good agreement with the long-time solution for Operation 1 in Figure 4.18. Finally, the increase in production rate is

$$\frac{\tilde{P}_2 - \tilde{P}_1}{\tilde{P}_1} \times 100\% = 22.5\%$$

By using the volume of the reactor more efficiently, the total polymer production increases 22.5%.  $\square$

## 4.6 The Plug-Flow Reactor (PFR)

Plug flow in a tube is an ideal-flow assumption in which the fluid is well mixed in the radial and angular directions. The fluid velocity is assumed to be a function of only the axial position in the tube. Plug flow is often used to approximate fluid flow in tubes at high Reynolds number. The turbulent flow mixes the fluid in the radial and angular directions. Also in turbulent flow, the velocity profile is expected to be reasonably flat in the radial direction except near the tube wall.

### 4.6.1 Thin Disk Volume Element

Because of the plug-flow assumption, it is natural to take a thin disk for the reactor volume element as shown in Figure 4.19. The concentration does not change over the volume element because there is complete mixing in the radial and angular directions and because the axial distance  $\Delta z$  is small. The element has volume  $\Delta V = A_c \Delta z$  in which  $A_c$  is the tube cross-sectional area. Writing Equation 4.2 in this situation gives

$$\frac{\partial (c_j \Delta V)}{\partial t} = c_j Q \Big|_z - c_j Q \Big|_{z+\Delta z} + R_j \Delta V \quad (4.68)$$

Dividing the above equation by  $\Delta V$  and taking the limit as  $\Delta V$  goes to zero yields,

$$\underbrace{\frac{\partial c_j}{\partial t}}_{\text{accumulation}} = - \underbrace{\frac{\partial (c_j Q)}{\partial V}}_{\text{convection}} + \underbrace{R_j}_{\text{reaction}} \quad (4.69)$$

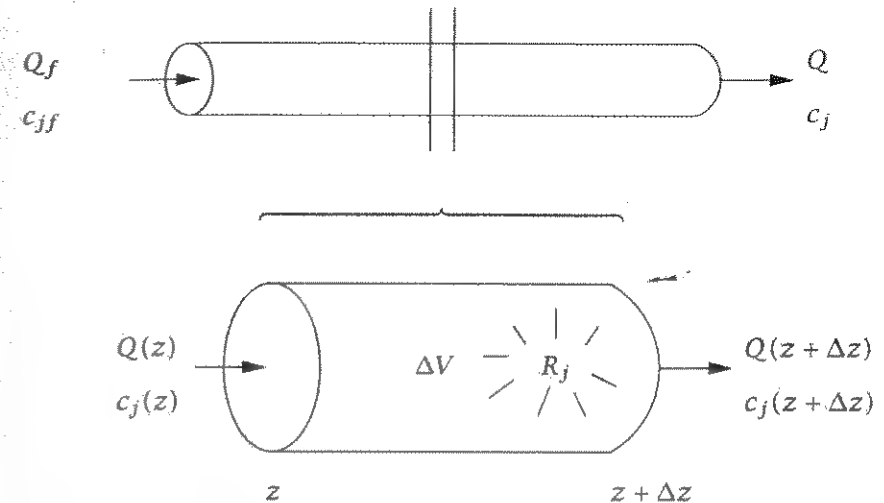


Figure 4.19: Plug-flow reactor volume element.

If the tube has constant cross section,  $A_c$ , then velocity,  $v$ , is related to volumetric flowrate by  $v = Q/A_c$ , and axial length is related to tube volume by  $z = V/A_c$ , so Equation 4.69 can be rearranged to the familiar form [1, p.584]

$$\frac{\partial c_j}{\partial t} = - \frac{\partial (c_j v)}{\partial z} + R_j \quad (4.70)$$

### 4.6.2 Steady-State Operation

One is often interested in the steady-state operation of a plug-flow reactor. Setting the time derivative in Equation 4.69 to zero gives,

$$\frac{d(c_j Q)}{dV} = R_j \quad (4.71)$$

The product  $c_j Q = N_j$  is the total molar flow of component  $j$ . One also can express the PFR mole balance in terms of the molar flow,

$$\frac{dN_j}{dV} = R_j \quad (4.72)$$

### 4.6.3 Volumetric Flowrate for Gas-Phase Reactions

To use Equation 4.72 for designing a gas-phase reactor, one has to be able to relate the volumetric flowrate,  $Q$ , to the molar flows,  $N_j, j =$

1, 2, ...,  $n_s$ . The important piece of information tying these quantities together is, again, the equation of state for the reaction mixture. The equation of state is a function relating the system temperature, pressure, and all molar concentrations,  $f(T, P, c_j) = 0$ . Because the molar flow and concentration are directly related,

$$N_j = c_j Q \quad (4.73)$$

the equation of state is also a relation between temperature, pressure, molar flows, and volumetric flowrate. For example, the ideal-gas equation of state,  $c = P/RT$ , can be stated in terms of molar concentrations,  $c_j$ , as

$$\sum_j c_j = \frac{P}{RT}$$

In terms of molar flows, the equation of state is

$$\frac{\sum_j N_j}{Q} = \frac{P}{RT}$$

One can solve the previous equation for the volumetric flowrate,

$$Q = \frac{RT}{P} \sum_j N_j \quad (4.74)$$

Equation 4.74 is used with the mole balance, Equation 4.72, to solve gas-phase PFR problems under the ideal-gas assumption. To evaluate the concentrations for use with the reaction rate expressions, one then rearranges Equation 4.73 to obtain

$$c_j = \frac{N_j}{Q} = \frac{P}{RT} \frac{N_j}{\sum_j N_j} \quad (4.75)$$

#### 4.6.4 Volumetric Flowrate for Liquid-Phase Reactions

Consider the equation of state for a liquid-phase system to be arranged in the form

$$\rho = f(T, P, c_j)$$

The mass density is related to the volumetric flowrate and total mass flow,  $M = \sum_j N_j M_j$ , via

$$M = \rho Q \quad (4.76)$$

Multiplying Equation 4.72 by  $M_j$  and summing on  $j$  produces

$$\frac{dM}{dV} = 0, \quad M(0) = M_f$$

in which  $M_f$  is the feed mass flowrate. The total mass balance indicates that the total mass flow is constant with respect to axial position in the PFR because chemical reaction cannot alter the total mass flow. We can solve for the volumetric flowrate by rearranging Equation 4.76

$$Q = \frac{M_f}{\rho} \quad (4.77)$$

If the liquid density is considered constant,  $\rho = \rho_f$ , then

$$Q = Q_f, \quad \text{constant density} \quad (4.78)$$

and the volumetric flowrate is constant and equal to the feed value. Equation 4.78 is used often for liquid-phase reactions. If the density variation is significant, however, Equation 4.77 should be used instead. Finally, if we denote the time spent in the tube by  $\theta = V/Q$ , if  $Q$  is constant, we can rewrite Equation 4.71 as

$$\frac{dc_j}{d\theta} = R_j, \quad \text{constant flowrate} \quad (4.79)$$

which is identical to the constant-volume batch reactor, Equation 4.6. In other words, for the constant-flowrate case, the steady-state profile in a PFR starting from a given feed condition is also the transient profile in a batch reactor starting from the equivalent initial condition.

#### 4.6.5 Single Reaction Systems

##### Example 4.4: Changing flowrate in a PFR

A pure vapor stream of A is decomposed in a PFR to form B and C



Determine the length of 2.5 cm inner-diameter tube required to achieve 35% conversion of A. The reactor temperature is 518°C and the pressure is 2.0 atm. Assume the pressure drop is negligible. The reaction rate is first order in A,  $k = 0.05 \text{ s}^{-1}$  at the reactor temperature, and the feed flowrate is 35 L/min.

##### Solution

Writing the mole balance for component A gives

$$\frac{dN_A}{dV} = R_A$$

The production rate of A is  $R_A = -r = -kc_A$ . Substituting the production rate into the above equation gives,

$$\frac{dN_A}{dV} = -kN_A/Q \quad (4.81)$$

The volumetric flowrate is not constant, so we use Equation 4.74, which assumes an ideal-gas equation of state,

$$Q = \frac{RT}{P} (N_A + N_B + N_C) \quad (4.82)$$

The ideal-gas assumption is reasonable at this reactor temperature and pressure. One can relate the molar flows of B and C to A using the reaction stoichiometry. The mole balances for B and C are

$$\frac{dN_B}{dV} = R_B = r, \quad \frac{dN_C}{dV} = R_C = r$$

Adding the mole balance for A to those of B and C gives

$$\frac{d(N_A + N_B)}{dV} = 0, \quad \frac{d(N_A + N_C)}{dV} = 0$$

The stoichiometry does not allow the molar flow  $N_A + N_B$  or  $N_A + N_C$  to change with position in the tube. Because both of these quantities are known at the tube entrance, one can relate  $N_B$  and  $N_C$  to  $N_A$ ,

$$\begin{aligned} N_A + N_B &= N_{Af} + N_{Bf} \\ N_A + N_C &= N_{Af} + N_{Cf} \end{aligned}$$

Rearranging the previous equations gives,

$$\begin{aligned} N_B &= N_{Af} + N_{Bf} - N_A \\ N_C &= N_{Af} + N_{Cf} - N_A \end{aligned}$$

Substituting the relations for  $N_B$  and  $N_C$  into Equation 4.82 gives

$$Q = \frac{RT}{P} (2N_{Af} + N_{Bf} + N_{Cf} - N_A)$$

Because the feed stream is pure A,  $N_{Bf} = N_{Cf} = 0$ , yielding

$$Q = \frac{RT}{P} (2N_{Af} - N_A)$$

Substituting this expression in Equation 4.81 gives the final mole balance,

$$\frac{dN_A}{dV} = -k \frac{P}{RT} \frac{N_A}{2N_{Af} - N_A}$$

The previous differential equation can be separated and integrated,

$$\int_{N_{Af}}^{N_A} \frac{2N_{Af} - N_A}{N_A} dN_A = \int_0^V -\frac{kP}{RT} dV$$

Performing the integration gives,

$$2N_{Af} \ln(N_A/N_{Af}) + (N_{Af} - N_A) = -\frac{kP}{RT} V$$

The conversion of component  $j$  for a plug-flow reactor operating at steady state is defined as

$$x_j = \frac{N_{jf} - N_j}{N_{jf}}$$

Because we are interested in the  $V$  corresponding to 35% conversion of A, we substitute  $N_A = (1 - x_A)N_{Af}$  into the previous equation and solve for  $V$ ,

$$V = -\frac{RT}{kP} N_{Af} [2 \ln(1 - x_A) + x_A]$$

Because  $Q_f = N_{Af}RT/P$  is given in the problem statement and the tube length is desired, it is convenient to rearrange the previous equation to obtain

$$z = -\frac{Q_f}{kA_c} [2 \ln(1 - x_A) + x_A]$$

Substituting in the known values gives

$$\begin{aligned} z &= -\left(\frac{35 \times 10^3 \text{ cm}^3/\text{min}}{0.05 \text{ s}^{-1} 60 \text{ s/min}}\right) \left(\frac{4}{\pi(2.5 \text{ cm})^2}\right) [2 \ln(1 - .35) + .35] \\ z &= 1216 \text{ cm} = 12.2 \text{ m} \end{aligned}$$

□

#### 4.6.6 Multiple-Reaction Systems

The modeler has some freedom in setting up the material balances for a plug-flow reactor with several reactions. The most straightforward method is to write the material balance relation for every component,

$$\frac{dN_j}{dV} = R_j, \quad j = 1, 2, \dots, n_s$$



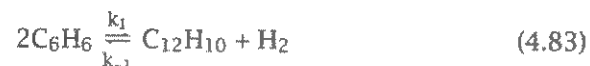
Recall the production rate of each species is a simple linear combination of the reaction rates times the appropriate stoichiometric coefficients,

$$R_j = \sum_{i=1}^{n_r} \nu_{ij} r_i, \quad j = 1, 2, \dots, n_s$$

The reaction rates are expressed in terms of the species concentrations. The  $c_j$  are calculated from the molar flows with Equation 4.73, and  $Q$  is calculated from Equation 4.74, if an ideal-gas mixture is assumed.

#### Example 4.5: Benzene pyrolysis in a PFR

Hougen and Watson [9] analyzed the rate data for the pyrolysis of benzene by the following two reactions. Diphenyl is produced by the dehydrogenation of benzene,



Triphenyl is formed by the secondary reaction,



The reactions are assumed to be elementary so that the rate expressions are

$$r_1 = k_1 \left( c_B^2 - \frac{c_D c_H}{K_1} \right) \quad r_2 = k_2 \left( c_B c_D - \frac{c_T c_H}{K_2} \right) \quad (4.85)$$

in which the subscripts, B, D, T and H represent benzene, diphenyl, triphenyl and hydrogen, respectively. Calculate the tube volume required to reach 50% total conversion of the benzene for a 60 kmol/hr feed stream of pure benzene. The reactor operates at 1033 K and 1.0 atm. Plot the mole fractions of the four components versus reactor volume. The rate and equilibrium constants at  $T = 1033$  K and  $P = 1.0$  atm are given in Hougen and Watson,

$$\begin{aligned} k_1 &= 7 \times 10^5 \text{ L/mol} \cdot \text{hr} & K_1 &= 0.31 \\ k_2 &= 4 \times 10^5 \text{ L/mol} \cdot \text{hr} & K_2 &= 0.48 \end{aligned}$$

#### Solution

The mole balances for the four components follow from the stoichiometry,

$$\begin{aligned} \frac{dN_B}{dV} &= -2r_1 - r_2 & \frac{dN_H}{dV} &= r_1 + r_2 \\ \frac{dN_D}{dV} &= r_1 - r_2 & \frac{dN_T}{dV} &= r_2 \end{aligned}$$

The initial condition for the ODEs are  $N_B(0) = N_{Bf}$  and  $N_D(0) = N_H(0) = N_T(0) = 0$ . Because the number of moles are conserved in both reactions, the total molar flux does not change with reactor volume. The volumetric flowrate can be evaluated from Equation 4.74, which in this case reduces to,

$$Q = \frac{RT}{P} N_{Bf} \quad (4.86)$$

The rate expressions are substituted into the four ODEs and they are solved numerically. The total conversion of benzene,  $x_B = (N_{Bf} - N_B)/N_{Bf}$ , is plotted versus reactor volume in Figure 4.20. A reactor volume of 403 L is required to reach 50% conversion. The composition of the reactor versus reactor volume is plotted in Figure 4.21.  $\square$

In the previous approach, a differential equation is required for each component. Therefore  $n_s$  differential equations are solved simultaneously in this approach. One can instead solve the problem by using one differential equation for each independent reaction for a total of  $n_i$  differential equations. Because  $n_i < n_s$ , this second approach produces fewer differential equations. Normally  $n_i$  is not much less than  $n_s$ , and the computational expenses of the two approaches are similar. Whether we write differential equations for the species or the reaction extents is largely a matter of taste.

In any case, the following procedure is guaranteed to give the minimal set of differential equations. Instead of writing the material balance for each component, define extents for each reaction that satisfy

$$\frac{d\varepsilon_i}{dV} = r_i, \quad i = 1, 2, \dots, n_i$$

The initial conditions for these extents can be defined to be zero,

$$\varepsilon_i(0) = 0, \quad i = 1, 2, \dots, n_i$$

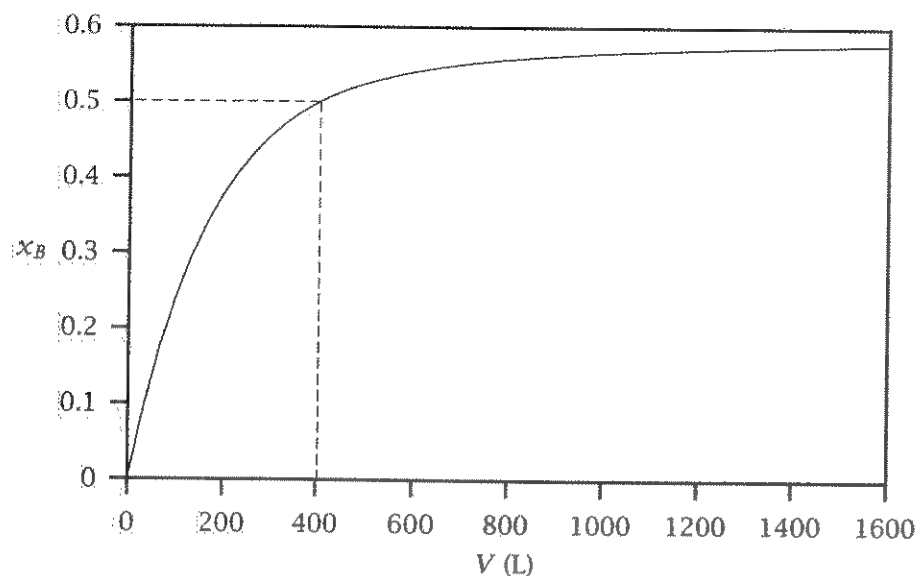


Figure 4.20: Benzene conversion versus reactor volume.

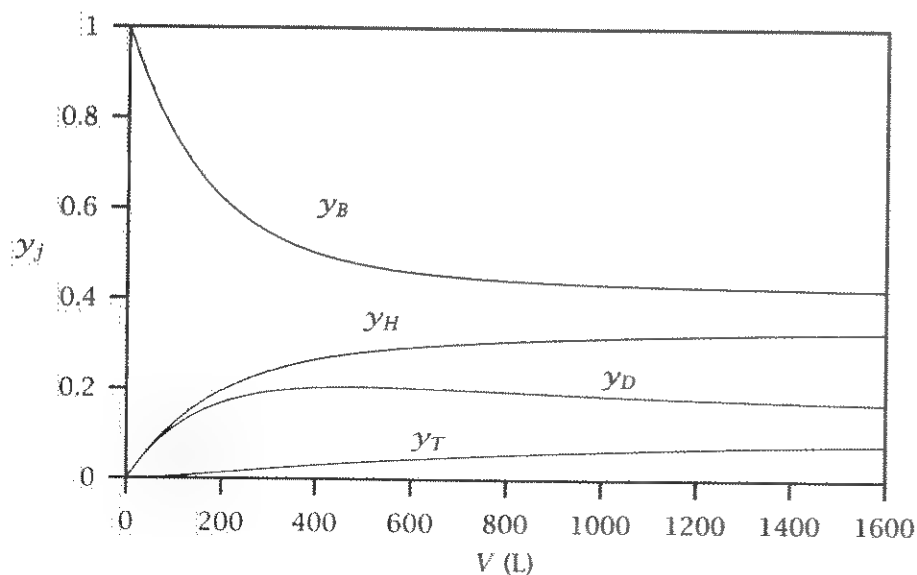


Figure 4.21: Component mole fractions versus reactor volume.

The molar flow for each species can be computed by summing the stoichiometric number times each extent,

$$N_j = N_{jf} + \sum_{i=1}^{n_r} \nu_{ij} \epsilon_i, \quad j = 1, 2, \dots, n_s$$

As with the previous method, the  $c_j$  are calculated from the molar flows with Equation 4.73, and  $Q$  is calculated from Equation 4.74.

#### Example 4.6: Benzene pyrolysis, revisited

Rework the previous example using the minimum number of differential equations.

#### Solution

Defining the extents to track the rates of the two reactions in Equations 4.83 and 4.84 gives

$$\frac{d\epsilon_1}{dV} = r_1, \quad \epsilon_1(0) = 0 \quad (4.87)$$

$$\frac{d\epsilon_2}{dV} = r_2, \quad \epsilon_2(0) = 0 \quad (4.88)$$

The stoichiometry allows all molar flows to be calculated from these two extents,

$$\begin{aligned} N_B + 2\epsilon_1 + \epsilon_2 &= N_{Bf} & N_H - \epsilon_1 - \epsilon_2 &= N_{Hf} \\ N_D - \epsilon_1 + \epsilon_2 &= N_{Df} & N_T - \epsilon_2 &= N_{Tf} \end{aligned}$$

Substituting in the feed conditions gives,

$$\begin{aligned} N_B &= N_{Bf} - 2\epsilon_1 - \epsilon_2 & N_H &= \epsilon_1 + \epsilon_2 \\ N_D &= \epsilon_1 - \epsilon_2 & N_T &= \epsilon_2 \end{aligned}$$

These relations can be used along with Equation 4.86 and Equations 4.85 to evaluate the rates  $r_1$  and  $r_2$  in Equations 4.87 and 4.88 in terms of  $\epsilon_1$  and  $\epsilon_2$ . In this approach one only has to solve two differential equations rather than the four required in the previous approach. The solutions are the same in the two approaches.  $\square$

**Example 4.7: Ethane pyrolysis in the presence of NO**

The thermal decomposition of ethane is inhibited by nitric oxide. Laidler and Wojciechowski [10] have suggested the following sequence of elementary reactions to describe pyrolysis.



The kinetic data for this reaction are, where  $k_i = A_i \exp(-E_i/RT)$

Reaction	$r_i$	$A_i$	$E_i$ (kJ/mol)
1	$r_1 = k_1 C_{\text{C}_2\text{H}_6} C_{\text{NO}}$	$1.0 \times 10^{14} \text{ cm}^3/\text{mol s}$	217.6
2	$r_2 = k_2 C_{\text{C}_2\text{H}_5}$	$3.0 \times 10^{14} \text{ s}^{-1}$	165.3
3	$r_3 = k_3 C_{\text{H}} C_{\text{C}_2\text{H}_6}$	$3.4 \times 10^{12} \text{ cm}^3/\text{mol s}$	28.5
4	$r_4 = k_4 C_{\text{H}} C_{\text{NO}}$	$1.0 \times 10^{12} \text{ cm}^3/\text{mol s}$	0
5	$r_5 = k_5 C_{\text{HNO}}$	$1.0 \times 10^{13} \text{ s}^{-1}$	200.8
6	$r_6 = k_6 C_{\text{C}_2\text{H}_5} C_{\text{HNO}}$	$1.0 \times 10^{12} \text{ cm}^3/\text{mol s}$	0

Assume the reaction takes place in an isothermal, 1500-cm<sup>3</sup> PFR operating at constant pressure (1.0 atm). The feed to the reactor consists of a mixture of ethane and NO with a molar ratio of 95% ethane and 5% NO. The inlet volumetric flowrate is 600 cm<sup>3</sup>/s. Consider a base-case reactor temperature of 1050 K.

1. Develop the necessary design equations to calculate the molar flow of ethane, NO and ethylene, and determine the effluent composition from the reactor. Be sure to specify the initial conditions.
2. Solve the design equations and calculate the molar flowrates of ethane, ethylene and NO as a function of reactor volume for a reactor temperature of 1050 K.
3. Examine the effect of increasing and decreasing the temperature 50 K on the molar flowrate of ethane. Why do you think a small

change in temperature creates such a large effect on the amount of ethane reacting?

**Solution**

The reaction rates are functions of molar concentrations of the components, and the concentrations can be found from the molar flowrates of each component.

$$c_j = \frac{N_j}{Q} = \frac{N_j}{\frac{RT}{P} \sum_j N_j} = \frac{P}{RT} \frac{N_j}{\sum_j N_j}$$

We need to know all the molar flows, and they can be found from the PFR mass balance

$$\frac{dN_j}{dV} = R_j = \sum_i \nu_{ij} r_i$$

which produces for this example

$$\begin{aligned} \frac{dN_{\text{C}_2\text{H}_6}}{dV} &= -r_1 - r_3 + r_6 & \frac{dN_{\text{H}}}{dV} &= r_2 - r_3 - r_4 + r_5 \\ \frac{dN_{\text{NO}}}{dV} &= -r_1 - r_4 + r_5 + r_6 & \frac{dN_{\text{C}_2\text{H}_4}}{dV} &= r_2 \\ \frac{dN_{\text{C}_2\text{H}_5}}{dV} &= r_1 - r_2 + r_3 - r_6 & \frac{dN_{\text{H}_2}}{dV} &= r_3 \\ \frac{dN_{\text{HNO}}}{dV} &= r_1 + r_4 - r_5 - r_6 \end{aligned} \quad (4.89)$$

The rates are calculated from the rate constants and concentrations; for example, the first rate is

$$r_1 = k_1 \left( \frac{P}{RT} \right)^2 \frac{N_{\text{C}_2\text{H}_6} N_{\text{NO}}}{N_{\text{tot}}^2}$$

in which

$$N_{\text{tot}} = N_{\text{C}_2\text{H}_6} + N_{\text{NO}} + N_{\text{C}_2\text{H}_5} + N_{\text{HNO}} + N_{\text{H}} + N_{\text{C}_2\text{H}_4} + N_{\text{H}_2}$$

The feed contains only C<sub>2</sub>H<sub>6</sub> and NO at molar flows given by

$$N_{j,f} = y_{j,f} \frac{P}{RT} Q_f$$

leading to  $N_{\text{C}_2\text{H}_6,f} = 6.62 \times 10^{-3} \text{ mol/s}$  and  $N_{\text{NO},f} = 3.48 \times 10^{-4} \text{ mol/s}$ .

The numerical solution of Equations 4.89 at 1050 K is shown in Figure 4.22. We see the consumption of ethane and the production of

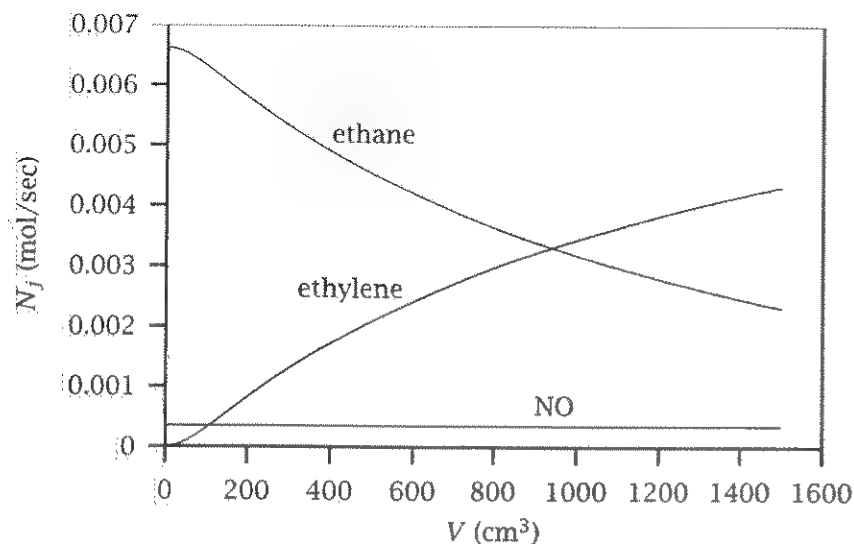


Figure 4.22: Molar flowrate of ethane, ethylene and NO versus reactor volume for ethane pyrolysis example.

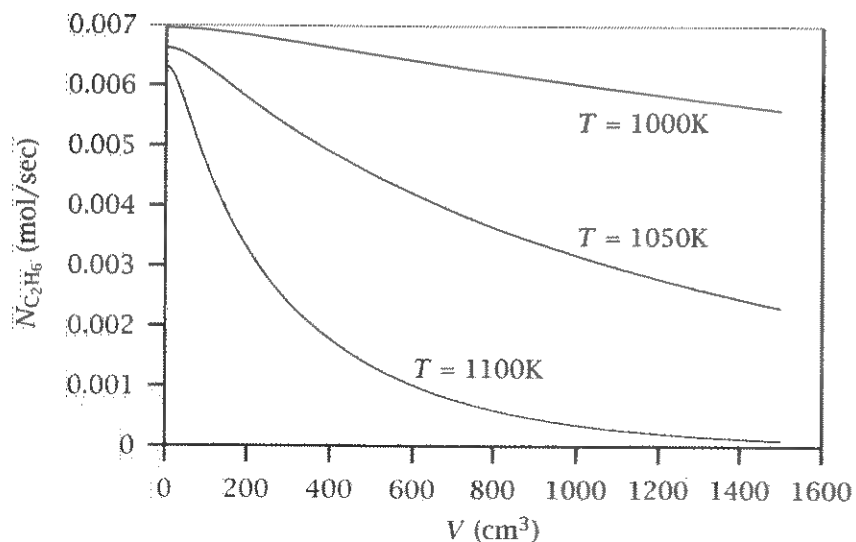


Figure 4.23: Molar flowrate of ethane versus reactor volume for inlet temperatures of 1000, 1050 and 1100 K.

ethylene. The ethane molar flows at the three different reactor temperatures are shown in Figure 4.23. The solution shows a small change in ethane molar flow initially, followed by a more rapid change of molar flow with increasing distance in the reactor. The rate is low initially until a sufficient amount of H is formed so the third reaction can begin to contribute to the ethane consumption. The rates are very sensitive to temperature because the activation energies are quite high for the reactions that lead to the loss of ethane. For example, the first reaction that initiates the pyrolysis and the second reaction that produces H, which contributes to the consumption of ethane, have high activation energies. □

#### 4.7 Some PFR-CSTR Comparisons

Given the two continuous reactors in this chapter, the CSTR and the PFR, it is natural to compare their steady-state efficiencies in converting reactants to products. For simplicity, consider a constant-density, liquid-phase reaction with  $n$ th-order, irreversible reaction rate



For this situation, the steady-state PFR material balance is given by Equation 4.79

$$\frac{dc_A}{d\theta} = -r(c_A)$$

We rearrange and solve for the time required to change from the feed condition  $c_{Af}$  to some exit concentration  $c_A$

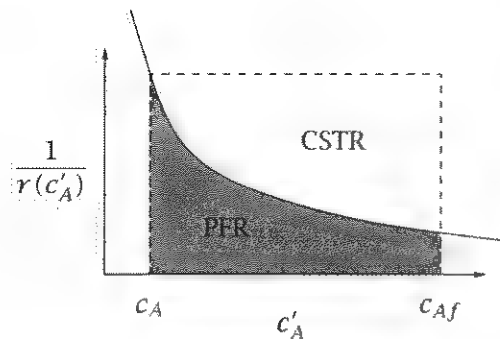
$$\theta = \int_{c_{Af}}^{c_A} \frac{1}{r(c'_A)} dc'_A$$

in which the prime is to remind us that this variable is a dummy variable of integration. Figure 4.24 shows the graphical interpretation of this result. The area under the curve  $1/r(c'_A)$  is the total time required to achieve the desired concentration change.

To achieve this same concentration change in the CSTR, we start with Equation 4.41, and solve for  $\theta$  giving

$$\theta = \frac{c_{Af} - c_A}{r(c_A)}$$

This result also can be interpreted as an area in Figure 4.24. Notice that this area is the height,  $1/r(c_A)$ , times the width,  $c_{Af} - c_A$ , of the

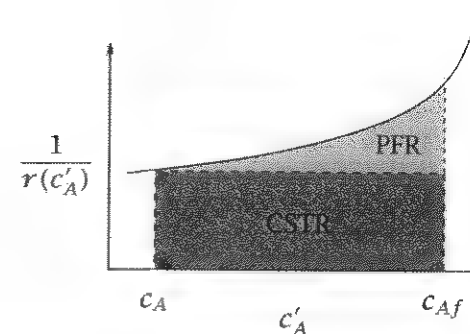


**Figure 4.24:** Inverse of reaction rate versus concentration; area under curve is the PFR residence time  $\theta$ ; area of the rectangle is the CSTR residence time; PFR is more efficient; irreversible,  $n$ th-order kinetics with positive order,  $n > 0$ .

rectangle shown in Figure 4.24. Obviously if  $1/r(c_A)$  is a *decreasing* function of  $c_A$ , or, equivalently,  $r(c_A)$  is an *increasing* function of  $c_A$ , to achieve the same conversion, the PFR time (or volume,  $V_R = Q_f \theta$ ) is less than the CSTR time (volume). We make a final comment to reinforce this point. The reaction rate for the PFR varies with length. The rate is high at the entrance to the tube where the concentration of A is equal to the feed value, and decreases with length as the concentration drops. At the exit of the PFR, the rate is the lowest of any location in the tube. Now considering that the entire volume of the CSTR is reacting at this *lowest* rate of the PFR, it seems intuitively clear that more volume is required for the CSTR to achieve the same conversion as the PFR.

So for this simple, single-reactant case, we only need to know if  $r(c_A)$  is an increasing or decreasing function of  $c_A$  to know which reactor is more efficient, i.e., requires less volume. If the reaction order is positive (the usual case), the PFR is more efficient, as shown in Figure 4.24. If the reaction order is negative, the CSTR is more efficient, as shown in Figure 4.25.

If the reaction rate is neither strictly increasing nor strictly decreasing, the reactor configuration with the smallest volume becomes a series of CSTRs and PFRs. Such an example is shown in Chapter 8, Example 8.4. Levenspiel provides further discussion of these interesting cases [11, pp. 128–156].



**Figure 4.25:** To achieve the same conversion, the CSTR is smaller than the PFR for irreversible,  $n$ th-order kinetics, negative order,  $n < 0$ .

#### Example 4.8: The PFR versus CSTR with separation

We have noticed that a PFR achieves higher conversion than an equivalent volume CSTR for the irreversible reaction with first-order kinetics



Consider the case in which we add separation. Find a single CSTR and separator combination that achieves the same conversion as the PFR. You may assume a perfect separation of A and B, the feed is a pure A stream, and  $k\theta = 1$  for the PFR.

#### Solution

The PFR achieves a fractional conversion of A

$$x_{\text{PFR}} = 1 - N_A/N_{A0} = 1 - \exp(-k\theta) = 0.632$$

For an equivalent volume CSTR without separation, the conversion of A is

$$x_{\text{CSTR}} = 1 - N_A/N_{A0} = k\theta/(1 + k\theta) = 0.5$$

The goal is to increase the achievable conversion in the CSTR using separation. Education in chemical engineering principles leads one immediately to consider recycle of the unreacted A as a means to increase this conversion. Consider the flowsheet depicted in Figure 4.26. A fraction of the outflow from the CSTR is recycled, the product B is removed and the unreacted A is combined with the feed as the inflow of the CSTR.

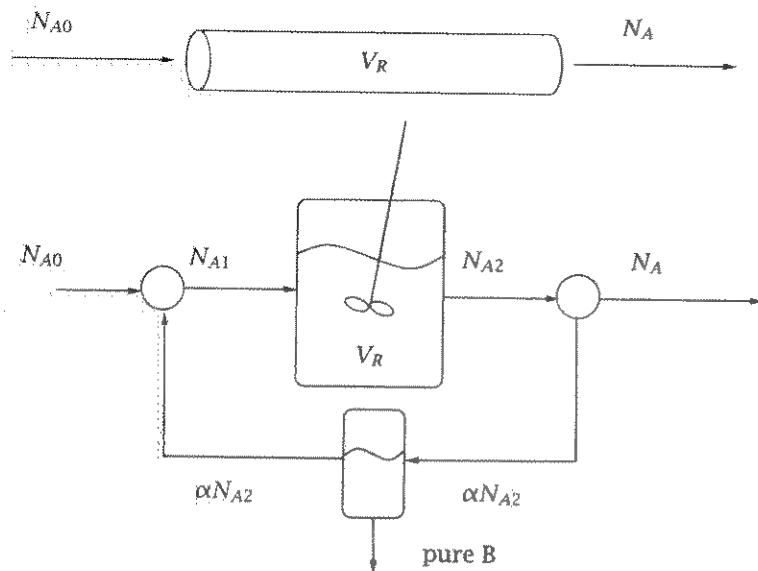


Figure 4.26: PFR versus CSTR with recycle and separation.

Given the assumption of perfect separation, we can achieve essentially complete conversion of A for  $k\theta = 1$  with complete recycle, so our goal here is to calculate the fractional recycle,  $\alpha$ , that achieves exactly the PFR conversion. For  $k\theta < 1$  the achievable conversion is less than one as discussed in Exercise 4.14.

Notice first that the mean residence time of the CSTR,  $\theta'$ , is less than that for the PFR,  $\theta$ , because the flowrate has increased due to the recycle. Notice that with perfect separation, pure A streams are combined at the mixer, and  $Q_0/Q_1 = N_{A0}/N_{A1}$  so

$$\theta' = V_R/Q_1 = (V_R/Q_0)(Q_0/Q_1) = \theta N_{A0}/N_{A1}$$

We may consider four variables to specify the state of the system:  $\alpha, N_{A1}, N_{A2}, N_A$ ; and we can write three component A material balances for the reactor, splitter at reactor exit and mixer at reactor inlet

$$\text{reactor: } N_{A2} = N_{A1}/(1 + k\theta N_{A0}/N_{A1})$$

$$\text{splitter: } N_A = (1 - \alpha)N_{A2}$$

$$\text{mixer: } N_{A1} = N_{A0} + \alpha N_{A2}$$

The separator balance is trivial because the separation of A is perfect, and, therefore, the molar flow of A is conserved across the recycle

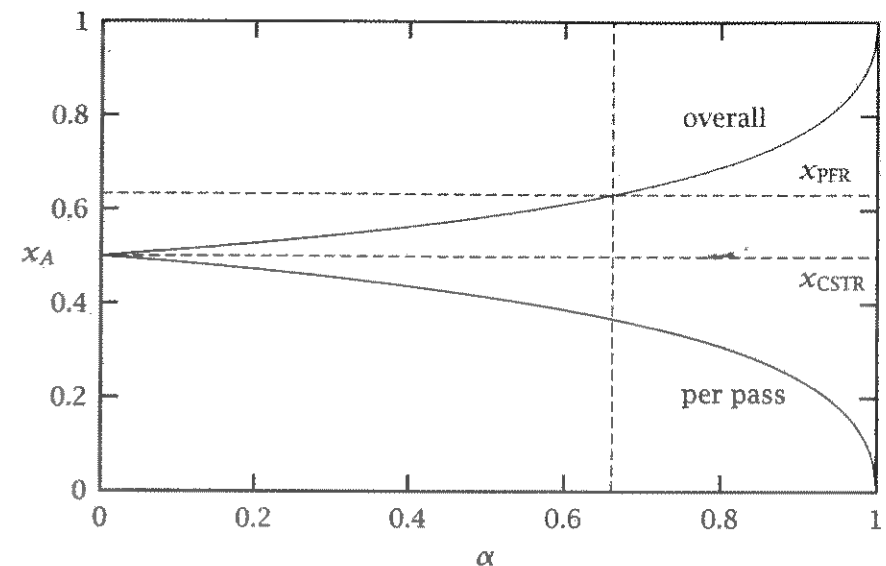


Figure 4.27: Overall and per-pass conversion of A as a function of fractional recycle,  $\alpha$ .

stream. Because the inlet flow of A is not specified, it is convenient to divide the preceding equations by  $N_{A0}$ , define dimensionless molar flows, and rearrange to obtain

$$\text{reactor: } \bar{N}_{A2}(1 + k\theta/\bar{N}_{A1}) - \bar{N}_{A1} = 0$$

$$\text{splitter: } \bar{N}_A - (1 - \alpha)\bar{N}_{A2} = 0$$

$$\text{mixer: } 1 + \alpha\bar{N}_{A2} - \bar{N}_{A1} = 0 \quad (4.91)$$

We can specify a single variable as known and solve for the remaining three with the three equations. For example, if we specify the recycle fraction,  $\alpha$ , we can solve Equations 4.91 for  $\bar{N}_A, \bar{N}_{A1}, \bar{N}_{A2}$ , and compute the conversion from  $x_A = 1 - \bar{N}_A$ . Figure 4.27 shows the resulting conversion of A plotted as a function of  $\alpha$ . We can see from Figure 4.27 that the PFR conversion is achieved at about  $\alpha = 0.65$ . If we want a more accurate answer, we can set  $\bar{N}_A = \exp(-k\theta) = 0.3678$  and solve



numerically for  $\alpha, \bar{N}_{A1}, \bar{N}_{A2}$ , with Equations 4.91<sup>1</sup>, and the result is

$$\alpha = 0.6613$$

□

**CSTR equivalence principle.** Example 4.8 is motivated by a recent result of Feinberg and Ellison called the CSTR equivalence principle of reactor-separator systems [4]. This surprising principle states:

For a given reaction network with  $n_i$  linearly independent reactions, any steady state that is achievable by any reactor-separator design with total reactor volume  $V$  is achievable by a design with *not more than*  $n_i + 1$  CSTRs, also of total reactor volume  $V$ . Moreover the concentrations, temperatures and pressures in the CSTRs are arbitrarily close to those occurring in the reactors of the original design.

Applying this principle to the last example, we know that any achievable concentration of the PFR for a single reaction is achievable with  $\blacksquare$  CSTR and separation. Note the number of CSTRs can be reduced from  $n_i + 1$  to  $n_i$  in certain situations, such as the one considered in Example 4.8. And we know the concentration in the CSTR will be achieved somewhere in the PFR.

## 4.8 Stochastic Simulation of Chemical Reactions

We wish to introduce next a topic of increasing importance to chemical engineers, stochastic (random) simulation. In stochastic models we simulate quite directly the random nature of the molecules. We will see that the deterministic rate laws and material balances presented in the previous sections can be captured in the stochastic approach by allowing the numbers of molecules in the simulation to become large. From this viewpoint, deterministic and stochastic approaches are complementary. Deterministic models and solution methods are quite efficient when the numbers of molecules are large and the random behavior is not important. The numerical methods for solution of the nonlinear differential equations of the deterministic models are

<sup>1</sup>Note one can solve this simple problem analytically as well. Eliminate  $\alpha \bar{N}_{A2}$  from the second and third equations in Equation 4.91. Substitute the result into the first equation and solve the resulting quadratic equation.

also highly developed. The stochastic modeling approach is appropriate if the random nature of the system is one of the important features to be captured in the model. These situations are becoming increasingly important to chemical engineers as we explore reactions at smaller and smaller length scales. For example, if we are modeling the chemical transformation by reaction of only a few hundreds or thousands of molecules at an interface, we may want to examine explicitly the random fluctuations taking place. In biological problems, we often consider the interactions of only several hundred or several thousand protein molecules and cells. In sterilization problems, we may wish to model the transient behavior until every last organism is eliminated.

It is perhaps best to illustrate features of the stochastic approach with a simple example. Instead of the common case in which we have on the order of Avogadro's number of reacting molecules, assume we have only a hundred molecules moving randomly in the gas phase and we wish to follow the reaction



in a constant-volume batch reactor. In this section we take reaction statements quite literally. We assume these reactions are not merely observed stoichiometries, but actual molecular events.

The *probability* of reaction is assumed proportional to the

$$r_1 = k_1 x_A, \quad r_2 = k_2 x_B$$

in which  $x_j$  is the *number* of component  $j$  molecules in the reactor volume. Note  $x_j$  is an integer, unlike  $c_j$  of the deterministic model, which is real. The reaction probabilities play the role of the rate expressions in the deterministic models. Given the stoichiometry and the reaction probabilities, we would like to simulate the expected behavior of the reaction network. One way to accomplish this task is the Gillespie algorithm, which we describe next. The basic idea of the Gillespie algorithm is to: (i) choose randomly the time at which the next reaction occurs, and (ii) choose randomly which reactions occurs at that time. Of course we do not choose completely randomly. If the total reaction probabilities are large, it is intuitively clear that the time interval until the next reaction should be small, and, if reaction probability  $r_1$  is much larger than  $r_2$ , the first reaction is more likely to occur at the next reaction

time. The beauty of the Gillespie algorithm is the simple and statistically correct manner in which these two random choices are made.

In a series of papers, Gillespie makes an elegant argument for the use of stochastic simulation in chemical kinetic modeling [7, 8] and provides the following simulation algorithm [7, p.2345].

1. Initialize. Set integer counter  $n$  to zero. Set the initial species numbers,  $x_j(0), j = 1, \dots, n_s$ . Determine stoichiometric matrix  $\mathbf{v}$  and reaction probability laws (rate expressions)

$$r_i = k_i h(x_j)$$

for all reactions.

2. Compute reaction probabilities,  $r_i = k_i h(x_j)$ . Compute total reaction probability  $r_{\text{tot}} = \sum_i r_i$ .
3. Select two random numbers,  $p_1, p_2$ , from a uniform distribution on the interval  $(0, 1)$ . Let the time interval until the next reaction be

$$\tau = -\ln(p_1)/r_{\text{tot}} \quad (4.94)$$

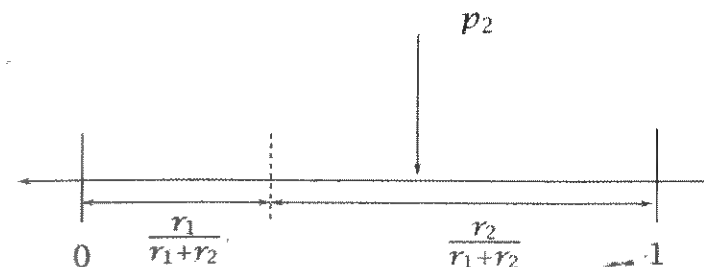
Determine reaction  $m$  to take place in this time interval. The idea here is to partition the interval  $(0, 1)$  by the relative sizes of each reaction probability and then "throw a dart" at the interval to pick the reaction that occurs. In this manner, all reactions are possible, but the reaction is selected in accord with its probability. See Figure 4.28.

4. Update the simulation time  $t(n+1) = t(n) + \tau$ . Update the species numbers for the single occurrence of the  $m$ th reaction via

$$x_j(n+1) = x_j(n) + v_{mj}, \quad j = 1, \dots, n_s$$

Let  $n = n + 1$ . Return to Step 2.

If  $r_{\text{tot}}$  is the total probability for reaction,  $e^{-r_{\text{tot}}\tau}$  is the probability that a reaction has not occurred during time interval  $\tau$ , which leads directly to Equation 4.94 for choosing the time of the next reaction. We will derive this fact in Chapter 8 when we develop the residence-time distribution for a CSTR. Shah, Ramkrishna and Borwanker call this time the "interval of quiescence," and use it to develop a stochastic simulation algorithm for particulate system dynamics rather than chemical kinetics [14].



**Figure 4.28:** Randomly choosing a reaction with appropriate probability; the interval is partitioned according to the relative sizes of the reaction rates; a random number  $p_2$  between zero and one is generated to determine the reaction; in this case,  $m = 2$  and the second reaction is selected.

Figure 4.29 shows the results of this algorithm when starting with  $x_A = 100$  molecules. Notice the random aspect of the simulation gives a rough appearance to the number of molecules versus time, which is quite unlike any of the deterministic simulations presented in Section 4.2. In fact, because the *number* of molecules is an integer, the simulation is actually discontinuous with jumps between simulation times. But in spite of the roughness, we already can make out the classic behavior of the series reaction: loss of starting material A, appearance and then disappearance of the intermediate species B, and slow increase in final product C. Note also that Figure 4.29 is only *one* simulation of the stochastic model. Unlike the deterministic models, if we repeat this simulation, we obtain a different sequence of random numbers and a different simulation. To talk about expected or average behavior of the system, we must perform many of these random simulations and then compute the averages of quantities we wish to report.

Next we explore the effect of increasing the initial number of A molecules on a single simulation. The results for 1000 and 4000 initial A molecules are shown in Figures 4.30 and 4.31, respectively. We see the random fluctuations become less pronounced. Notice that even with only 4000 starting molecules, Figure 4.31 compares very favorably with the deterministic simulation shown in Figure 4.11 of Section 4.2.

Another striking feature of the stochastic approach is the trivial

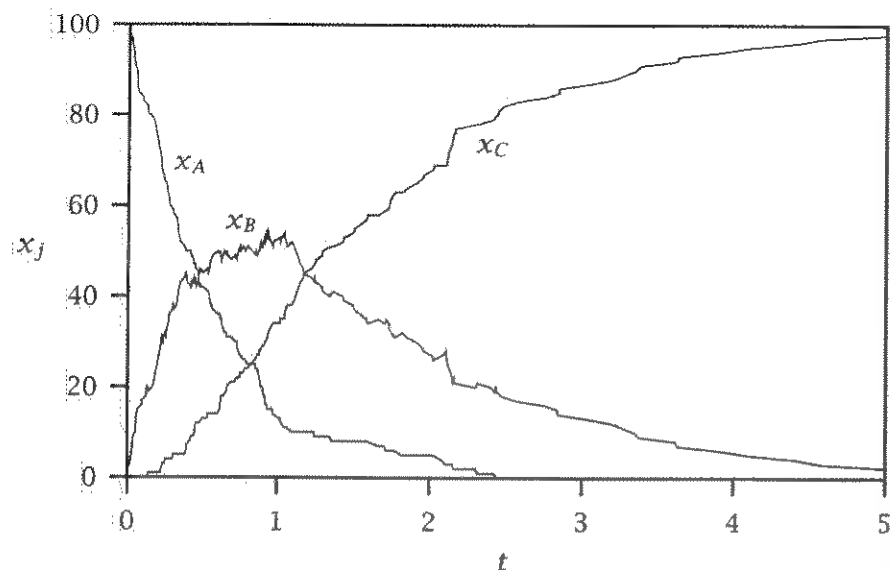


Figure 4.29: Stochastic simulation of the first-order reactions  $A \rightarrow B \rightarrow C$  starting with 100 A molecules.

level of programming effort required to make the simulations. In fact, the biggest numerical challenge is producing the random numbers,<sup>2</sup> and many well-developed algorithms are available for that task. The computational time required for performing the stochastic simulation may, however, be large. The solution time depends on the number of simulation steps, and also on whether or not we must repeat the simulations to calculate averages. Usually large numbers of simulation steps are chosen when one has large numbers of initial molecules. If reliable deterministic rate laws are available, at some point it becomes more efficient to use the deterministic models as the number of molecules increases.

But the stochastic approach is invaluable in several ways. It builds a clear intuitive connection between the microscopic and the macroscopic. The microscopic level is characterized by discontinuous, random molecular motion and the probability of collision as the basis for chemical reaction rate. The macroscopic level is characterized by

<sup>2</sup>It is more accurate to use the term pseudo-random number here to distinguish something we compute from a truly random number.

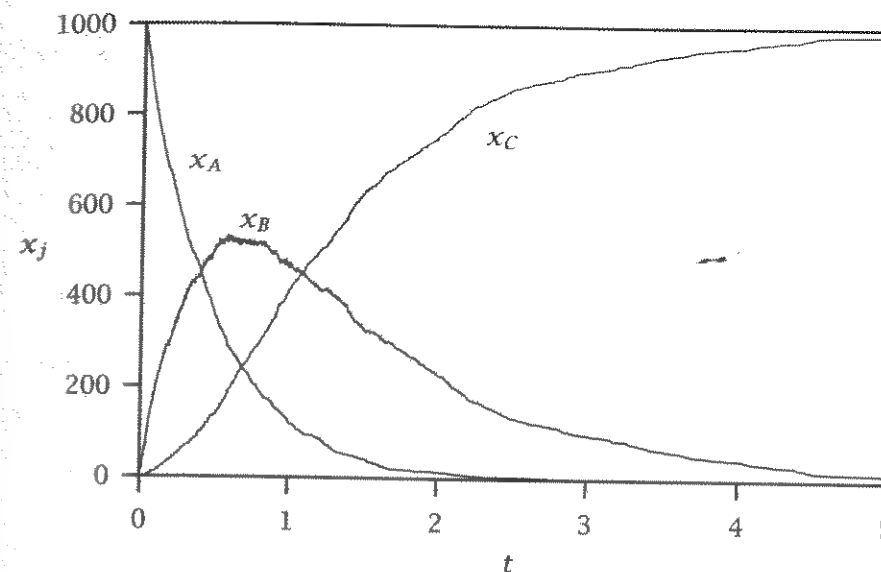


Figure 4.30: Stochastic simulation of the first-order reactions  $A \rightarrow B \rightarrow C$  starting with 1000 A molecules.

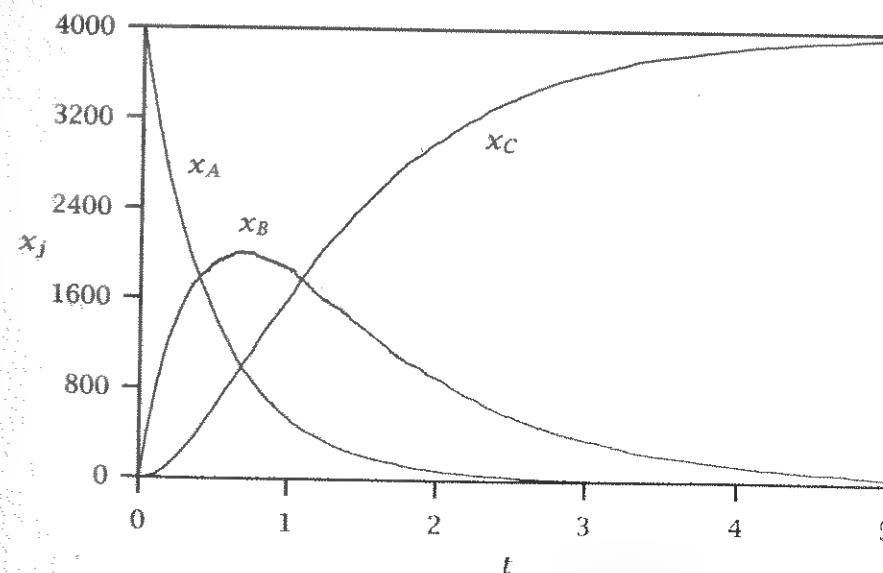


Figure 4.31: Stochastic simulation of the first-order reactions  $A \rightarrow B \rightarrow C$  starting with 4000 A molecules.

smoothly varying concentrations, and deterministic rate laws and material balances. Watching the transition in Figures 4.29–4.31 and then finally to the deterministic Figure 4.11 is a nice illustration of this connection and provides logical support for the construction of the deterministic rate laws. It is possible to prove that the average of stochastic simulations converges to the deterministic simulation as the number of molecules becomes large, which is known as the thermodynamic limit.

As stressed earlier, the random fluctuations may be an important physical behavior to include in the model. In this situation, the stochastic approach is essential and a deterministic approach cannot be substituted. We illustrate with the hepatitis B virus model introduced in Chapter 1.

#### Example 4.9: Stochastic versus deterministic simulation of a virus model

Consider the hepatitis B virus model given in Chapter 1.



Assume the system starts with a single cccDNA molecule, and no rcDNA and no envelope protein, and use the following rate constants

$$\begin{bmatrix} x_A & x_B & x_C \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \quad (4.101)$$

$$\mathbf{k}^T = \begin{bmatrix} 1 & 0.025 & 1000 & 0.25 & 2 & 7.5 \times 10^{-6} \end{bmatrix} \quad (4.102)$$

Compare the results of a deterministic simulation to the average of 500 stochastic simulations. If these results are not the same, explain why not.

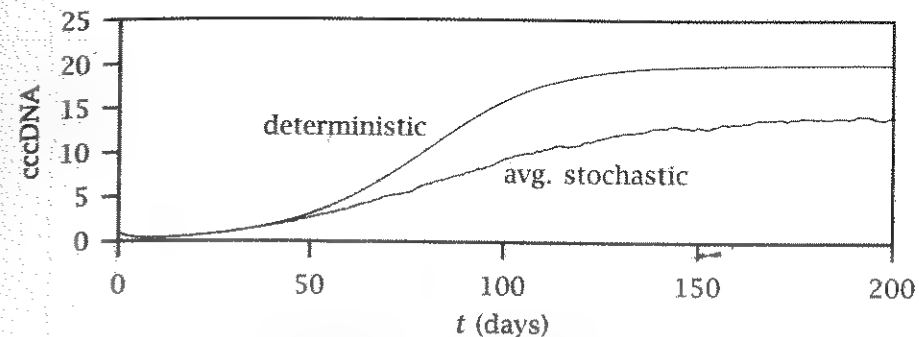


Figure 4.32: Species cccDNA versus time for hepatitis B virus model; deterministic and average stochastic models.

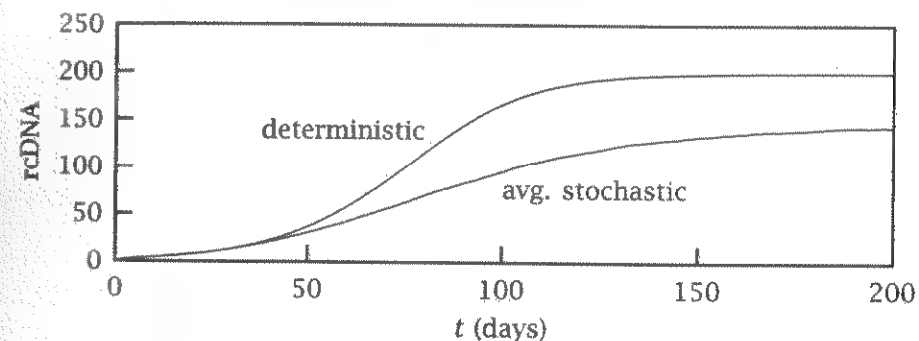


Figure 4.33: Species rcDNA versus time for hepatitis B virus model; deterministic and average stochastic models.

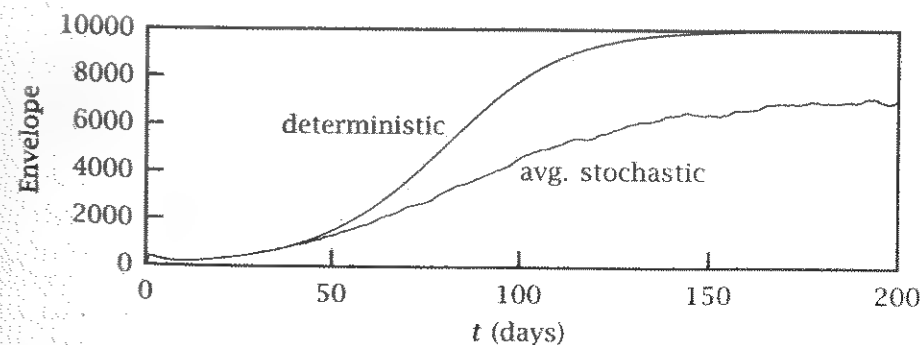


Figure 4.34: Envelope versus time for hepatitis B virus model; deterministic and average stochastic models.

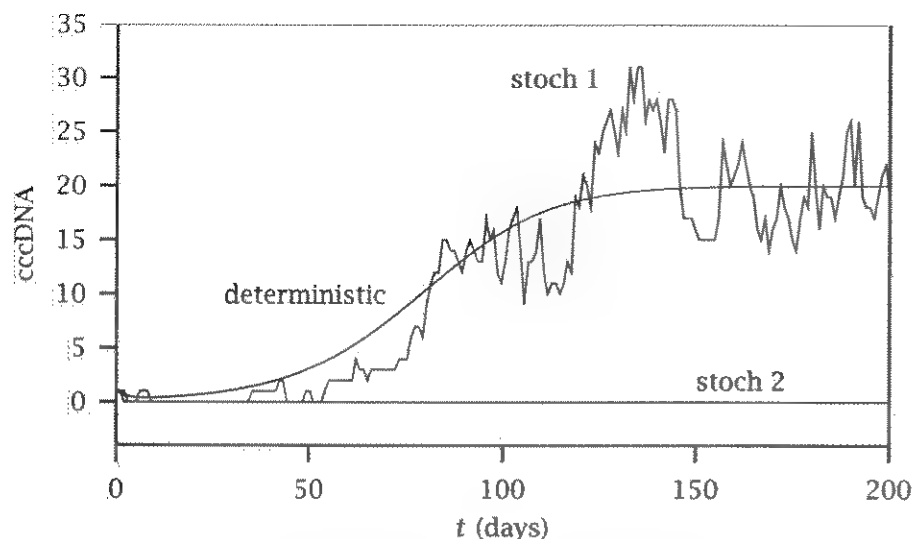


Figure 4.35: Species cccDNA versus time for hepatitis B virus model; two representative stochastic trajectories.

### Solution

The reaction rates and production rates for Reactions 4.95–4.100 are given by

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{bmatrix} = \begin{bmatrix} k_1 x_A \\ k_2 x_B \\ k_3 x_A \\ k_4 x_A \\ k_5 x_C \\ k_6 x_B x_C \end{bmatrix} \quad \begin{bmatrix} R_A \\ R_B \\ R_C \end{bmatrix} = \begin{bmatrix} r_2 - r_4 \\ r_1 - r_2 - r_6 \\ r_3 - r_5 - r_6 \end{bmatrix} \quad (4.103)$$

in which A is cccDNA, B is rcDNA, and C is envelope.

Figures 4.32–4.34 show the deterministic model simulation and an average of 500 stochastic simulations. Notice these results are *not* the same, and we should investigate why not. Figure 4.35 shows *two* representative stochastic simulations for only the cccDNA species. Notice the first stochastic simulation does fluctuate around the deterministic simulation as expected. The second stochastic simulation, however, shows complete extinction of the virus. That is another possible steady state for the stochastic model. In fact, it occurs for 125 of the 500

BATCH	$\frac{d(c_j V_R)}{dt} = R_j V_R$	(4.104)
constant volume	$\frac{dc_j}{dt} = R_j$	(4.105)
CSTR	$\frac{d(c_j V_R)}{dt} = Q_f c_{jf} - Q c_j + R_j V_R$	(4.106)
constant density	$\frac{dc_j}{dt} = \frac{1}{\theta} (c_{jf} - c_j) + R_j$	(4.107)
steady state	$c_j = c_{jf} + R_j \theta$	(4.108)
SEMI-BATCH	$\frac{d(c_j V_R)}{dt} = Q_f c_{jf} + R_j V_R$	(4.109)
PFR	$\frac{\partial c_j}{\partial t} = -\frac{\partial(c_j Q)}{\partial V} + R_j$	(4.110)
steady state	$\frac{d(c_j Q)}{dV} = R_j$	(4.111)
constant flowrate	$\frac{dc_j}{d\theta} = R_j, \quad \theta = V/Q_f$	(4.112)

Table 4.3: Summary of mole balances for several ideal reactors.

simulations. So the *average* stochastic simulation in Figures 4.32–4.34 consist of 75% trajectories that fluctuate about the deterministic trajectory and 25% trajectories that go to zero. The two types of stochastic trajectories therefore explain why the average stochastic model is not equal to the deterministic model. We should bear this feature in mind when using deterministic models with small numbers of molecules. □

## 4.9 Summary

We have introduced four main reactor types in this chapter: the batch reactor, the continuous-stirred-tank reactor (CSTR), the semi-batch reactor, and the plug-flow reactor (PFR). Table 4.3 summarizes the mole balances for these four reactors. We also have introduced some of the basic reaction-rate expressions:

- first order, irreversible

- first order, reversible
- second order, irreversible
- $n$ th order, irreversible
- two first-order reactions in series
- two first-order reactions in parallel
- two second-order, reversible reactions

We developed the equations required to compute the volume of the reactor if there is a significant volume change upon reaction. We require an equation of state for this purpose. Tables 4.1 and 4.2 describe the appropriate balances for a constant-density mixture, an ideal mixture, and a mixture with a general equation of state.

Several of these simple mass balances with basic rate expressions were solved analytically. In the case of multiple reactions with nonlinear rate expressions (i.e., not first-order reaction rates), the balances must be solved numerically. A high-quality ordinary differential equation (ODE) solver is indispensable for solving these problems. For a complex equation of state and nonconstant-volume case, a differential-algebraic equation (DAE) solver may be convenient.

We showed that the PFR achieves higher conversion than the CSTR of the same volume if the reaction rate is an increasing function of component composition ( $n > 0$  for an  $n$ th-order rate expression). Conversely, the CSTR achieves higher conversion than the same-volume PFR if the rate is a decreasing function of a component composition ( $n < 0$ ).

Finally, we introduced stochastic simulation to model chemical reactions occurring with *small* numbers of molecules. Each of these random simulation trajectories has a rough appearance and the average of many of these simulations is required to show the expected system behavior. The stochastic model uses basic probability to compute reaction rate. The probability of occurrence of a given reaction is assumed proportional to the number of possible combinations of reactants for the given stoichiometry. Two pseudo-random numbers are chosen to determine: (i) the time of the next reaction and (ii) the reaction that occurs at that time. The smooth behavior of the macroscopic ODE models is recovered by the random simulations in the limit of large numbers of reacting molecules. With small numbers of molecules, however, the average of the stochastic simulation does not have to be equal to the deterministic simulation. We demonstrated this fact with the simple, nonlinear hepatitis B virus model.

## Notation

$A_c$	reactor tube cross-sectional area
$A_i$	preexponential factor for rate constant $i$
$c_j$	concentration of species $j$
$c_{jf}$	feed concentration of species $j$
$c_{js}$	steady-state concentration of species $j$
$c_{j0}$	initial concentration of species $j$
$E_i$	activation energy for rate constant $i$
$k_i$	reaction rate constant for reaction $i$
$K_i$	equilibrium constant for reaction $i$
$M$	total mass flow, $N_j M_j$
$M$	total monomer mass
$M_f$	total feed mass flow, $N_j M_j$
$M_j$	molecular weight of species $j$
$n$	reaction order
$n_j$	moles of species $j$ , $V_R c_j$
$n_r$	number of reactions in the reaction network
$n_s$	number of species in the reaction network
$N_j$	molar flow of species $j$ , $Q c_j$
$N_{jf}$	feed molar flow of species $j$ , $Q c_j$
$p$	uniformly distributed random number on $(0, 1)$
$P$	pressure
$\hat{P}$	total polymer mass
$Q$	volumetric flowrate
$Q_f$	feed volumetric flowrate
$r_i$	reaction rate for $i$ th reaction
$r_{\text{tot}}$	total reaction rate, $\sum_i r_i$
$R$	gas constant
$R_j$	production rate for $j$ th species
$t$	time
$T$	temperature
$V$	reactor volume variable
$V_j$	partial molar volume of species $j$
$V_j^\circ$	specific molar volume of species $j$
$V_R$	reactor volume
$\Delta V_i$	change in volume upon reaction $i$ , $\sum_j \nu_{ij} V_j$
$x_j$	number of molecules of species $j$ in a stochastic simulation
$x_j$	molar conversion of species $j$
$y_j$	mole fraction of gas-phase species $j$



$z$	reactor length variable
$\varepsilon_i$	extent of reaction $i$
$\theta$	reactor residence time, $V_R/Q_f$
$\nu_{ij}$	stoichiometric coefficient for species $j$ in reaction $i$
$\rho$	mass density
$\tau$	simulation time in a stochastic simulation

## 4.10 Exercises

### Exercise 4.1: Order versus rate of reaction

Liquid A decomposes with  $n$ th-order kinetics in a batch reactor



The conversion of A reaches 50% in a five-minute run.

What is the order of the reaction if it takes 10 minutes to reach 75% conversion? What is the order of the reaction if it takes 20 minutes to reach 75% conversion? What is the order of the reaction if it takes 30 minutes to reach 75% conversion?

### Exercise 4.2: Constant volume versus constant pressure batch reactor

Consider the following two well-mixed, isothermal gas-phase batch reactors for the elementary and irreversible decomposition of A to B,



**reactor 1:** The reactor volume is held constant (reactor pressure therefore changes).

**reactor 2:** The reactor pressure is held constant (reactor volume therefore changes).

Both reactors are charged with pure A at 1.0 atm and  $k = 0.35 \text{ min}^{-1}$ .

- What is the fractional decrease in the *concentration* of A in reactors 1 and 2 after five minutes?
- What is the *total molar conversion* of A in reactors 1 and 2 after five minutes?

### Exercise 4.3: CSTR performance

A liquid-phase reaction



takes place in a CSTR of volume  $V_R$  in the presence of a large excess of reactant B. Assume the reactor achieves 50% conversion of A at steady state.

- What is the steady-state conversion if the original reactor is replaced by two CSTRs of volume  $V_R/2$  in series?
- What is the conversion if the original reactor is replaced by three CSTRs of volume  $V_R/3$  in series?
- What is the conversion if the original reactor is replaced by a plug-flow reactor of volume  $V_R$ ?

### Exercise 4.4: Catalyst deactivation in a batch reactor

Consider the irreversible, liquid-phase isomerization reaction carried out in a solvent containing dissolved catalyst at 25°C in a batch reactor



The *apparent* first-order reaction-rate constant,  $k_a$ , decreases with time because of catalyst deterioration. A chemist colleague of yours has studied the catalyst deactivation process and has proposed that it can be modeled by

$$k_a = \frac{k}{1 + k_d t}$$

in which  $k$  is the fresh catalyst rate constant and  $k_d$  is the deactivation-rate constant.

- Write down the mole balance for this reactor.
- Solve the mole balance for  $c_A(t)$ . Sketch your solution.
- If it takes two hours to reach 50% conversion and the fresh catalyst has a rate constant of  $0.6 \text{ hr}^{-1}$ , what is  $k_d$ ?
- How long does it take to reach 75% conversion?

**Exercise 4.5: Dynamic CSTR**

A CSTR is used to convert A to products B and C via the following liquid-phase reaction



The reaction is first order in A and irreversible. The tank initially is charged with species A at concentration  $c_{A0}$ . At time zero, the feed pump is turned on and delivers constant flowrate,  $Q_f$ . The feed concentration of A is  $c_{Af}$ , which is also constant. The tank volume is  $V_R$ . Liquid density change due to reaction may be neglected.

- Write down and solve the dynamic material balance for component A.
- Sketch the solution,  $c_A(t)$  versus  $t$ , for the tank initially filled with solvent,  $c_{A0} = 0$ . On the same plot, sketch the solution for the tank initially filled with feed,  $c_{A0} = c_{Af}$ . Clearly label on your plot the initial and steady-state concentrations for both curves.
- For a  $50 \text{ m}^3$  tank with flowrate of  $7 \text{ L/s}$  and rate constant  $k = 0.02 \text{ min}^{-1}$ , what is the steady-state conversion of A?

**Exercise 4.6: Linear differential equations and Laplace transforms**

Consider the linear, first-order, constant-coefficient differential equation

$$\frac{dx}{dt} = Ax + b$$

$$x(0) = x_0 \quad (4.113)$$

The Laplace transform is a handy method for solving linear differential equations.

- Take the Laplace transform of Equation 4.113 and show

$$\bar{x}(s) = \frac{x_0}{s - A} + \frac{b}{s(s - A)} \quad (4.114)$$

- Invert the transform and show

$$x(t) = x_0 e^{At} + \frac{b}{A} e^{At} \quad (4.115)$$

You may want to use the following partial fraction formula

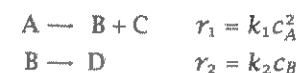
$$\frac{1}{s(s - A)} = \frac{1}{A} \left[ \frac{1}{s} - \frac{1}{s - A} \right]$$

- Substitute Equation 4.115 into Equation 4.113 and show that it satisfies the differential equation and initial condition.

- Identify  $A, b$ , so that Equation 4.113 applies to the reversible, first-order reaction case given in Equation 4.13. Substitute these values into Equation 4.115 and verify Equation 4.15.
- Identify  $A, b$ , so that Equation 4.113 applies to the reversible, first-order reaction case given in Equation 4.42. Substitute these values into Equation 4.115 and verify Equation 4.43.

**Exercise 4.7: Multiple reactions in CSTRs in series**

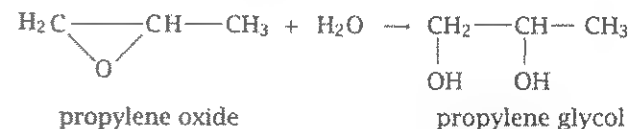
The following liquid-phase reactions take place in two identical CSTRs in series,



in which  $k_1 = 0.05 \text{ L/mol} \cdot \text{min}$  and  $k_2 = 0.022 \text{ min}^{-1}$ . The feed to the first reactor is pure A,  $c_{Af} = 1.5 \text{ mol/L}$ , and the residence time in each reactor is  $15 \text{ min}$ . Determine the overall conversion of A and the yield of B for the series of reactors. Recall yield is the fraction of reactant converted into a specified product.

**Exercise 4.8: Nonconstant density with a liquid-phase reaction**

Propylene glycol is produced by the hydrolysis of propylene oxide according to the following reaction



In the presence of excess water, the reaction has been found to be first-order in propylene oxide

$$r = k c_{\text{PO}}$$

and the rate constant is [6]

$$k = k_0 e^{-E_a/RT} \quad k_0 = 4.71 \times 10^9 \text{ s}^{-1} \quad E_a = 18.0 \text{ kcal/mol K}$$

Methanol is added as a solvent, and the reaction is performed in a  $1000 \text{ L}$  CSTR operating at  $60^\circ\text{C}$ . The feed conditions and physical properties are as follows [15]:

Component	Density (g/cm <sup>3</sup> )	Mol. wt. (g/mol)	Inlet feedrate (L/hr)
propylene oxide	0.859	58.08	1300
water	1.000	18.02	6600
propylene glycol	1.0361	76.11	0
methanol	0.7914	32.04	1300

Assume the mixture is ideal so that

$$1 = \sum_j c_j V_j^*$$

in which  $V_j^* = M_j/\rho_j^*$  are the pure component specific molar volumes. Neglect any change in the pure component densities with temperature in the temperature range 25–60°C.

(a) Compute the steady-state concentrations of all components,  $Q$ , and  $V_R$  for the following two situations.

1. A float in the top of the tank is used to adjust  $Q$  to maintain reactor volume constant at 1000 L.
2. The reactor is initially charged with pure solvent, and a differential pressure measurement is used to adjust  $Q$  to maintain constant reactor mass.

Which operation do you recommend, constant volume or constant mass? Look at the conversion of propylene oxide and production rate of propylene glycol for the two cases. What are you wasting in constant mass operation?

(b) Resolve the constant reactor volume operation under the assumption that all densities are equal to water. How much error in the conversion and production rate do you commit under this assumption?

#### Exercise 4.9: Linear density versus concentration assumption

Consider the important case of liquid-phase reaction in an excess of solvent. Let component  $j = s$  be the solvent and consider the solution density at some reference concentrations,  $c_j = c_{j0}$ ,

$$\rho_0 = M_s c_{s0} + \sum_{j \neq s} M_j c_{j0}$$

In this situation, a useful approximation for the density-concentration relationship is the linear form [13]

$$\rho = \rho_0 + \sum_{j \neq s} M_j (c_j - c_{j0})$$

or, equivalently, upon substitution of the previous equation

$$\rho = M_s c_{s0} + \sum_{j \neq s} M_j c_j \quad (4.116)$$

(a) Show that even though the density is not assumed constant in Equation 4.116, one still obtains for the reactor volume in a CSTR

$$\frac{dV_R}{dt} = Q_f - Q$$

(b) Show that the approximation error in the density in Equation 4.116 is

$$M_s (c_s - c_{s0})$$

in which  $c_s$  is the actual solvent concentration in the mixture, which also changes, in general, upon reaction even though the solvent does not take part in any reactions.

#### Exercise 4.10: Maximizing an intermediate

We considered the following two first-order, irreversible reactions in Reaction 4.29



and derived Equation 4.31 for the intermediate B concentration

$$c_B = c_{B0} e^{-k_2 t} + c_{A0} \frac{k_1}{k_2 - k_1} [e^{-k_1 t} - e^{-k_2 t}], \quad k_1 \neq k_2$$

- (a) What is the optimal stopping time in order to maximize the concentration of intermediate B?
- (b) Check your result against Figure 4.11.
- (c) How does this result simplify for the case  $k_1 = k_2$ ?

#### Exercise 4.11: Some limits

(a) For the second-order, irreversible reaction



in the batch reactor, we showed solving the material balance produces Equation 4.24

$$c_A = (c_{A0} - c_{B0}) \left[ 1 - \frac{c_{B0}}{c_{A0}} e^{(c_{B0} - c_{A0}) k t} \right]^{-1} \quad c_{A0} \neq c_{B0}$$

Take the limit  $c_{A0} - c_{B0} \rightarrow 0$  in this equation and show it produces the other second-order reaction result given in Equation 4.19

$$c_A = \left( \frac{1}{c_{A0}} + kt \right)^{-1}$$

(b) For the  $n$ -order, irreversible chemical reaction



in the batch reactor, we showed solving the material balance produces Equation 4.26

$$\frac{c_A}{c_{A0}} = [1 + (n-1)k_0t]^{-\frac{1}{n-1}}, \quad n \neq 1$$

in which

$$k_0 = kc_{A0}^{n-1}$$

Take the limit  $n \rightarrow 1$  in this equation and show that it produces the first-order reaction result

$$\frac{c_A}{c_{A0}} = e^{-kt}, \quad n = 1$$

#### Exercise 4.12: Constant gas flowrate; large or small error?

One of the interesting features of (ideal) gas molecules is that they demand a certain amount of "territory." It doesn't matter if they are heavy or light, big or small — all gas molecules occupy the same amount of space at a given temperature and pressure; hence the idea gas law.

Consider the following hypothetical reaction taking place in the gas phase in an isothermal, constant pressure PFR with a pure A feed stream



Derive an expression for the tube volume required to achieve molar conversion of A,  $x_A$ , for the following two cases.

- Ideal gas equation of state, which we call the correct answer,  $V_{Rc}$ .
- Constant gas flowrate, which we call the dubious assumption,  $V_{Rd}$ .
- Compute the fractional error committed when using the dubious assumption

$$e = \frac{V_{Rc} - V_{Rd}}{V_{Rc}}$$

Your result should be a function of only  $x_A$  and  $b$ .

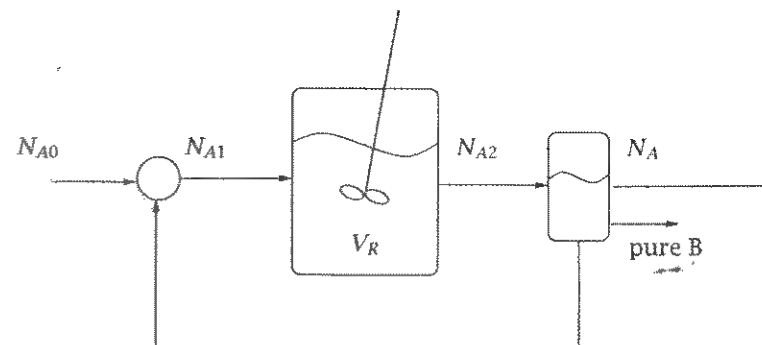


Figure 4.36: An alternative CSTR-separator design.

- Plot  $e$  versus  $x_A$  for  $b = 1, 2, 5, 100$ . Note for  $b = 100$ , Reaction 4.117 is very hypothetical.
- If you are unwilling to commit errors larger than 10%, for what  $b$  and  $x_A$  values must you object to the dubious assumption?

Bottom line: are you generally making large or small errors when you neglect the change in the number of moles when modeling a gas-phase reaction in a PFR?

#### Exercise 4.13: Ethane pyrolysis, revisited

Using the reaction chemistry and kinetics presented in Example 4.7, change the feed concentration of NO from 5 ppm to 5%, and examine the effect on the concentration profile for ethane in the PFR at 1050 K. Assume the feed consists only of ethane and NO, and that the pressure and volumetric flowrate are the same as in Example 4.7. Explain the effect of the feed NO concentration on the ethane conversion.

#### Exercise 4.14: Recycle effects

Consider again Example 4.8, and compute the limiting conversion for a CSTR with recycle for  $k\theta = 0.5$ . What is the corresponding limiting fractional recycle,  $\alpha$ ? What happens if one attempts to recycle more material than this limiting value?

#### Exercise 4.15: An alternative reactor-separator design

Feinberg and Ellison considered an alternative design for Example 4.8 shown in Figure 4.36. Notice the separator is placed ahead of the recycle split in this design, and after the recycle split in Figure 4.26.

- Calculate the flowrate of the recycle stream that achieves the PFR conversion given in Example 4.8.
- Compare the two reactor-separator designs. What are the advantages and disadvantages of each design.

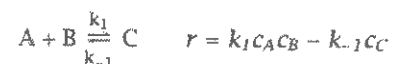
#### Exercise 4.16: Stochastic simulation of a virus model

Perform several random simulations of the virus model given in Equations 4.95–4.102. Perform enough simulations so that some of your trajectories lead to extinction of all three species: cccDNA, rcDNA and envelope protein.

- Compare and contrast a simulation that leads to a steady-state infection, and a simulation that leads to extinction.
- During the first 25–50 days, is it possible to tell by looking at the trajectory whether it will lead to infection or extinction? About how many days are required before you have a reliable idea of where a trajectory will finish?
- Of the trajectories that lead to extinction, what is the longest time required to reach extinction? What fraction of your trajectories lead to extinction?
- Which parameters in the model have the largest affect on the behavior?

#### Exercise 4.17: Stochastic simulation for nonlinear kinetics

Consider the reversible, second-order reaction

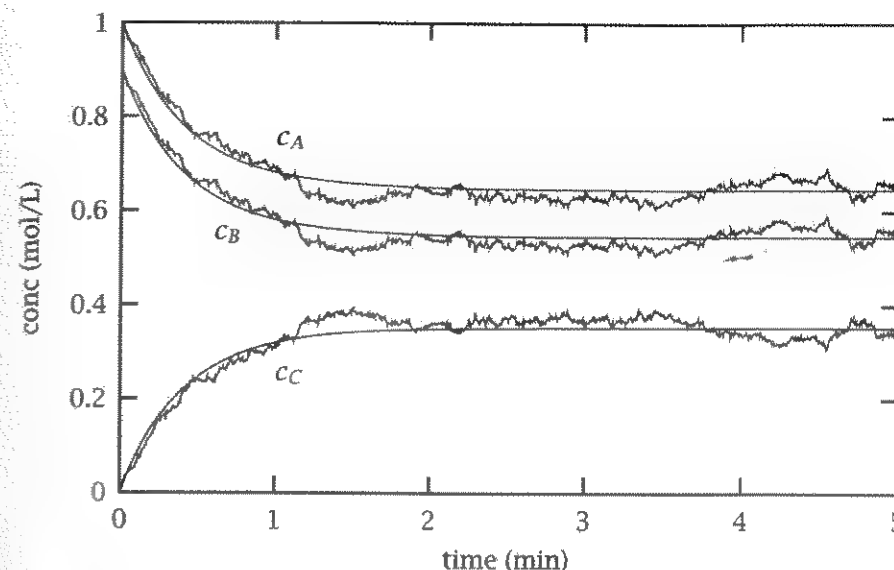


- Solve the deterministic material balance for a constant-volume batch reactor with

$$k_1 = 1 \text{ mol/L} \cdot \text{min} \quad k_{-1} = 1 \text{ min}^{-1} \\ c_A(0) = 1 \text{ mol/L} \quad c_B(0) = 0.9 \text{ mol/L} \quad c_C(0) = 0 \text{ mol/L}$$

Plot the A, B and C concentrations out to  $t = 5$  min.

- Compare the result to a stochastic simulation using an initial condition of 400 A, 360 B and zero C molecules. Notice from the units of the rate constants that  $k_1$  should be divided by 400 to compare simulations. Figure 4.37 is a representative comparison for one sequence of pseudo-random numbers.



**Figure 4.37:** Deterministic simulation of reaction  $A + B = C$  compared to stochastic simulation starting with 400 A molecules.

- Repeat the stochastic simulation for an initial condition of 4000 A, 3600 B, zero C molecules. Remember to scale  $k_1$  appropriately. Are the fluctuations noticeable with this many starting molecules?

#### Exercise 4.18: From microscopic to macroscopic

It is often easier to provide a correct explanation than it is to find the error in someone else's incorrect explanation. Consider the following situation from an instructor's point of view. A student in your class asks you to explain the following difficulty he is having. "I start with the continuity equation for species  $j$ ,"

$$\frac{\partial c_j}{\partial t} + \nabla \cdot (c_j \mathbf{v}_j) - R_j = 0 \quad (4.118)$$

"Since this equation is true at all points in the volume element, I integrate it over the CSTR reactor volume element (as depicted in Figure 4.13),"

$$\int_V \left[ \frac{\partial c_j}{\partial t} + \nabla \cdot (c_j \mathbf{v}_j) - R_j \right] d\Omega = 0 \quad (4.119)$$

"I then use the divergence theorem that we learned in the Transport Phenomena course to convert the divergence term to a surface integral and obtain,"

$$\int_V \left[ \frac{\partial c_j}{\partial t} - R_j \right] d\Omega + \int_S c_j (\mathbf{v}_j \cdot \mathbf{n}) d\sigma = 0 \quad (4.120)$$

"It is clear to me that the surface integral picks up the flux of species  $j$  into and out of the reactor with the flow streams,"

$$\int_S c_j (\mathbf{v}_j \cdot \mathbf{n}) d\sigma = -Q_f c_{jf} + Q c_j \quad (4.121)$$

"Substituting this into Equation 4.120 gives"

$$\int_V \left[ \frac{\partial c_j}{\partial t} - R_j \right] d\Omega - Q_f c_{jf} + Q c_j = 0 \quad (4.122)$$

"Finally since  $c_j$  and  $R_j$  don't vary over the well-mixed reactor, I simplify to"

$$V_R \frac{dc_j}{dt} = Q_f c_{jf} - Q c_j + R_j V_R \quad (4.123)$$

"But this doesn't agree with Equation 4.106!" How do you respond to this student? Remember, he does not really want to hear a restatement of why Equation 4.106 is correct; he really wants to know what is wrong with his derivation.

#### Exercise 4.19: Index and DAEs

We saw in Section 4.5 that it was convenient to specify reactor models for nonconstant-density cases by using DAEs instead of ODEs. Compare, for example, the complexity of the ODE and DAE models appearing in Table 4.2. Some care should be exercised, however, to avoid creating a DAE that is difficult to solve. The DAEs in Table 4.2 are of the following form in which the time derivatives of only the  $\mathbf{x}$  variables appear explicitly in the model,

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{y}) \quad (4.124)$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{y}) \quad (4.125)$$

The total moles of the species,  $n_j$ , are the  $\mathbf{x}$  variables and  $V_R$  and  $Q$  are the  $\mathbf{y}$  variables.

$$\mathbf{x}^T = [n_1 \ n_2 \ \cdots \ n_{n_s}] \quad \mathbf{y}^T = [V_R \ Q]$$

The index of a DAE is the number of times that the algebraic equations must be differentiated with respect to  $t$  to determine  $d\mathbf{y}/dt$  [3]. An ODE, with no algebraic relations, has index zero. Differentiating Equation 4.125 with respect to time produces

$$\mathbf{0} = \frac{\partial \mathbf{g}(\mathbf{x}, \mathbf{y})}{\partial \mathbf{x}^T} \frac{d\mathbf{x}}{dt} + \frac{\partial \mathbf{g}(\mathbf{x}, \mathbf{y})}{\partial \mathbf{y}^T} \frac{d\mathbf{y}}{dt}$$

in which  $\partial \mathbf{g} / \partial \mathbf{y}^T$  is a Jacobian matrix with elements

$$\left( \frac{\partial \mathbf{g}}{\partial \mathbf{y}^T} \right)_{ij} = \frac{\partial g_i}{\partial y_j}$$

If  $\partial \mathbf{g} / \partial \mathbf{y}^T$  is a nonsingular matrix, then we can solve this equation for  $d\mathbf{y}/dt$  and the DAE has index one. DAEs with index zero and one are generally much easier to solve than those with index two or higher [3]. Notice if the  $\mathbf{y}$  variables do not appear explicitly in the algebraic equations, for example, then  $\partial \mathbf{g} / \partial \mathbf{y}^T = \mathbf{0}$  and the DAE has index two or higher.

Consider the algebraic equations appearing in Table 4.2 for the nonconstant-density cases: 1. reactor volume constant, 2. reactor mass constant, and 3.  $Q$  specified. Show that, as expressed in Table 4.2, cases 1. and 2. are high-index DAEs and case 3. is index one. Without differentiating the equation of state, how can case 2. be modified to be index one? Can you find a simple way to modify case 1. to be index one?

#### Exercise 4.20: Matrix exponential

Consider the general linear kinetic expression written in vector-matrix form

$$\mathbf{R}(\mathbf{c}) = \mathbf{K}\mathbf{c}$$

in which  $\mathbf{R}$  is an  $n_s$ -vector of production rates,  $\mathbf{c}$  is an  $n_s$ -vector of concentrations, and  $\mathbf{K}$  is an  $n_s \times n_s$  matrix of rate constants. The batch reactor material balance can be written as

$$\begin{aligned} \frac{d\mathbf{c}}{dt} &= \mathbf{K}\mathbf{c} \\ \mathbf{c}(0) &= \mathbf{c}_0 \end{aligned} \quad (4.126)$$

The solution to Equations 4.126 can be written as  $\mathbf{c} = e^{\mathbf{K}t} \mathbf{c}_0$ , in which  $e^{\mathbf{K}t}$  is the matrix exponential. The matrix exponential can be defined using the power series

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2!} \mathbf{A}^2 + \frac{1}{3!} \mathbf{A}^3 + \cdots$$

This series converges for all  $\mathbf{A}$ .

(a) Write  $\mathbf{K}$  for two first-order series reactions in a batch reactor

$$\mathbf{A} = \mathbf{B} \quad \mathbf{B} = \mathbf{C}$$

(b) Write out the series expansion for  $e^{\mathbf{K}t}$  and verify the solution derived in Section 4.2.1

(c) Repeat for two first-order parallel reactions in a batch reactor

$$\mathbf{A} = \mathbf{B} \quad \mathbf{A} = \mathbf{C}$$



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## 5

## Chemical Kinetics

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### 5.1 Introduction

The purpose of this chapter is to provide a framework for determining the reaction rate given a detailed statement of the reaction chemistry. We use several concepts from the subject of chemical kinetics to illustrate two key points:

1. The stoichiometry of an elementary reaction defines the concentration dependence of the rate expression.
2. The quasi-steady-state assumption (QSSA) and the reaction equilibrium assumption allow us to generate reaction-rate expressions that capture the details of the reaction chemistry with a minimum number of rate constants.

The concepts include:

- the elementary reaction
- Tolman's principle of microscopic reversibility
- elementary reaction kinetics
- the quasi-steady-state assumption
- the reaction equilibrium assumption

This chapter complements Chapter 9 on data analysis and empirical reaction-rate expressions in which the goal is to find a mathematical relation that describes the reaction rate over the range of experimental conditions. The goals here are to develop a chemical kinetics basis for the empirical expression, and to show that kinetic analysis can be used to take mechanistic insight and describe reaction rates from first principles.

In this chapter we also discuss heterogeneous catalytic adsorption and reaction kinetics. Catalysis has a significant impact on the United States economy<sup>1</sup> and many important reactions employ catalysts.<sup>2</sup> We describe the kinetic principles that are needed for rate studies and demonstrate how the concepts for homogeneous reactions apply to heterogeneously catalyzed reactions with the added constraint of surface-site conservation. The physical characteristics of catalysts are discussed in Chapter 7.

## 5.2 Elementary Reactions and Microscopic Reversibility

Stoichiometric statements such as



are used to represent the changes that occur during a chemical reaction. These statements can be interpreted in *two* ways. The reaction statement may represent the change in the relative amounts of species that is observed when the reaction proceeds. Or the reaction statement may represent the actual molecular events that are presumed to occur as the reaction proceeds. The former is referred to as an overall stoichiometry or an overall reaction statement. The latter is referred to as an elementary reaction. The elementary reaction is characterized by a change from reactants to products that proceeds without identifiable intermediate species forming. We show subsequently that for an elementary reaction, the reaction rates for the forward and reverse paths are proportional to the concentration of species taking part in the reaction raised to the absolute value of their stoichiometric coefficients. The reaction order in all species is determined directly from the stoichiometry. Elementary reactions are usually unimolecular or bimolecular because the probability of collision between several species is low and is not observed at appreciable rates. For an overall stoichiometry, on the other hand, any correspondence between the stoichiometric coefficients and the reaction order is purely coincidental.

We use three examples to illustrate overall reaction stoichiometry and elementary reactions. The first involves the mechanism proposed

<sup>1</sup>Catalysis is at the heart of the chemical industry, with sales in 1990 of \$292 billion and employment of 1.1 million, and the petroleum refining industry, which had sales of \$140 billion and employment of 0.75 million in 1990 [27].

<sup>2</sup>Between 1930 and the 1980s, 63 major products and 34 major process innovations were introduced by the chemical industry. More than 60% of the products and 90% of these processes were based on catalysis[27].

for the thermal decomposition of acetone at 900 K to ketene and methyl-ethyl ketone (2-butanone) [33]. The overall reaction can be represented by



and is proposed to proceed by the following elementary reactions



In the first elementary reaction, acetone undergoes unimolecular decomposition to a methyl radical and to an acetyl radical. These radicals continue to react: the methyl radical reacts to methane and a third radical ( $\text{CH}_3\text{COCH}_2$ ) in elementary Reaction 5.4, and the methyl radical combines with the third radical to produce methyl ethyl ketone in elementary Reaction 5.6. The acetyl radical undergoes unimolecular decomposition to a methyl radical and carbon monoxide in the second elementary reaction. The third radical also undergoes unimolecular decomposition in elementary Reaction 5.5 to ketene and a methyl radical. The thermal decomposition of acetone generates four stable molecules that can be removed from the reaction vessel: CO,  $\text{CH}_4$ ,  $\text{CH}_2\text{CO}$  and  $\text{CH}_3\text{COC}_2\text{H}_5$ . Three radicals are formed and consumed during the thermal decomposition of acetone:  $\text{CH}_3$ ,  $\text{CH}_3\text{CO}$  and  $\text{CH}_3\text{COCH}_2$ . These three radicals are reaction intermediates and cannot be isolated outside of the reaction vessel.

The overall reaction stoichiometry, Reaction 5.1, is explained as a linear combination of the five reactions in the proposed mechanism. By balancing the species in the overall stoichiometry, one can determine that the overall stoichiometry is produced by adding twice the third reaction to the remaining reactions. If the elementary reactions cannot be combined to form the overall stoichiometry, then the mechanism is not a valid description of the observed stoichiometry.

If we assign the species to the A vector as follows

Species	Formula	Name
A <sub>1</sub>	CH <sub>3</sub> COCH <sub>3</sub>	acetone
A <sub>2</sub>	CH <sub>3</sub>	methyl radical
A <sub>3</sub>	CH <sub>3</sub> CO	acetyl radical
A <sub>4</sub>	CO	carbon monoxide
A <sub>5</sub>	CH <sub>3</sub> COCH <sub>2</sub>	acetone radical
A <sub>6</sub>	CH <sub>2</sub> CO	ketene
A <sub>7</sub>	CH <sub>4</sub>	methane
A <sub>8</sub>	CH <sub>3</sub> COC <sub>2</sub> H <sub>5</sub>	methyl ethyl ketone

then the stoichiometric matrix is

$$\mathbf{v} = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

If we multiply  $\mathbf{v}$  by  $\begin{bmatrix} 1 & 1 & 2 & 1 & 1 \end{bmatrix}$  we obtain

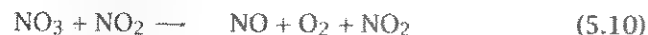
$$\begin{bmatrix} -3 & 0 & 0 & 1 & 0 & 1 & 2 & 1 \end{bmatrix}$$

which is the overall stoichiometry given in Reaction 5.1.

The second example involves one of the major reactions responsible for the production of photochemical smog. The overall reaction is



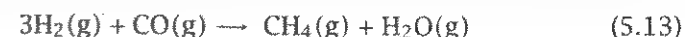
and one possible mechanism is [12, 10]



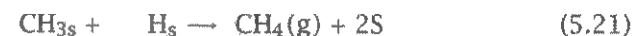
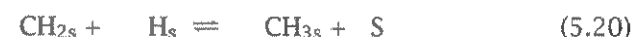
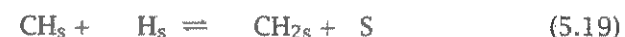
In this reaction mechanism, nitrogen dioxide is activated by absorbing photons and decomposes to nitric oxide and oxygen radicals (elementary Reaction 5.8). As in the previous example, stable molecules are formed, in this case NO and O<sub>2</sub>, and radicals, O and NO<sub>3</sub>, are generated and consumed during the photochemical reaction. The student

should work through the steps to determine the linear combination of the mechanistic steps that produces the overall reaction.

The third example involves the synthesis of methane from synthesis gas, CO and H<sub>2</sub>, over a ruthenium catalyst [8]. The overall reaction is



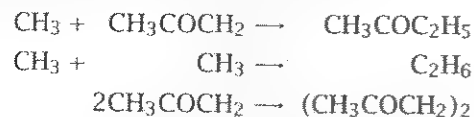
and one possible mechanism is



in which subscripts g and s refer to gas phase and adsorbed species, respectively, and S refers to a vacant ruthenium surface site. During the overall reaction, the reagents adsorb (elementary Reactions 5.14 and 5.17), and the products form at the surface and desorb (elementary Reactions 5.16 and 5.21). Adsorbed CO (CO<sub>s</sub>) either occupies surface sites or dissociates to adsorbed carbon and oxygen in elementary Reaction 5.15. The adsorbed carbon undergoes a sequential hydrogenation to methyne, methylene, and methyl, all of which are adsorbed on the surface. Hydrogenation of methyl leads to methane in elementary Reaction 5.21. In this example, the overall reaction is twice the fourth reaction added to the remaining reactions.

In each of these examples, the elementary reactions describe the detailed pathway between reactants and products. Numerous factors must be considered in proposing, developing and verifying a mechanism and a complete discussion of these factors is beyond the scope of this text. Keep in mind that the elementary reactions must be possible chemically, that is to say the reactants could form the products (and the products could form the reactants), and the elementary reactions should be kinetically significant, that is to say the reaction contributes to the appearance of intermediates or products. For example, during

acetone pyrolysis several radicals are produced and these could recombine in a variety of processes that are chemically possible such as



However, only Reaction 5.6 is listed in the mechanism presented above because the other two radical recombination reactions occur at rates too small to be significant. Methyl recombination was ruled out because the methyl concentration is anticipated to be much lower than the concentration of  $\text{CH}_3\text{COCH}_2$ ; therefore the probability of methyl recombining is much lower than the methyl being consumed in the other steps. Furthermore, the recombination of  $\text{CH}_3\text{COCH}_2$  can be discounted because the chemical instability of the bond formed in the recombination product  $(\text{CH}_3\text{COCH}_2)_2$  would cause this product to decompose rapidly back into the radical  $\text{CH}_3\text{COCH}_2$ . Laidler [21] and Hill [14] provide a good discussion of some of these kinetic arguments and present additional examples of complex reactions.

One criterion for a reaction to be elementary is that as the reactants transform into the products they do so without forming intermediate species that are chemically identifiable. A second aspect of an elementary reaction is that the reverse reaction also must be possible on energy, symmetry and steric bases, using only the products of the elementary reaction. This reversible nature of elementary reactions is the essence of Tolman's principle of microscopic reversibility [37, p. 699].

This assumption (at equilibrium the number of molecules going in unit time from state 1 to state 2 is equal to the number going in the reverse direction) should be recognized as a distinct postulate and might be called the principle of microscopic reversibility. In the case of a system in thermodynamic equilibrium, the principle requires not only that the total number of molecules leaving a given quantum state in unit time is equal to the number arriving in that state in unit time, but also that the number leaving by any one particular path, is equal to the number arriving by the reverse of that particular path.

Various models or theories have been postulated to describe the rate of an elementary reaction. Transition state theory (TST) is reviewed

briefly in the next section to describe the flow of systems (reactants  $\rightarrow$  products) over potential-energy surfaces. Using Tolman's principle, the most probable path in one direction must be the most probable path in the opposite direction. Furthermore, the Gibbs energy difference in the two directions must be equal to the overall Gibbs energy difference — the ratio of rate constants for an elementary reaction must be equal to the equilibrium constant for the elementary reaction.

### 5.3 Elementary Reaction Kinetics

In this section we outline the transition-state theory (TST), which can be used to predict the rate of an elementary reaction. Our purpose is to show how the rate of an elementary reaction is related to the concentration of reactants. The result is

$$r_i = k_i \prod_{j \in \mathcal{R}_i} c_j^{-\nu_{ij}} - k_{-i} \prod_{j \in \mathcal{P}_i} c_j^{\nu_{ij}} \quad (5.22)$$

in which  $j \in \mathcal{R}_i$  and  $j \in \mathcal{P}_i$  represent the reactant species and the product species in the  $i$ th reaction, respectively.

Equation 5.22 forms the basis for predicting reaction rates and is applied to homogeneous and heterogeneous systems. Because of its wide use, the remainder of this section describes the concepts and assumptions that underlie Equation 5.22. Transition state theory is based on the principles of statistical mechanics and, for the purposes here, you need only an understanding of molecular partition functions at the level presented in undergraduate physical chemistry texts.

Before describing TST it is necessary to develop the concept of the reaction coordinate. We use a two-body reaction example to calculate and illustrate a potential energy surface from which a reaction coordinate diagram is constructed. Elementary reaction kinetics is a rich field [26, 21, 40] and the material outlined here is far from comprehensive — it is intended to provide a framework for developing a physical picture and mathematical model of a chemical reaction.

Molecules containing  $N$  atoms require  $3N$  Cartesian coordinates to specify the locations of all the nuclei. For convenience, we may use three coordinates to locate the center of mass of the molecule, and three more coordinates to orient a chosen axis passing through the molecule. Only two are needed to describe the orientation if the molecule is diatomic. The remaining  $3N - 6$  coordinates are used to specify the relative positions of the atoms. The potential energy  $V(r)$

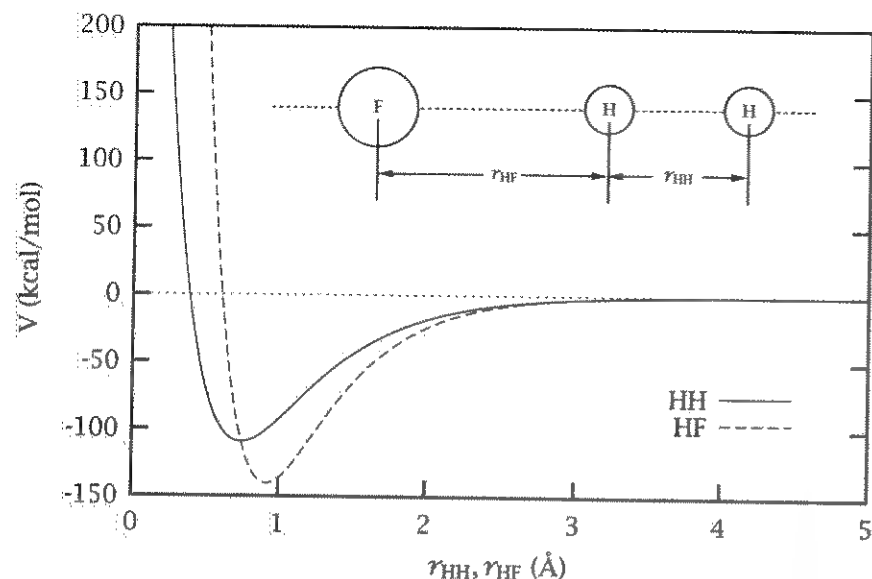


Figure 5.1: Morse potential for H<sub>2</sub> and HF.

of the molecule is related to the relative position of the atoms, and for a diatomic molecule the potential energy can be represented with a Morse function

$$V(r) = D \left[ e^{-2\beta r} - 2e^{-\beta r} \right]$$

in which  $D$  is the dissociation energy,  $r$  is the displacement from the equilibrium bond length, and  $\beta$  is related to the vibrational frequency of the molecule. Figure 5.1 presents the Morse functions for HF and H<sub>2</sub> molecules. The depth of the well is governed by the magnitude of  $D$  and the curvature at the bottom is governed by the vibrational frequency. The potential energy is a strong function of the distance of separation between the atoms.

If we increase the number of atoms in the molecule to three, we require three coordinates to describe the internuclear distances. We might choose the two distances from the center atom and the bond angle as the three coordinates. The potential energy is a function of all three coordinate values. A plot of the energy would be a three-dimensional surface, requiring four dimensions to plot the surface. If we fix the bond angle by choosing a collinear molecule, say, only the two distances relative to a central atom are required to describe

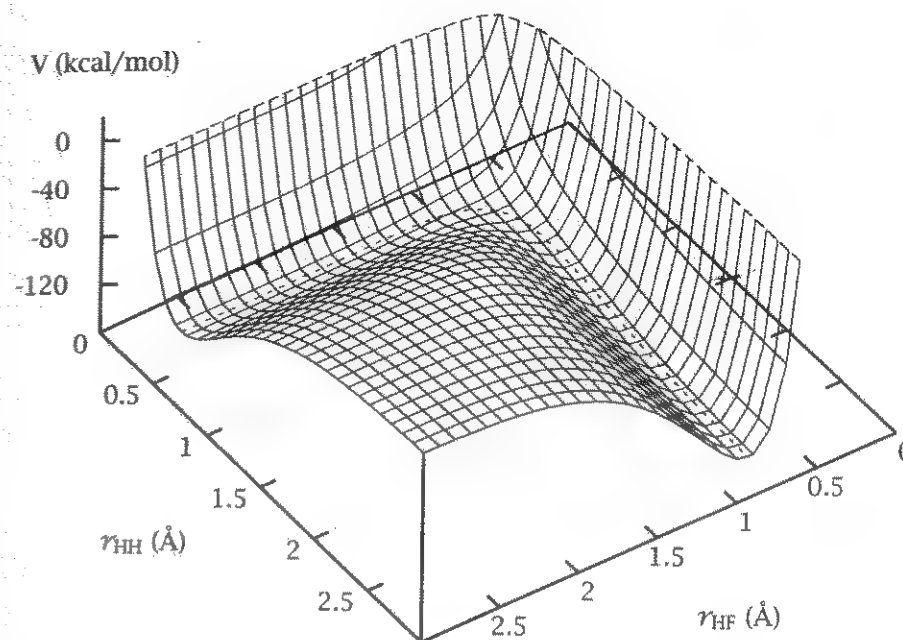


Figure 5.2: Potential-energy surface for the F, H<sub>2</sub> reaction.

the molecule. The potential energy can be expressed as a function of these two distances, and we can view this case as a three-dimensional plot. Figure 5.2 shows a representative view of such a surface for the collinear collision between F and H<sub>2</sub>.



Several sources provided the data used to calculate this potential-energy surface [30, 32, 31]. Moore and Pearson provide a more detailed discussion of computing potential-energy surfaces [26].

Figure 5.3 gives a contour representation of Figure 5.2. The contour lines are isoenergetic. Figures 5.2 and 5.3 illustrate, for a collinear collision between a diatomic molecule and an atom, the potential-energy surface consists of two valleys connected by a pass. A slice through the energy surface of Figure 5.2 at large values of  $r_{\text{HF}}$  — in which F atoms exist along with H<sub>2</sub> molecules — reproduces the curve for H<sub>2</sub> in Figure 5.1. A slice at large  $r_{\text{HH}}$  — in which H atoms exist along with HF molecules — reproduces the curve for HF in Figure 5.1. As the F atom is brought into contact with H<sub>2</sub> in a collinear fashion, a surface of energies

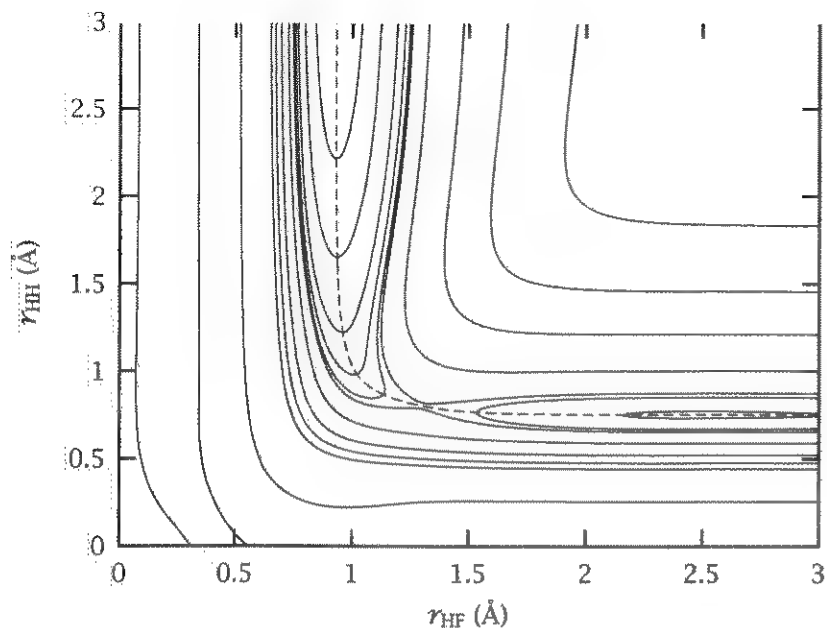


Figure 5.3: Contour representation of the potential-energy surface.

is possible that depends on the values of  $r_{\text{HF}}$  and  $r_{\text{HH}}$ . There is a minimum energy path along the valley of decreasing  $r_{\text{HF}}$  and constant  $r_{\text{HH}}$ . This path is shown by the dashed line in Figures 5.2 and 5.3. The reaction starts in the  $\text{H}_2$  valley at large  $r_{\text{HF}}$ , proceeds along the minimum energy path and ends in the  $\text{HF}$  valley at large  $r_{\text{HH}}$ . Because of the repulsive and attractive forces present during the collision, the maximum value of the energy along the minimum energy path is a saddle point. The dashed line is referred to as the reaction-coordinate pathway, and movement along this pathway is the reaction coordinate,  $\xi$ .

Figure 5.4 is a reaction-coordinate diagram, which displays the energy change during this reaction. The reaction coordinate represents travel along the minimum energy path from reactants to products. For this example, there is a one-to-one correspondence between the energy associated with a  $r_{\text{HF}}-r_{\text{HH}}$  coordinate pair in Figure 5.3 and the energy presented in Figure 5.4. The difference in energies between reactants and products is the heat of reaction; in this case it is exothermic by 34 kcal/mol. The barrier height of 4 kcal/mol between reactants and products is used to calculate the activation energy for the reaction.

Potential-energy surfaces such as Figures 5.2 and 5.3 cannot be

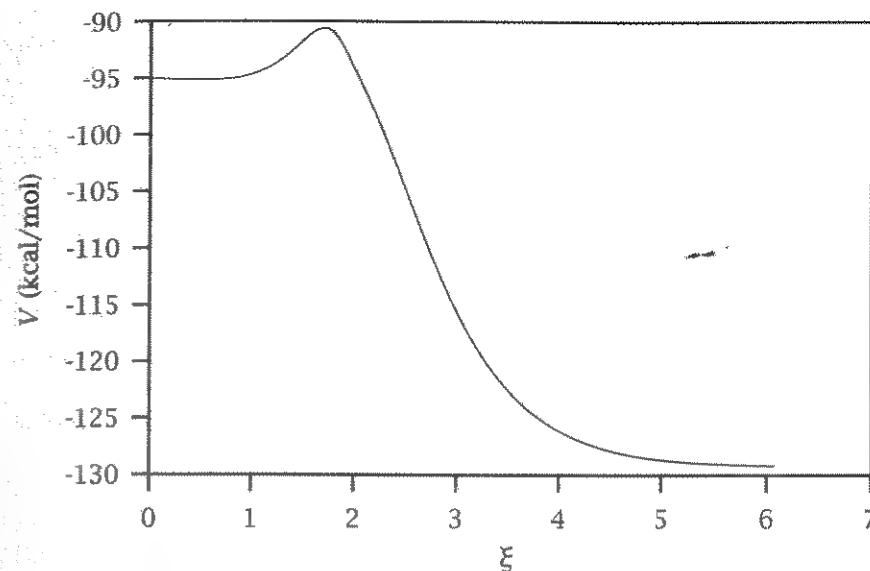


Figure 5.4: Reaction-coordinate diagram.

depicted for polyatomic systems with  $N > 3$  because more variables would be needed to describe all the relative positions of the nuclei. However, a reaction-coordinate diagram can always be constructed from a minimum energy path along a potential-energy surface. The saddle-point location defines the relative positions of all nuclei in the system, just as in Figure 5.3.

The principle of microscopic reversibility for this elementary reaction implies that the same structure at the saddle point must be realized if the reaction started at  $\text{HF} + \text{H}$  and went in reverse. The molecular structure (the relative positions of F, and two H's) at the saddle point is called the transition-state complex. This transition-state complex is not a chemically identifiable reaction intermediate; it is the arrangement of atoms at the point in energy space between reactants and products. Any change of relative positions of F and the two H's for the transition-state complex (as long as the motions are collinear) results in the complex reverting to reactants or proceeding to products.

Statistical mechanics uses the properties of individual molecules to describe the thermodynamic properties of the bulk system such as the energy, which is the average energy of all molecules in the system. This energy is found by summing over the product of all possible en-



ergies and the probability of finding each energy in the system. The individual probabilities are normalized with a quantity known as the partition function,  $Q$ . Therefore, the partition function is a normalization function and we use a microscopic description of the molecules in the system to calculate the partition function. All thermodynamic properties can be expressed in terms of the partition function, including the internal energy, entropy, Helmholtz energy, pressure, enthalpy and Gibbs energy. It is also possible to compute the chemical potential of a component in a mixture from the partition function. This ability to take a microscopic description of the molecules and determine macroscopic properties is exploited to calculate equilibrium constants for a reaction, and rate constants using TST.

The partition function of the system  $Q$  is related to the molecular partition function of the individual molecules in the system. In our development of rate constants we make use of the molecular partition functions. The molecular partition function per unit volume for an ideal gas is the product of the translational, rotational, vibrational and electronic energy states in the molecule

$$\left(\frac{q}{V}\right)_j = \left(\frac{q}{V}\right)_{\text{tran}} q_{\text{rot}} q_{\text{vib}} q_{\text{elec}}$$

Table 5.1 lists relationships needed to calculate the various terms of the molecular partition function. Further, the activity for any species  $j$  is

$$a_j = \left(\frac{q}{V}\right)_j \frac{k_B T}{f_j^\circ}$$

The molecular partition functions can be used to calculate the equilibrium constant for the reaction between F and H<sub>2</sub>. For this gas-phase reaction

$$K = \frac{a_{\text{HF}} a_{\text{H}}}{a_{\text{F}} a_{\text{H}_2}} = \frac{\left(\frac{q}{V}\right)_{\text{HF}} \left(\frac{q}{V}\right)_{\text{H}} f_{\text{F}}^\circ f_{\text{H}_2}^\circ}{\left(\frac{q}{V}\right)_{\text{F}} \left(\frac{q}{V}\right)_{\text{H}_2} f_{\text{HF}}^\circ f_{\text{H}}^\circ} \quad (5.23)$$

This example serves to illustrate how a microscopic description — the bond distances in HF and H<sub>2</sub>, the vibrational frequencies of HF and H<sub>2</sub>, the degeneracies of the electronic states of HF, H<sub>2</sub>, F and H, and the electronic energy levels of HF, H<sub>2</sub>, F and H — can be used to determine a thermodynamic property. This same microscopic description can equally well be used to calculate the enthalpy or any other thermodynamic property of the system.

The reaction-coordinate diagram illustrates the idea that, at some particular orientation of atoms, the reacting molecules (atoms) are in

Component	Units	Expression
translation	length <sup>-3</sup>	$\left(\frac{q}{V}\right)_{\text{tran}} = \frac{(2\pi m k_B T)^{3/2}}{h^3}$
rotation (linear)	—	$q_{\text{rot}} = \frac{8\pi^2 I k_B T}{\sigma h^2}$
rotation (nonlinear)	—	$q_{\text{rot}} = \frac{8\pi^2 8\pi^3 (I_A I_B I_C)^{1/2} (k_B T)^{3/2}}{\sigma h^3}$
vibration	—	$q_{\text{vib}} = \prod_i^{3N-6} \frac{\exp(-h\nu_i/2k_B T)}{1 - \exp(-h\nu_i/k_B T)}$
electronic	—	$q_{\text{elec}} = \sum_i^{\text{states}} g_{ei} \exp(-\epsilon_i/k_B T)$

Table 5.1: Molecular partition function terms.

a configuration that either goes forward to products or reverts back to reactants. TST is based on the assumption that the transition-state complex is in equilibrium with the reactants forming it. We use X to designate the transition-state complex, A to denote one reactant, and B to denote the other reactant



Using the principles of statistical mechanics we can write for this gas-phase reaction

$$K = \frac{a_X}{a_A a_B} = \frac{\left(\frac{q}{V}\right)_X}{\left(\frac{q}{V}\right)_A \left(\frac{q}{V}\right)_B} \frac{f_A^\circ f_B^\circ}{f_X^\circ} \frac{1}{k_B T} \quad (5.24)$$

The concentration of the transition-state complex is required to express the rate as a function of concentration. If activity is defined as

$$a_j = \frac{f_j}{f_j^\circ} = \frac{\phi_j P_j}{f_j^\circ} \quad (5.25)$$

and the equation of state is,

$$P_j V = z k_B T n_j \quad (5.26)$$



then

$$\frac{a_X}{a_A a_B} = \frac{\phi_X}{\phi_A \phi_B} \frac{f_A^\circ f_B^\circ}{f_X^\circ} \frac{1}{z k_B T} \frac{c_X}{c_A c_B} \quad (5.27)$$

Combining Equations 5.24 and 5.27 and solving for  $c_X$  gives

$$c_X = \frac{\left(\frac{q}{V}\right)_X}{\left(\frac{q}{V}\right)_A \left(\frac{q}{V}\right)_B} \frac{\phi_A \phi_B z}{\phi_X} c_A c_B \quad (5.28)$$

TST invokes the idea that the rate in the forward direction ( $\text{H}_2 + \text{F} \rightarrow \text{HF} + \text{H}$ ) is equal to the number of transition-state complexes moving to the right of on the reaction coordinate diagram. For the example worked here, the rate has units of molecules/time·volume. Since movement along the reaction coordinate describes molecules (or atoms) coming together and bonds rearranging, TST assumes that the complex either moves to the right or to the left within the frequency of the single vibration (molecular motion) that best represents the bond to be made (or broken). We refer to this special frequency as  $\nu^*$ , and we can write

$$r = \nu^* c_X = \nu^* \frac{\left(\frac{q}{V}\right)_X}{\left(\frac{q}{V}\right)_A \left(\frac{q}{V}\right)_B} \frac{\phi_A \phi_B z}{\phi_X} c_A c_B \quad (5.29)$$

The vibrational partition function is

$$q_{\text{vib}} = \prod_i^{3N-6} \frac{\exp(-h\nu_i/2k_B T)}{1 - \exp(-h\nu_i/k_B T)}$$

in which  $3N - 6$  vibrational modes are needed to describe a polyatomic, nonlinear molecule comprised of  $N$  atoms ( $3N - 5$  modes describe a linear molecule). In the limit of a very weak vibration (small  $\nu_i$ ), the quantity

$$\frac{\exp(-h\nu_i/2k_B T)}{1 - \exp(-h\nu_i/k_B T)} \rightarrow \frac{k_B T}{h\nu_i} \quad (5.30)$$

This approximation is valid for  $h\nu_i/k_B T \leq 0.1$ . This limit is sometimes referred to as the classical high-temperature limit. The argument  $h\nu/k_B T$  equals 0.1 at a temperature of 28,800 K for a stretching frequency of  $2000 \text{ cm}^{-1}$ , and it equals 0.1 at a stretching frequency of  $20.8 \text{ cm}^{-1}$  for a temperature of 300 K. A frequency of  $20.8 \text{ cm}^{-1}$  is reasonable for a rocking motion or a hindered rotation about a bond.

At this point, we assume a weak bond is made (or broken) as the transition-state complex transforms to products. We then rewrite the molecular partition function for the transition-state complex as

$$\left(\frac{q}{V}\right)_X = \left(\frac{q}{V}\right)_{\text{tran}} q_{\text{rot}} q_{\text{elec}} q_{\text{vib}}^* \frac{\exp(-h\nu^*/2k_B T)}{1 - \exp(-h\nu^*/k_B T)} \quad (5.31)$$

in which  $q_{\text{vib}}^*$  now has  $3N - 7$  (or  $3N - 6$  linear) vibrational modes remaining because one was used in the argument containing  $\nu^*$ . At the classical high-temperature limit (shown in Equation 5.30) Equation 5.31 becomes

$$\left(\frac{q}{V}\right)_X^* = \left(\frac{q}{V}\right)_{\text{tran}} q_{\text{rot}} q_{\text{elec}} q_{\text{vib}}^* \frac{k_B T}{h\nu^*}$$

Combining this result with Equation 5.29 gives

$$r = \frac{k_B T}{h} \frac{\left(\frac{q}{V}\right)_X^*}{\left(\frac{q}{V}\right)_A \left(\frac{q}{V}\right)_B} \frac{\phi_A \phi_B z}{\phi_X} c_A c_B \quad (5.32)$$

The rate constant contains all the terms on the right-hand side of Equation 5.32 except for the concentration of reactants,

$$k = \frac{k_B T}{h} \frac{\left(\frac{q}{V}\right)_X^*}{\left(\frac{q}{V}\right)_A \left(\frac{q}{V}\right)_B} \frac{\phi_A \phi_B z}{\phi_X} \quad (5.33)$$

which leads to

$$r = k c_A c_B \quad (5.34)$$

Equation 5.33 can be rewritten as

$$k = k^\circ f(c_j) \quad (5.35)$$

in which

$$k^\circ = \frac{k_B T}{h} \frac{\left(\frac{q}{V}\right)_X^*}{\left(\frac{q}{V}\right)_A \left(\frac{q}{V}\right)_B} \quad (5.36)$$

represents all the terms that are composition independent. For a reaction involving ideal gases,  $f(c_j) = 1$ .

Most rate expressions use the form of Equation 5.34. The variable  $k$  is the rate constant and many texts consider this to be a composition-independent term. The rate "constant" is not strictly constant. It changes with temperature and even composition. The composition dependence is absent for ideal gases and it is reasonable to neglect

compositional dependence of  $k$  in many situations. The temperature dependence is rigorously found by accounting for the effect of temperature on each term in Equation 5.33. The Arrhenius rate expression is one very common way of representing the temperature dependence.

$$k = k_0 \exp(-E_a/k_B T) \quad (5.37)$$

where  $k_0$  is temperature independent. Often  $k_0$  varies weakly with temperature

$$k_0 = AT^B$$

with  $-2 \leq B \leq 1$ . The Arrhenius expression is accurate provided  $E_a \gg Bk_B T$ .

With Equation 5.32 and a potential-energy surface, it is possible, in principle, to calculate the rate of any elementary reaction. This is a useful result because, as computational chemistry develops better empirical methods for determining the potential-energy surface, it becomes possible to predict reaction rates from first principles and compare them against experimental information or possibly avoid experimental determination of the reaction rate.

TST teaches us that elementary reactions depend on the amount (concentration) of each reactant raised to the absolute value of its stoichiometric coefficient because of the equilibrium assumed between the transition-state complex and the reactants. This allows us to generalize the results illustrated above to all elementary reactions<sup>3</sup> and adopt the convention

$$r_i = k_i \prod_{j \in \mathcal{R}_i} c_j^{-\nu_{ij}} - k_{-i} \prod_{j \in \mathcal{P}_i} c_j^{\nu_{ij}} \quad (5.38)$$

Equation 5.38 cannot be set to zero to define the equilibrium concentration, nor is the ratio of the forward and reverse rate constants equal to the equilibrium constant for the elementary reaction. The equilibrium constant  $K$  is given by

$$K = \frac{k_i^\circ}{k_{-i}^\circ} \prod_j \left( \frac{k_B T}{f_j^\circ} \right)^{\nu_{ij}}$$

The equilibrium constant is equal to a ratio of activities. The composition at equilibrium is found by expressing the activities in terms of concentrations.

<sup>3</sup>There is an exception for unimolecular reactions in the low-pressure regime.

Note that  $k_{-i}$  may be small compared to  $k_i$ , in which case the reverse reaction is insignificant. The reversibility condition, although true in general, may not be important in a particular reaction. Recall also that the production rate of any species  $j$  is

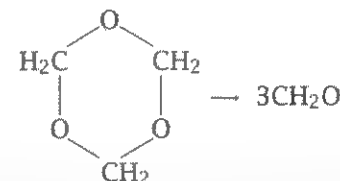
$$R_j = \sum_{i=1}^{n_r} \nu_{ij} r_i \quad (5.39)$$

Applying Equations 5.38 and 5.39 to  $\text{NO}_3$  in the photochemical smog example gives

$$R_{\text{NO}_3} = k_2 \text{CO} \text{CNO}_2 - k_{-2} \text{CNO}_3 - k_3 \text{CNO}_3 \text{CNO}_2 - k_4 \text{CNO}_3 \text{CNO}$$

### Example 5.1: Computing a rate constant for trioxane decomposition

Using TST, predict the value of the rate constant for the unimolecular decomposition of 1,3,5-trioxane at 750 K. The decomposition proceeds by the concerted rupture of three C—O bonds in the ring to form  $\text{CH}_2\text{O}$ .



The ring expands during the reaction. The vibrational frequencies and moments of inertia of the molecule and the transition complex are listed in Table 5.2, and the experimentally observed rate constant is [16].

$$k = k_0 e^{-E_a/RT} \quad k_0 = 10^{15.28 \pm 0.06} \text{ s}^{-1} \quad E_a = 47.5 \pm 2.4 \text{ kcal/mol}$$

### Solution

It is reasonable to assume the reacting mixture is ideal and  $k$  is given by Equation 5.36.

$$k = \frac{k_B T}{h} \frac{\left( \frac{q}{V} \right)_{\text{complex}}^*}{\left( \frac{q}{V} \right)_{\text{trioxane}}} \quad (5.40)$$

Because the masses of the transition-state complex and the trioxane are equal,  $\left( \frac{q}{V} \right)_{\text{tran}}$  is the same for the complex and trioxane molecule.

Parameter	Value
frequencies, $\text{cm}^{-1}$	
trioxane	296, 296, 524, 524, 945, 945, 1070, 1070, 1178, 1178, 1305, 1305, 1410, 1410, 1481, 1481, 2850, 2850, 3025, 3025, 1122, 1242, 1383, 466, 752, 978, 1235, 1495, 2850, 3025
transition complex	100, 100, 200, 200, 945, 945, 1400, 1400, 1178, 1178, 1200, 1200, 1200, 1200, 1481, 1481, 2850, 2850, 3025, 3025, 1000, 1242, 1100, 200, 700, 1200, 1495, 2850, 3025
moments of inertia, $\text{amu}\cdot\text{\AA}^2$	
trioxane	96.4, 96.4, 173.0
transition complex	125.3, 120.5, 249.2
potential-energy barrier, kcal/mol	51.4

Table 5.2: Parameters for the trioxane reaction.

A ratio of the  $q_{\text{rot}}$  reduces to

$$\frac{(q_{\text{rot}})_{\text{complex}}}{(q_{\text{rot}})_{\text{oxirane}}} = \frac{(\sqrt{I_A I_B I_C})_{\text{complex}}}{(\sqrt{I_A I_B I_C})_{\text{oxirane}}}$$

$$\frac{(q_{\text{rot}})_{\text{complex}}}{(q_{\text{rot}})_{\text{oxirane}}} = \frac{\sqrt{(125.3)(120.5)(249.2)}}{\sqrt{(96.4)(96.4)(173.0)}} = 1.53$$

The vibrational terms use the frequency  $\nu$  in reciprocal time and we are given the frequency in wave numbers ( $\text{cm}^{-1}$ ). To use wave number units, the following is used in place of  $h/k_B T$

$$\frac{hc}{k_B T} = \frac{(6.626 \times 10^{-34} \text{ J s})(2.998 \times 10^{10} \text{ cm/s})}{(1.381 \times 10^{-23} \text{ J/K})(750 \text{ K})} = 1.92 \times 10^{-3} \text{ cm}$$

Using the vibrational frequencies listed in Table 5.2, the vibrational partition functions are

$$q_{\text{vib, complex}} = \prod_{i=1}^{29} \frac{\exp(-1/2(1.92 \times 10^{-3} \text{ cm})(\nu_i \text{ cm}^{-1}))}{1 - \exp(-(1.92 \times 10^{-3} \text{ cm})(\nu_i \text{ cm}^{-1}))} = 1.94 \times 10^{-13}$$

$$q_{\text{vib, oxirane}} = \prod_{i=1}^{30} \frac{\exp(-1/2(1.92 \times 10^{-3} \text{ cm})(\nu_i \text{ cm}^{-1}))}{1 - \exp(-(1.92 \times 10^{-3} \text{ cm})(\nu_i \text{ cm}^{-1}))} = 3.25 \times 10^{-16}$$

$$\frac{q_{\text{vib, complex}}}{q_{\text{vib, oxirane}}} = 596$$

The reaction frequency is

$$\frac{k_B T}{h} = \frac{(1.381 \times 10^{-23} \text{ J/K})(750 \text{ K})}{6.626 \times 10^{-34} \text{ J s}} = 1.56 \times 10^{13} \text{ s}^{-1}$$

Assuming degeneracies of unity the electronic terms are

$$q_{\text{elec, complex}} = (1) \exp\left(\frac{-51400 \text{ cal/mol}}{(1.987 \text{ cal/mol K})(750 \text{ K})}\right) = 1.05 \times 10^{-15}$$

$$q_{\text{elec, oxirane}} = 1$$

and the ratio of electronic terms is  $1.05 \times 10^{-15}$ .

Substituting all the individual terms into Equation 5.40 yields

$$k = (1.56 \times 10^{13} \text{ s}^{-1})(1.53)(596)(1.05 \times 10^{-15}) = 14.9 \text{ s}^{-1}$$

This value compares reasonably well to the experimentally determined value of  $27.4 \text{ s}^{-1}$  at this temperature. A factor of two is not a large disagreement if you consider the temperature dependence of the rate constant. Figure 5.5 compares the experimental and calculated values of the rate constant over the temperature range 700–800 K. □

## 5.4 Fast and Slow Time Scales

One of the characteristic features of many chemically reacting systems is the widely disparate time scales at which reactions occur. It is not unusual for complex reaction mechanisms to contain rate constants that differ from each other by several orders of magnitude. Moreover, the

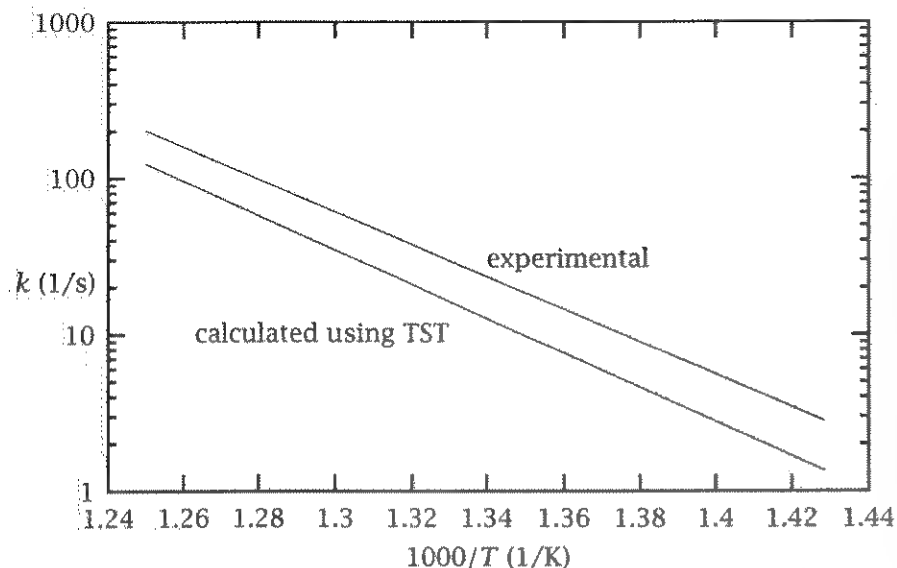


Figure 5.5: Comparison of measured and calculated rate constant versus temperature for oxirane decomposition.

concentrations of highly reactive intermediates may differ by orders of magnitude from the concentrations of relatively stable reactants and products. These widely different time and concentration scales present challenges for accurate estimation of rate constants, measurement of low-concentration species, and even numerical solution of complex models.

On the other hand, these disparate scales often allow us to approximate the complete mechanistic description with simpler rate expressions that retain the essential features of the full problem on the time scale or in the concentration range of interest. Although these approximations were often used in earlier days to allow easier model solution, that is not their primary purpose today. Most models, even stiff differential equation models with fairly disparate time scales, can be solved efficiently with modern ODE solvers. On the other hand, the physical insight provided by these approximations remains valuable. Moreover the reduction of complex mechanisms removes from consideration many parameters that would be difficult to estimate from available data. The next two sections describe two of the most widely

used methods of model simplification: the equilibrium assumption and the quasi-steady-state assumption.

#### 5.4.1 The Reaction Equilibrium Assumption

In the reaction equilibrium assumption, we reduce the full mechanism on the basis of fast and slow reactions. In a given mechanism consisting of multiple reactions, some reactions may be so much faster than others, that they equilibrate after any displacement from their equilibrium condition. The remaining, slower reactions then govern the rate at which the amounts of reactants and products change. If we take the extreme case in which all reactions except one are assumed at equilibrium, this remaining slow reaction is called the **rate-limiting step**. But the equilibrium approximation is more general and flexible than this one case. We may decompose a full set of reactions into two sets consisting of any number of slow and fast reactions, and make the equilibrium assumption on the set of fast reactions.

We illustrate the main features with the simple series reaction



Assume that the rate constants  $k_2, k_{-2}$  are much larger than the rate constants  $k_1, k_{-1}$ , so the second reaction equilibrates quickly. By contrast, the first reaction is slow and can remain far from equilibrium for a significant period of time. Because the only other reaction in the network is at equilibrium, the slow, first reaction is called the rate-limiting step. We show that the rate of this slow reaction does indeed determine or limit the rate at which the concentrations change.

It is perhaps clearest to start with the full model and demonstrate what happens if  $k_2, k_{-2} \gg k_1, k_{-1}$ . Consider the solution to the full model with rate constants  $k_1 = 1, k_{-1} = 0.5, k_2 = k_{-2} = 1$  and initial conditions  $c_A(0) = 1.0, c_B(0) = 0.4, c_C(0) = 0$  shown in Figure 5.6. Now consider what happens as  $k_2$  and  $k_{-2}$  become larger. Figure 5.7 shows the solution for  $k_2 = k_{-2} = 10$ . Notice the characteristic feature is that equilibrium is now quickly established between species B and C. The time scale for this equilibration is about 0.1 min. The remaining slow part of the system takes about 4 min to reach equilibrium. Figure 5.8 shows the rapid equilibration of  $c_B$  and  $c_C$  at small times for increasing  $k_2$  holding  $K_2 = k_2/k_{-2} = 1$ .

We now analyze the kinetic model. We can express the mole balances in terms of either the extents of the two reactions or the con-

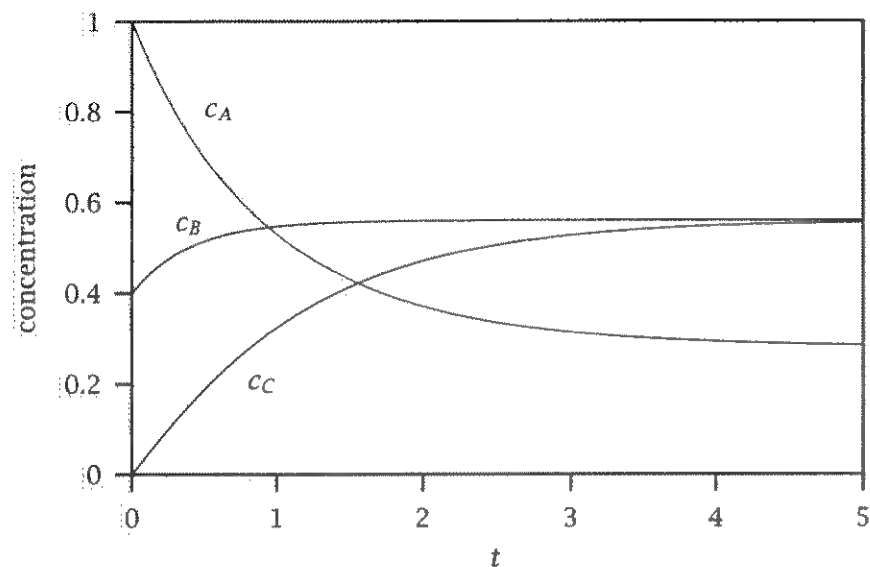


Figure 5.6: Full model solution for  $k_1 = 1, k_{-1} = 0.5, k_2 = k_{-2} = 1$ .

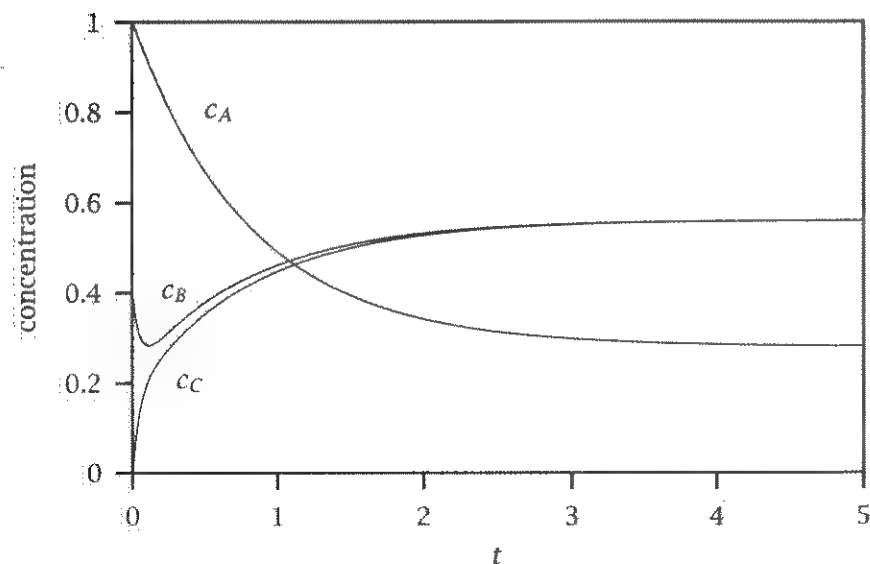


Figure 5.7: Full model solution for  $k_1 = 1, k_{-1} = 0.5, k_2 = k_{-2} = 10$ .

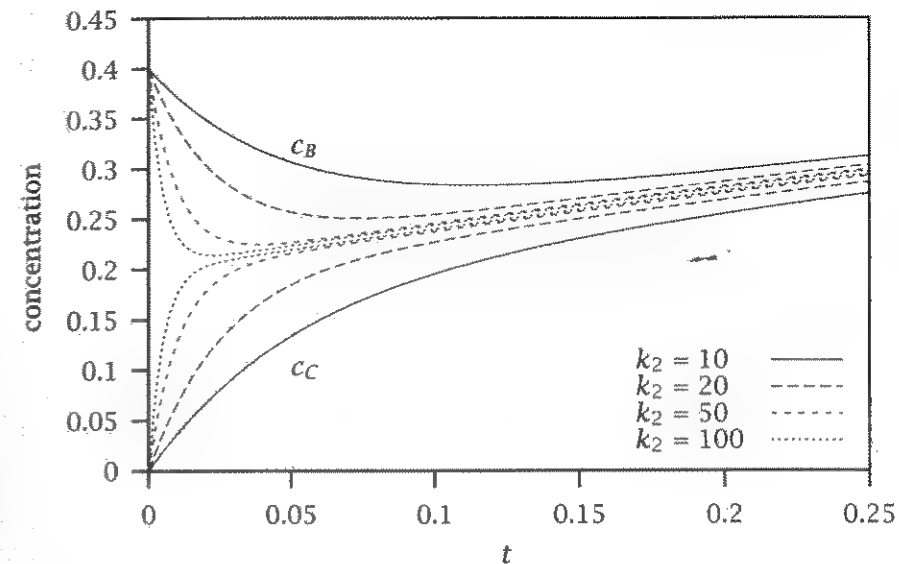


Figure 5.8: Concentrations of B and C versus time for full model with increasing  $k_2$  with  $K_2 = k_2/k_{-2} = 1$ .

concentrations of the three species. Both approaches are instructive so we examine both, but we start with the two reaction extents because there are only two independent reactions. The full model can be expressed as

$$\begin{aligned}\frac{d\varepsilon_1}{dt} &= r_1 = k_1 c_A - k_{-1} c_B, & \varepsilon_1(0) &= 0 \\ \frac{d\varepsilon_2}{dt} &= r_2 = k_2 c_B - k_{-2} c_C, & \varepsilon_2(0) &= 0\end{aligned}$$

in which, for the batch reactor,

$$c_A = c_{A0} - \varepsilon_1, \quad c_B = c_{B0} + \varepsilon_1 - \varepsilon_2, \quad c_C = c_{C0} + \varepsilon_2 \quad (5.42)$$

To describe the fast time-scale behavior displayed in Figure 5.8, let  $\tau$  denote a fast time variable via

$$\tau = k_{-2} t$$

so a small change in  $t$  is equivalent to a large change in  $\tau$ . If we divide

both full model equations by  $k_{-2}$  we produce

$$\frac{d\varepsilon_1}{d\tau} = \frac{1}{k_{-2}} (k_1 c_A - k_{-1} c_B), \quad \frac{d\varepsilon_2}{d\tau} = K_2 c_B - c_C$$

in which  $K_2 = k_2/k_{-2}$  is the equilibrium constant of the fast reaction. Now if we take the limit as  $k_{-2} \rightarrow \infty$ , these equations reduce to

$$\frac{d\varepsilon_1}{d\tau} = 0, \quad \frac{d\varepsilon_2}{d\tau} = K_2 c_B - c_C$$

which shows the extent of the slow reaction does not change,  $\varepsilon_1(\tau) = 0$ , over the small time during which the fast reaction equilibrates. We compute the steady state achieved by  $\varepsilon_2$  on this fast time scale by substituting Equations 5.42 into the right-hand side of the differential equation for  $\varepsilon_2$ , and setting the equation to zero, which yields

$$\varepsilon_{1s} = 0, \quad \varepsilon_{2s} = \frac{K_2}{1 + K_2} c_{B0} - \frac{1}{1 + K_2} c_{C0}$$

Given this steady state achieved on the fast time scale, we can compute the relaxation on the slow time scale. On the slow time scale, we wish to impose the equilibrium condition  $K_2 c_B - c_C = 0$ . In terms of extents, this equation is equivalent to

$$\varepsilon_2 = \frac{K_2}{1 + K_2} \varepsilon_1 + \frac{K_2}{1 + K_2} c_{B0} - \frac{1}{1 + K_2} c_{C0} \quad (5.43)$$

This equilibrium relationship takes the place of the differential equation for  $\varepsilon_2$  on the slow time scale. Notice the time derivative of  $\varepsilon_2$  should not be set to zero on the long time scale because

$$\frac{d\varepsilon_2}{dt} = k_{-2} (K_2 c_B - c_C)$$

Although the term in parentheses approaches zero for a fast second reaction,  $k_{-2}$  approaches infinity, and the product does not go to zero. To determine a differential equation for  $\varepsilon_2$ , if one is desired, we instead differentiate the equilibrium condition, Equation 5.43, to obtain<sup>4</sup>

$$\frac{d\varepsilon_2}{dt} = \frac{K_2}{1 + K_2} \frac{d\varepsilon_1}{dt} = \frac{K_2}{1 + K_2} r_1$$

<sup>4</sup>Notice we again are choosing between a DAE and an ODE description of the model as in Section 4.5 and Exercise 4.19.

Equations	Fast Time Scale ( $\tau = k_{-2}t$ )	Slow Time Scale ( $t$ )
ODEs	$\frac{d\varepsilon_1}{d\tau} = 0$ $\frac{d\varepsilon_2}{d\tau} = K_2 c_B - c_C$ $\varepsilon_1(0) = 0$ $\varepsilon_2(0) = 0$	$\frac{d\varepsilon_1}{dt} = r_1$ $\frac{d\varepsilon_2}{dt} = \frac{K_2}{1 + K_2} r_1$ $\varepsilon_1(0) = 0$ $\varepsilon_2(0) = \varepsilon_{2s} = \frac{K_2 c_{B0} - c_{C0}}{1 + K_2}$
DAEs	$\varepsilon_1 = 0$ $\frac{d\varepsilon_2}{d\tau} = K_2 c_B - c_C$ $\varepsilon_2(0) = 0$	$\frac{d\varepsilon_1}{dt} = r_1$ $\varepsilon_2 = \frac{K_2}{1 + K_2} \varepsilon_1 + \frac{K_2 c_{B0} - c_{C0}}{1 + K_2}$ $\varepsilon_1(0) = 0$
	$c_A = c_{A0} - \varepsilon_1, \quad c_B = c_{B0} + \varepsilon_1 - \varepsilon_2, \quad c_C = c_{C0} + \varepsilon_2$	

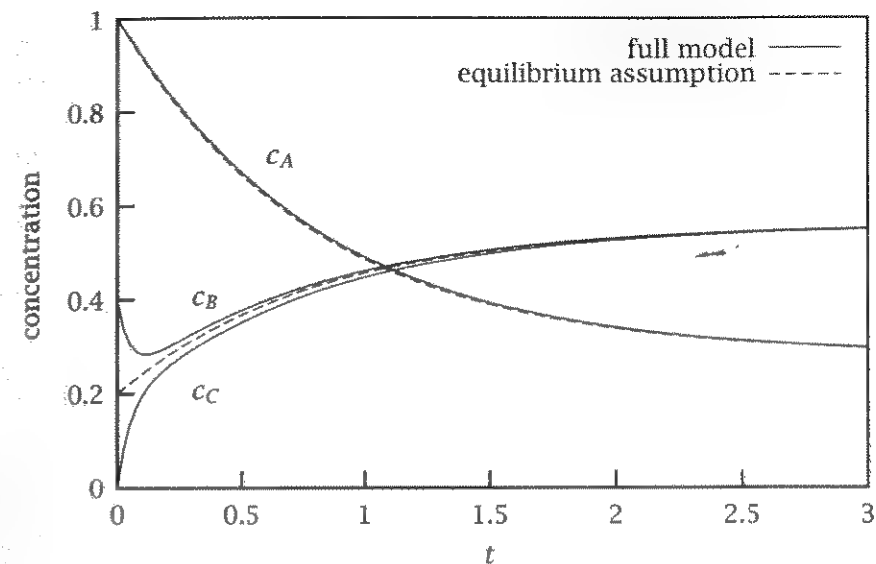
Table 5.3: Fast and slow time-scale models in extents of reactions.

Table 5.3 summarizes the results so far. A network with fast and slow reactions leads to fast and slow time-scale reduced models, shown in the left- and right-hand columns of Table 5.3, respectively. We can express the model either as a set of two ODEs or a DAE consisting of one ODE and one algebraic equation, shown in the top and bottom portions of the table. In either case, the dynamics of the slow time scale are determined only by  $r_1$ , verifying that the slow first reaction is the rate-limiting step. The steady state of the fast model becomes the initial condition for the slow model. If we solve the slow time-scale model, we produce the results in Figure 5.9. Notice the concentrations of all components are in excellent agreement after a short time. The big advantage of the reduced model is that values of  $k_2$  and  $k_{-2}$ , which are large and hard to estimate from data, are not required. In the reduced model, only their ratio,  $K_2$ , appears.

Extents of reaction are convenient variables for a batch reactor, but are inadequate to describe systems with flow terms such as the CSTR. We require species balances for reactors with flow terms, so we reexamine the equilibrium assumption in terms of the concentration variables. The mole balances for the full model are

Equations	Fast Time Scale ( $\tau = k_{-2}t$ )	Slow Time Scale ( $t$ )
ODEs	$\frac{dc_A}{d\tau} = 0,$ $\frac{dc_B}{d\tau} = -(K_2c_B - c_C),$ $\frac{dc_C}{d\tau} = K_2c_B - c_C,$	$\frac{dc_A}{dt} = -r_1,$ $\frac{dc_B}{dt} = \frac{1}{1+K_2}r_1,$ $\frac{dc_C}{dt} = \frac{K_2}{1+K_2}r_1,$
DAEs	$c_A = c_{A0}$ $\frac{dc_B}{d\tau} = -(K_2c_B - c_C),$ $0 = c_A - c_{A0} + c_B - c_{B0} + c_C - c_{C0}$	$\frac{dc_A}{dt} = -r_1,$ $0 = K_2c_B - c_C$ $0 = c_A - c_{A0} + c_B - c_{B0} + c_C - c_{C0}$
	$c_{B\tau} = \frac{1}{1+K_2}(c_{B0} + c_{C0}),$	$c_{Cs} = \frac{K_2}{1+K_2}(c_{B0} + c_{C0})$

Table 5.4: Fast and slow time-scale models in species concentrations.

Figure 5.9: Comparison of equilibrium assumption to full model for  $k_1 = 1, k_{-1} = 0.5, k_2 = k_{-2} = 10$ .

$$\frac{dc_A}{dt} = -r_1, \quad \frac{dc_B}{dt} = r_1 - r_2, \quad \frac{dc_C}{dt} = r_2$$

We know that only two of these differential equations are independent; we could add the three expressions to deduce the implied algebraic constraint, which could serve as a replacement for any one of the ODEs above

$$\frac{d(c_A + c_B + c_C)}{dt} = 0, \quad c_A + c_B + c_C = c_{A0} + c_{B0} + c_{C0} \quad (5.44)$$

The short time-scale model is similar to the case with reaction extents. Table 5.4 shows the short time-scale result in the left hand column. Deriving the correct slow time-scale model is less obvious and sometimes leads to confusion. The equilibrium assumption is made by adding the algebraic constraint that  $r_2 = 0$  or  $K_2c_B - c_C = 0$ . This algebraic equation coupled with Equation 5.44 and the mole balance for component A, in which the second reaction does not appear, then constitute a complete set of equations for the slow time scale, shown in the lower, right portion of Table 5.4. If we prefer differential equations, we can add the



mole balances for components B and C to eliminate  $r_2$

$$\frac{d(c_B + c_C)}{dt} = r_1 \quad (5.45)$$

Notice we have *not* set  $r_2 = 0$  to obtain this result. Equation 5.45 involves no approximation. If we differentiate the equilibrium condition, we obtain a second relation,  $K_2 dc_B/dt - dc_C/dt = 0$ , which allows us to solve for both  $dc_B/dt$  and  $dc_C/dt$ ,

$$\frac{dc_B}{dt} = \frac{1}{1 + K_2} r_1, \quad \frac{dc_C}{dt} = \frac{K_2}{1 + K_2} r_1$$

This result also is listed in Table 5.4. Notice that this result is not derivable by setting  $r_2 = 0$  in the full model. We see immediately from the slow time-scale differential equations that all concentrations are driven by  $r_1$ , again showing the first reaction is the rate-limiting step.

Normally one is interested in the solution to only the slow time-scale model and ignores the small errors at early times. However, one can also knit the slow and fast time-scale models together and obtain a more accurate reduced model valid at all times. This knitting procedure is known as matching the inner (fast) and outer (slow) solutions in the mathematical theory of singular perturbations [29, 23, 35]. The inverse of the large rate constant plays the role of the perturbation parameter in this theory. The inner (fast) solution is valid in a boundary layer (small time) that decreases in size as the perturbation parameter goes to zero ( $k_{-2}$  goes to infinity) as shown in Figure 5.8. The slow and fast time-scale models presented in Tables 5.3 and 5.4 are zero-order terms in a full perturbation series solution. Higher-order corrections also can be computed as shown by O'Malley [29], for example. Although this series solution may be of interest to gain analytical insight into nonlinear kinetic models, the higher-order terms depend on the values of  $k_2$  and  $k_{-2}$ . In other words, the higher-order corrections require as much information as the solution of the full model. It is usually simpler to compute numerically the solution to the full model if these rate constants are known. Kumar, Christofides and Daoutidis provide further discussion of the application of the singular perturbation method, and also discuss the resulting control problems arising with chemical reactors having both fast and slow time-scale kinetics [20].

Another interesting extension is the case of larger sets of reactions. The equilibrium assumption for larger sets of reactions is elegantly handled by finding the null space of the stoichiometric matrix of the

fast reactions. Ramkrishna and coworkers [11] provide a more complete discussion of this approach.

Finally, if the fast reactions are irreversible, further simplification is possible. In the simple series reactions, let the second reaction be fast and irreversible in Equation 5.41. We start with the slow time-scale model given in Table 5.4 and take the limit as  $K_2 \rightarrow \infty$  giving

$$\begin{aligned} \frac{dc_A}{dt} &= -r_1, & c_A(0) &= c_{A0} \\ \frac{dc_B}{dt} &= 0, & c_B(0) &= 0 \\ \frac{dc_C}{dt} &= r_1, & c_C(0) &= c_{B0} + c_{C0} \end{aligned}$$

In this limit the B disappears,  $c_B(t) = 0$ , and  $r_1 = k_1 c_A$ , which describes the irreversible reaction of A directly to C with rate constant  $k_1$



We see in the next section that this case is well described also by making the quasi-steady-state assumption on species B.

If we take the second reaction as irreversible in the backward direction,  $K_2 = 0$ , we obtain from Table 5.4

$$\begin{aligned} \frac{dc_A}{dt} &= -r_1, & c_A(0) &= c_{A0} \\ \frac{dc_B}{dt} &= r_1, & c_B(0) &= c_{B0} + c_{C0} \\ \frac{dc_C}{dt} &= 0, & c_C(0) &= 0 \end{aligned}$$

which describes the reversible reaction between A and B with no second reaction,  $c_C(t) = 0$ ,



Under the equilibrium assumption, we see that the two series reactions in Equation 5.41 may reduce to A going directly and irreversibly to C, Equation 5.46, or A going reversibly to B, Equation 5.47, depending on the magnitude of the equilibrium constant of the second, fast reaction.

### 5.4.2 The Quasi-Steady-State Assumption

In the quasi-steady-state assumption, we reduce the full mechanism on the basis of rapidly equilibrating *species* rather than reactions as in the reaction equilibrium assumption. We have seen in Section 5.2 that reaction networks can involve the formation and consumption of intermediate species. In some cases the intermediates are transitory, highly reactive species that are chemically identifiable but unlikely to exist outside the reaction mixture. Examples of these transitory species include atoms, radicals, ions and molecules in excited states. The examples in Section 5.2 contained radicals ( $\text{CH}_3$ ,  $\text{CH}_3\text{CO}$ ,  $\text{CH}_3\text{COCH}_2$  and  $\text{NO}_3$ ) and an atom (O). When the transitory species (reaction intermediates) have certain kinetic properties that we discuss next, their net rate of formation can be set equal to zero, which enables their concentration to be determined in terms of reactants and products using algebraic equations. After solving for these small concentration species, one is then able to construct reaction-rate expressions for the stable reactants and stable products in terms of reactant and product concentrations only.

The idea to set the production rate of a reaction intermediate equal to zero has its origins in the early 1900s [2, 4] and has been termed the Bodenstein-steady-state, pseudo-steady-state, or quasi-steady-state assumption. We use the term quasi-steady-state assumption (QSSA). The intermediate species are referred to as QSSA species [38]. The conditions are: the error in the concentration of the important species (reactants and products) calculated with the QSSA and without the QSSA (the exact solution) be small, and the time domain over which the QSSA is applied be selected to ensure the calculated error is minimized. A number of other qualifying conditions have been applied, which may be valid in certain circumstances, but as pointed out by Turányi et al. [38, p.172] are not universally applicable.

There have been several empirical observations or conclusions based on the investigation of small model reaction systems that showed that the rates of consuming reactions of QSSA species are unusually high, that the concentrations, and the net rates of reaction of QSSA species are unusually low, that the induction period is usually short, and that most QSSA species are radicals. These observations are simple consequences of the physical pictures presented above and the error formulas derived from them.

Having stated that certain reactive characteristics (high rates of con-

sumption, short lifetimes, short induction times, low concentrations, radicals, etc.) do not necessarily make a good QSSA species, we still use these characteristics to identify prospective QSSA species, but must then test to determine if these reaction intermediates are QSSA species [38, 26]. The key issue is how well the approximate solution that invokes the QSSA describes the exact solution. A detailed discussion of the error analysis is given by Turányi et al. [38]. Here we illustrate the features of the QSSA and its application to a QSSA species.

For a spatially homogeneous reaction system, such as a constant-volume batch reactor, we write the following differential equation for each component

$$\frac{dc_j}{dt} = R_j = \sum_{i=1}^{n_r} \nu_{ij} r_i = f(T, c_j) \quad (5.48)$$

A system composed of  $n_s$  species leads to  $n_s$  differential equations, and the exact solution is found by solving Equations 5.48. The QSSA is applied to  $k$  QSSA species, where  $k < n_s$ . This leads to the following set of algebraic equations for the QSSA species

$$\frac{dc_{js}}{dt} = 0 = \sum_{i=1}^{n_r} \nu_{ij} r_i = g(T, c_s) \quad (5.49)$$

The system of differential-algebraic equations is solved subject to the same initial conditions. We show in the next section how the solution of Equations 5.49 leads to reaction-rate expressions in terms of only the non-QSSA species concentrations, which is the goal. Given today's computing capability, not much has been gained by reducing the set of ODEs required for the model solution. If one knows all the  $k_i$  values, it is just as easy to simulate the entire set of ODEs as it is to solve the reduced set of ODEs. If experimental data are being fit to a kinetic model, however, it is advantageous to reduce the number of estimated kinetic parameters. A valuable feature of the QSSA is that it eliminates hard-to-estimate rate constants from the model as demonstrated in Example 5.5.

To illustrate the QSSA we consider two simple elementary reaction schemes. Scheme I is given by



and Scheme II is given by



Let the initial concentration in a constant-volume, isothermal, batch reactor be  $c_A = c_{A0}$ ,  $c_B = c_C = 0$ . The exact solution to the set of Equations 5.48 for Scheme I is

$$c_A = c_{A0}e^{-k_1t} \quad (5.50)$$

$$c_B = c_{A0} \frac{k_1}{k_2 - k_1} (e^{-k_1t} - e^{-k_2t}) \quad (5.51)$$

$$c_C = c_{A0} \frac{1}{k_2 - k_1} (k_2(1 - e^{-k_1t}) - k_1(1 - e^{-k_2t})) \quad (5.52)$$

The exact solution to the set of Equations 5.48 for Scheme II is<sup>5</sup>

$$c_A = c_{A0} \left( \frac{k_1(\alpha - k_2)}{\alpha(\alpha - \beta)} e^{-\alpha t} + \frac{k_1(k_2 - \beta)}{\beta(\alpha - \beta)} e^{-\beta t} \right) \quad (5.53)$$

$$c_B = c_{A0} \left( \frac{-k_1}{\alpha - \beta} e^{-\alpha t} + \frac{k_1}{\alpha - \beta} e^{-\beta t} \right) \quad (5.54)$$

$$c_C = c_{A0} \left( \frac{k_1 k_2}{\alpha \beta} + \frac{k_1 k_2}{\alpha(\alpha - \beta)} e^{-\alpha t} - \frac{k_1 k_2}{\beta(\alpha - \beta)} e^{-\beta t} \right) \quad (5.55)$$

$$\alpha = \frac{1}{2} \left( k_1 + k_{-1} + k_2 + \sqrt{(k_1 + k_{-1} + k_2)^2 - 4k_1 k_2} \right) \quad (5.56)$$

$$\beta = \frac{1}{2} \left( k_1 + k_{-1} + k_2 - \sqrt{(k_1 + k_{-1} + k_2)^2 - 4k_1 k_2} \right) \quad (5.57)$$

We next examine the effect of increasing  $k_2$  on the concentration of C. Figure 5.10 shows the normalized concentration of C given in Equation 5.52 versus time for different values of  $k_2/k_1$ . The curve  $c_C/c_{A0} = (1 - e^{-k_1t})$  is shown also (the condition where  $k_2/k_1 = \infty$ ). Figure 5.10 illustrates that as  $k_2 \gg k_1$ , the exact solution is equivalent to

$$c_C = c_{A0} (1 - e^{-k_1t}) \quad (5.58)$$

For  $k_2 \gg k_1$  the rate of decomposition of B is much greater than its rate of formation, which is a necessary condition for B to be a QSSA species.

If component B is defined to be a QSSA species, we write

$$\frac{dc_{Bs}}{dt} = 0 = k_1 c_{As} - k_2 c_{Bs}$$

which leads to

$$c_{Bs} = \frac{k_1}{k_2} c_{As} \quad (5.59)$$

<sup>5</sup>The Laplace transform is a good approach for solving these differential equations. Also see Exercise 4.6.

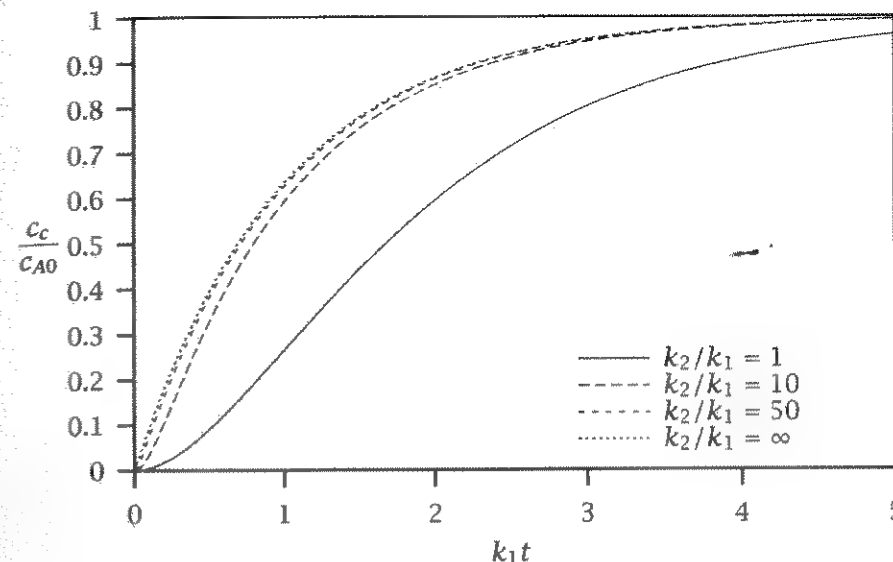


Figure 5.10: Normalized concentration of C versus dimensionless time for the series reaction  $A \rightarrow B \rightarrow C$  for different values of  $k_2/k_1$ .

for Scheme I. Note  $c_{Bs}$  is not constant; it changes linearly with the concentration of A. The issue is not if  $c_B$  is a constant, but whether or not the B equilibrates quickly to its quasi-steady value.

Substitution of Equation 5.59 into Equation 5.48 for components A and C results in

$$\frac{dc_{As}}{dt} = -k_1 c_{As} \quad (5.60)$$

$$\frac{dc_{Cs}}{dt} = k_1 c_{As} \quad (5.61)$$

for Scheme I. The solutions to Equations 5.60 and 5.61 are presented in Table 5.5 for the initial condition  $c_{As} = c_{A0}$ ,  $c_{Bs} = c_{Cs} = 0$ .

Similarly, for Scheme II, when component B is defined to be a QSSA species,

$$c_{Bs} = \frac{k_1}{k_{-1} + k_2} c_{As} \quad (5.78)$$

Substitution of Equation 5.78 into Equation 5.48 for components A and

Scheme I:	$A \xrightarrow{k_1} B \xrightarrow{k_2} C$		(5.62)
	Exact Solution	QSSA Solution	
	$C_A = C_{A0} e^{-k_1 t}$	$C_{As} = C_{A0} e^{-k_1 t}$	(5.66)
	$C_B = C_{A0} \frac{k_1}{k_2 - k_1} (e^{-k_1 t} - e^{-k_2 t})$	$C_{Bs} = C_{A0} \frac{k_1}{k_2} e^{-k_1 t}$	(5.67)
	$C_C = C_{A0} \frac{1}{k_2 - k_1} (k_2 (1 - e^{-k_1 t}) - k_1 (1 - e^{-k_2 t}))$	$C_{Cs} = C_{A0} (1 - e^{-k_1 t})$	(5.68)
Scheme II:	$A \xrightleftharpoons[k_{-1}]{k_1} B \xrightarrow{k_2} C$		(5.69)
	Exact Solution	QSSA Solution	
	$C_A = C_{A0} \left( \frac{k_1(\alpha - k_2)}{\alpha(\alpha - \beta)} e^{-\alpha t} + \frac{k_1(k_2 - \beta)}{\beta(\alpha - \beta)} e^{-\beta t} \right)$	$C_{As} = C_{A0} e^{-\frac{k_1 k_2}{k_{-1} + k_2} t}$	(5.75)
	$C_B = C_{A0} \left( \frac{-k_1}{\alpha - \beta} e^{-\alpha t} + \frac{k_1}{\alpha - \beta} e^{-\beta t} \right)$	$C_{Bs} = C_{A0} \frac{k_1}{k_{-1} + k_2} e^{-\frac{k_1 k_2}{k_{-1} + k_2} t}$	(5.76)
	$C_C = C_{A0} \left( \frac{k_1 k_2}{\alpha \beta} + \frac{k_1 k_2}{\alpha(\alpha - \beta)} e^{-\alpha t} - \frac{k_1 k_2}{\beta(\alpha - \beta)} e^{-\beta t} \right)$	$C_{Cs} = C_{A0} \left( 1 - e^{-\frac{k_1 k_2}{k_{-1} + k_2} t} \right)$	(5.77)
	$\alpha = \frac{1}{2} (k_1 + k_{-1} + k_2 + \sqrt{(k_1 + k_{-1} + k_2)^2 - 4k_1 k_2})$		(5.73)
	$\beta = \frac{1}{2} (k_1 + k_{-1} + k_2 - \sqrt{(k_1 + k_{-1} + k_2)^2 - 4k_1 k_2})$		(5.74)

Table 5.5: Exact and QSSA solutions for kinetic Schemes I and II.

## C results in

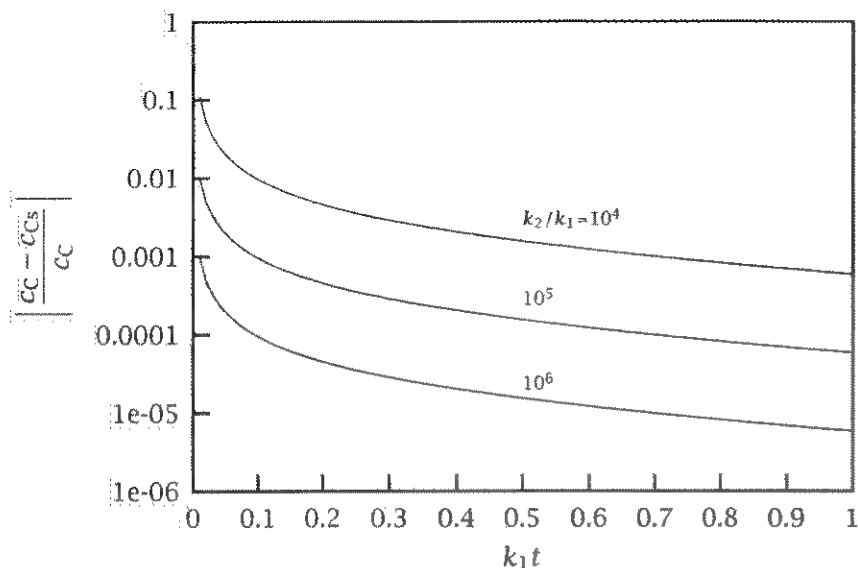
$$\frac{dc_{As}}{dt} = -\frac{k_1 k_2}{k_{-1} + k_2} c_{As} \quad (5.79)$$

$$\frac{dc_{Cs}}{dt} = \frac{k_1 k_2}{k_{-1} + k_2} c_{As} \quad (5.80)$$

The solutions to Equations 5.79 and 5.80 are presented in Table 5.5 for the initial condition  $C_{As} = C_{A0}$ ,  $C_{Bs} = C_{Cs} = 0$ .

For the QSSA to be accurate, the error in the predicted concentrations of the reactants and products must be acceptable and the induction time should be small. The results in Table 5.5 permit us to examine the error in the predicted concentration of an important species, the reaction product C, and the induction time to reach the quasi-steady state. Equation 5.65 is used in Figure 5.10 to show  $C_C/C_{A0}$  versus dimensionless time ( $k_1 t$ ) for different values of  $k_2/k_1$ . The dotted curve shows  $C_{Cs}/C_{A0}$  (Equation 5.68). The fractional error between the QSSA and exact values for  $C_C$  is shown in Figure 5.11 for much larger ratios of  $k_2/k_1$  and for shorter dimensionless times than are presented in Figure 5.10. Figure 5.11 illustrates several points. As the lifetime of the intermediate product decreases (the reciprocal of  $k_2$ ), the error decreases and the induction time (dimensionless because we plotted  $k_1 t$ ) to reach an acceptable error, such as  $10^{-2}$ , decreases. Turányi et al. [38] have shown that the induction period to reach the QSSA is likely to be several times the lifetime of the longest-lived QSSA species. Figure 5.11 illustrates this change in induction time with decreasing lifetime. The induction time should be such that the quasi-steady state is established rapidly (i.e., before a significant amount of reactant A has disappeared). An error of 1% is reached at  $k_1 t = 0.1$  for  $k_2/k_1 = 10^4$ , which corresponds to a conversion of about 10% for A. As  $k_2/k_1$  increases, the amount of A that reacts before an error of 1% is reached decreases as indicated in Figure 5.11.

If component B were a radical or atom, it could be treated as a QSSA species provided  $k_2/k_1$  were large enough because large  $k_2/k_1$  leads to small errors in the predicted concentration of product. In this simple example the net rate of production of the intermediate never reached zero over the values of  $k_1 t$  examined, demonstrating it is not necessary for the net rate of production of QSSA species to be zero. At sufficiently large values of  $k_2/k_1$  the intermediate B can be considered a QSSA species because the error in the actual and approximate concentration of the QSSA reaches an acceptable level.



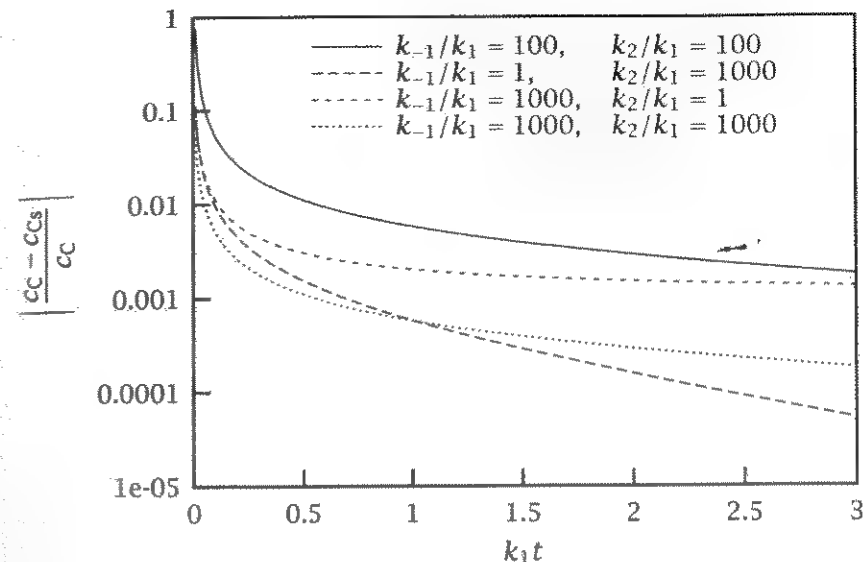
**Figure 5.11:** Fractional error in the QSSA concentration of C versus dimensionless time for the series reaction  $A \rightarrow B \rightarrow C$  for different values of  $k_2/k_1$ .

Similar concentration plots can be developed for the components in Scheme II to examine the effect of increasing the rate of decomposition of intermediate B relative to its rate of formation. Now both  $k_{-1}$  and  $k_2$  need to be varied. Figure 5.12 was generated using Equations 5.72 and 5.77 to illustrate the effect of decreasing the lifetime  $\tau$  of B, here  $\tau = (1/k_{-1} + 1/k_2)$ . As the lifetime decreases, the error in  $c_C$  reaches an acceptable level such as  $10^{-2}$  more rapidly. This effect can be seen by comparing the two curves for  $k_{-1} = k_2 = 100k_1$  and  $k_{-1} = k_2 = 1000k_1$ . Note also that the error at long times does depend on the relative values of  $k_{-1}$  and  $k_2$ , and there are subtle differences in the induction time for comparable  $\tau$ . Then provided  $\tau$  is sufficiently small and  $k_{-1}$  and/or  $k_2$  are sufficiently large, we would be justified in letting B be a QSSA species.

In summary, we write for a QSSA species

$$R_j = 0, \quad j = \text{QSSA species} \quad (5.81)$$

The QSSA was developed for a constant-volume batch reactor, but can



**Figure 5.12:** Fractional error in the QSSA concentration of C versus dimensionless time for the series-parallel reaction,  $A \rightarrow B \rightarrow C$ .

be applied equally well to a PFR. For this reason, it is best to define the QSSA based on the production rate of species  $j$ , rather than the time derivative of the concentration of species  $j$ .

The QSSA is a useful tool in reaction analysis. Material balances for batch and plug-flow reactors are ordinary differential equations. By applying Equation 5.81 to the components that are QSSA species, their material balances become algebraic equations. These algebraic equations can be used to simplify the reaction expressions and reduce the number of equations that must be solved simultaneously. In addition, appropriate use of the QSSA can eliminate the need to know several difficult-to-measure rate constants. The required information can be reduced to ratios of certain rate constants, which can be more easily estimated from data. In the next section we show how the QSSA is used to develop a rate expression for the production of a component from a statement of the elementary reactions, and illustrate the kinetic model simplification that results from the QSSA model reduction.

## 5.5 Developing Rate Expressions from Complex Mechanisms

In this section we develop rate expressions for the mechanistic schemes presented in Section 5.1. The goal is to determine the rate in terms of measurable quantities, such as the concentrations of the reactants and products, and the temperature. Heterogeneous problems are considered in Section 5.6. The approach taken for homogeneous and heterogeneous reactions is similar, but additional constraints are placed on heterogeneous problems so the synthesis reaction example is presented after discussing these constraints.

The following procedure enables one to develop a reaction-rate expression from a set of elementary reactions.

1. Identify the species that do not appear in equilibrium reactions and write a statement for the rate of production of these species using Equations 5.38 and 5.39.
2. If the rate of production statement developed in step 1 contains the concentration of reaction intermediates that are QSSA species, their concentrations can be found in terms of reactant and product concentrations by writing a necessary number of algebraic statements. These algebraic statements come from applying the QSSA to reaction intermediates. In some cases, either by assumption or because of kinetic insight, an elementary reaction is treated as being at equilibrium.
3. Perform the necessary algebraic manipulations and substitute the resulting intermediate concentrations into the rate of production statement from step 1.

### Example 5.2: Production rate of acetone

The thermal decomposition of acetone is represented by the following stoichiometry



Use the following mechanism to determine the production rate of acetone [33]. You may assume the methyl, acetyl and acetone radicals are

QSSA species.



### Solution

Let the species be designated as:

Species	Formula	Name
A <sub>1</sub>	CH <sub>3</sub> COCH <sub>3</sub>	acetone
A <sub>2</sub>	CH <sub>3</sub>	methyl radical
A <sub>3</sub>	CH <sub>3</sub> CO	acetyl radical
A <sub>4</sub>	CO	carbon monoxide
A <sub>5</sub>	CH <sub>3</sub> COCH <sub>2</sub>	acetone radical
A <sub>6</sub>	CH <sub>2</sub> CO	ketene
A <sub>7</sub>	CH <sub>4</sub>	methane
A <sub>8</sub>	CH <sub>3</sub> COC <sub>2</sub> H <sub>5</sub>	methyl ethyl ketone

Write the production rate of acetone using Reactions 5.82 and 5.84

$$R_1 = -k_1 c_1 - k_3 c_1 c_2 \quad (5.87)$$

Species 2, 3 and 5 are QSSA species. Apply the QSSA to each of these species

$$R_2 = 0 = k_1 c_1 + k_2 c_3 - k_3 c_1 c_2 + k_4 c_5 - k_5 c_2 c_5 \quad (5.88)$$

$$R_3 = 0 = k_1 c_1 - k_2 c_3 \quad (5.89)$$

$$R_5 = 0 = k_3 c_1 c_2 - k_4 c_5 - k_5 c_2 c_5 \quad (5.90)$$

From Equation 5.89

$$c_3 = \frac{k_1}{k_2} c_1 \quad (5.91)$$

Adding Equations 5.88, 5.89 and 5.90 gives

$$c_5 = \frac{k_1 c_1}{k_5 c_2} \quad (5.92)$$

Inserting the concentrations shown in Equations 5.91 and 5.92 into Equation 5.88 gives

$$k_3 c_2^2 - k_1 c_2 - \frac{k_1 k_4}{k_5} = 0$$

$$c_2 = \frac{k_1}{2k_3} + \sqrt{\frac{k_1^2}{4k_3^2} + \frac{k_1 k_4}{k_3 k_5}} \quad (5.93)$$

in which the positive sign is chosen to obtain a positive concentration. This result can be substituted into Equation 5.87 to give the rate in terms of measurable species.

$$R_1 = - \left( \frac{3}{2} k_1 + \sqrt{\frac{k_1^2}{4} + \frac{k_1 k_3 k_4}{k_5}} \right) c_1 \quad (5.94)$$

Equation 5.94 can be simplified to

$$R_1 = -k_{\text{eff}} c_1 \quad (5.95)$$

by defining an effective rate constant

$$k_{\text{eff}} = \frac{3}{2} k_1 + \sqrt{\frac{k_1^2}{4} + \frac{k_1 k_3 k_4}{k_5}}$$

This simplified form illustrates the rate is first order in the acetone concentration. If, based on kinetic theories, we knew all the individual rate constants, we could calculate the rate of acetone production. Alternately, we can use Equation 5.95 as the basis for designing experiments to determine if the rate of production is first order in acetone and to determine the magnitude of the first-order rate constant,  $k_{\text{eff}}$ . □

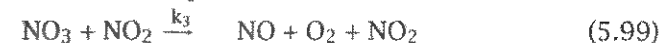
### Example 5.3: Production rate of oxygen

Given the following mechanism for the overall stoichiometry



derive an expression for the production rate of  $\text{O}_2$ . Assume atomic O and  $\text{NO}_3$  radicals are QSSA species. The production rate should be in

terms of the reactant,  $\text{NO}_2$ , and products, NO and  $\text{O}_2$ .



### Solution

Let the species be designated as:



The production rate of molecular oxygen is

$$R_5 = k_3 c_1 c_4 + k_5 c_1 c_3 \quad (5.102)$$

Species 3 and 4 are QSSA species. Applying the QSSA to these reaction intermediates gives

$$R_3 = 0 = k_1 c_1 - k_2 c_1 c_3 + k_{-2} c_4 - k_5 c_1 c_3 \quad (5.103)$$

$$R_4 = 0 = k_2 c_1 c_3 - k_{-2} c_4 - k_3 c_1 c_4 - k_4 c_2 c_4 \quad (5.104)$$

Adding Equations 5.103 and 5.104

$$0 = k_1 c_1 - k_3 c_1 c_4 - k_4 c_2 c_4 - k_5 c_1 c_3 \quad (5.105)$$

This result can be used to simplify the rate expression. Adding Equations 5.102 and 5.105

$$R_5 = k_1 c_1 - k_4 c_2 c_4 \quad (5.106)$$

The intermediate  $\text{NO}_3$  concentration is found by solving Equation 5.103 for  $c_3$ .

$$c_3 = \frac{k_1}{k_2 + k_5} + \frac{k_{-2}}{k_2 + k_5} \frac{c_4}{c_1} \quad (5.107)$$

Substituting Equation 5.107 into Equation 5.104 and rearranging gives

$$c_4 = \frac{k_1 k_2 c_1}{(k_2 k_3 + k_3 k_5) c_1 + (k_2 k_4 + k_4 k_5) c_2 + k_{-2} k_5} \quad (5.108)$$



The rate expression now can be found using Equations 5.106 and 5.108

$$R_5 = k_1 c_1 - \frac{k_1 k_2 k_4 c_1 c_2}{(k_2 k_3 + k_3 k_5) c_1 + (k_2 k_4 + k_4 k_5) c_2 + k_{-2} k_5} \quad (5.109)$$

Equation 5.109 is rather complex; under reaction conditions several terms in the denominator are expected to be kinetically unimportant leading to a simple power-law reaction-rate expression.  $\square$

#### Example 5.4: Free-radical polymerization kinetics

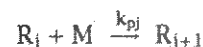
Polymers are economically important and many chemical engineers are involved with some aspect of polymer manufacturing during their careers. Polymerization reactions raise interesting kinetic issues because of the long chains that are produced. Consider free-radical polymerization reaction kinetics as an illustrative example. A simple polymerization mechanism is represented by the following set of elementary reactions.

##### Initiation:



##### Propagation:



$$\vdots$$


$$\vdots$$

##### Termination:



in which  $M$  is monomer,  $M_j$  is a dead polymer chain of length  $j$ , and  $R_j$  is a growing polymer chain of length  $j$ . In free-radical polymerizations, the initiation reaction generates the free radicals, which initiate the polymerization reactions. An example is the thermal dissociation

of benzoyl peroxide as initiator to form a benzyl radical that subsequently reacts with styrene monomer to form the first polymer chain. The termination reaction presented here is a termination of two growing polymer chains by a combination reaction.

Develop an expression for the rate of monomer consumption using the assumptions that  $k_{pj} = k_p$  for all  $j$ , and  $k_{tmn} = k_t$  is independent of the chain lengths of the two terminating chains. You may make the QSSA for all growing polymer radicals.

##### Solution

From the propagation reactions we can see the rate of monomer consumption is given by

$$R_M = - \sum_{j=1}^{\infty} r_{pj} = -k_p c_M \sum_{j=1}^{\infty} c_{Rj} \quad (5.110)$$

in which the rate of the  $j$ th propagation reaction is given by

$$r_{pj} = k_p c_M c_{Rj}$$

Making the QSSA for all polymer radicals, we set their production rates to zero

$$\begin{array}{rclcl} r_1 & - & k_p c_1 c_M & - & k_t c_{R1} \sum_{j=1}^{\infty} c_{Rj} & = & 0 \\ k_p c_1 c_M & - & k_p c_{R2} c_M & - & k_t c_{R2} \sum_{j=1}^{\infty} c_{Rj} & = & 0 \\ k_p c_{R2} c_M & - & k_p c_{R3} c_M & - & k_t c_{R3} \sum_{j=1}^{\infty} c_{Rj} & = & 0 \\ + & \vdots & \vdots & & \vdots & & \\ & r_i & & - & k_t \sum_{i=1}^{\infty} c_{Ri} \sum_{j=1}^{\infty} c_{Rj} & = & 0 \end{array} \quad (5.111)$$

The first term is the formation of growing chain  $i$  by propagation (initiation for  $i = 1$ ), the second term is loss of chain  $i$  by propagation with monomer, and the third term is loss of chain  $i$  by termination reaction with chains of all other lengths. Notice if we add these equations, propagation terms on each line cancel corresponding entries on the lines directly above and below. It is valid to perform the infinite sum in Equation 5.111 because the polymer concentration for long chains goes to zero,  $\lim_{j \rightarrow \infty} c_{pj} = 0$ . We solve Equation 5.111 for the total growing polymer concentration and obtain

$$\sum_{j=1}^{\infty} c_{Rj} = \sqrt{r_1 / k_t}$$

Substituting this result into Equation 5.110 yields the monomer consumption rate

$$R_M = -k_p c_M \sqrt{r_I / k_t}$$

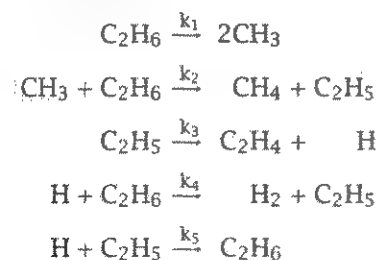
From the initiation reaction, the initiation rate is given by  $r_I = k_I c_I$ , which upon substitution gives the final monomer consumption rate

$$R_M = -k_p \sqrt{k_I / k_t} \sqrt{c_I} c_M \quad (5.112)$$

Notice that this result also provides a mechanistic justification for the production rate used in Example 4.3 in which monomer consumption rate was assumed linear in monomer concentration.  $\square$

### Example 5.5: Ethane pyrolysis

This example illustrates how to apply the QSSA to a flow reactor. We are interested in determining the effluent concentration from the reactor and in demonstrating the use of the QSSA to simplify the design calculations. Ethane pyrolysis to produce ethylene and hydrogen also generates methane as an unwanted reaction product. The overall stoichiometry for the process is not a simple balance of ethane and the products. The following mechanism and their kinetics have been proposed for ethane pyrolysis [22]



The rate constants are listed in Table 5.6 for the elementary reactions, in which  $k = A_0 \exp(-E/RT)$ . The preexponential factor  $A_0$  has units of  $\text{s}^{-1}$  or  $\text{cm}^3/\text{mol s}$  for first- and second-order reactions, respectively.

The pyrolysis is performed in a  $100 \text{ cm}^3$  isothermal PFR, operating at a constant pressure of 1.0 atm and at 925 K. The feed consists of ethane in a steam diluent. The inlet partial pressure of ethane is 50 torr and the partial pressure of steam is 710 torr. The feed enters at a flowrate of  $35 \text{ cm}^3/\text{s}$ . The exact solution of this problem uses the methods

Reaction	$A_0$	$E(\text{kJ/mol})$
1	$1.0 \times 10^{17}$	356
2	$2.0 \times 10^{11}$	44
3	$3.0 \times 10^{14}$	165
4	$3.4 \times 10^{12}$	28
5	$1.6 \times 10^{13}$	0

Table 5.6: Ethane pyrolysis kinetics.

developed in Chapter 4 and involves solving a set of eight initial-value ODEs of the form

$$\frac{dN_j}{dV} = R_j \quad (5.113)$$

subject to the initial conditions. In this numerical solution, each of the molar flows are computed as the volume increases from 0 to 100, with the concentrations of each component given by

$$c_j = \frac{N_j}{\sum_j N_j} \frac{P}{RT}$$

The results for  $\text{C}_2\text{H}_6$ ,  $\text{C}_2\text{H}_4$  and  $\text{CH}_4$  are plotted in Figure 5.13. The  $\text{H}_2$  concentration is not shown because it is almost equal to the  $\text{C}_2\text{H}_4$  concentration. Note the molar flowrate of  $\text{CH}_4$  is only 0.2% of the molar flowrate of the other products,  $\text{C}_2\text{H}_4$  and  $\text{H}_2$ . The concentrations of the radicals,  $\text{CH}_3$ ,  $\text{C}_2\text{H}_5$  and  $\text{H}$ , are on the order of  $10^{-6}$  times the ethylene concentration.

Assuming the radicals  $\text{CH}_3$ ,  $\text{C}_2\text{H}_5$  and  $\text{H}$  are QSSA species, develop an expression for the rate of ethylene formation. Verify that this approximation is valid.

### Solution

The rate of ethylene formation is

$$R_{\text{C}_2\text{H}_4} = k_3 c_{\text{C}_2\text{H}_5} \quad (5.114)$$

Next use the QSSA to relate  $\text{C}_2\text{H}_5$  to stable reactants and products

$$R_{\text{CH}_3} = 0 = 2k_1 c_{\text{C}_2\text{H}_6} - k_2 c_{\text{CH}_3} c_{\text{C}_2\text{H}_6} \quad (5.115)$$

$$R_{\text{H}} = 0 = k_3 c_{\text{C}_2\text{H}_5} - k_4 c_{\text{H}} c_{\text{C}_2\text{H}_6} - k_5 c_{\text{H}} c_{\text{C}_2\text{H}_5} \quad (5.116)$$

$$R_{\text{C}_2\text{H}_5} = 0 = k_2 c_{\text{CH}_3} c_{\text{C}_2\text{H}_6} - k_3 c_{\text{C}_2\text{H}_5} + k_4 c_{\text{C}_2\text{H}_6} c_{\text{H}} - k_5 c_{\text{H}} c_{\text{C}_2\text{H}_5} \quad (5.117)$$

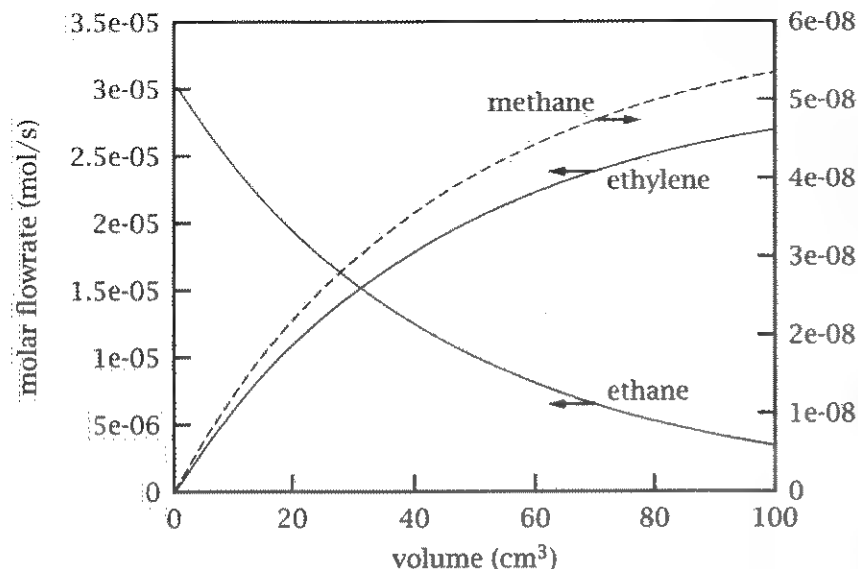


Figure 5.13: Molar flowrates of  $C_2H_6$ ,  $C_2H_4$  and  $CH_4$  corresponding to the exact solution.

Adding Equations 5.115, 5.116 and 5.117 gives

$$c_H = \frac{k_1 c_{C_2H_6}}{k_5 c_{C_2H_5}} \quad (5.118)$$

Inserting Equation 5.118 into Equation 5.116 yields

$$0 = k_3 k_5 c_{C_2H_5}^2 - k_4 k_1 c_{C_2H_6}^2 - k_1 k_5 c_{C_2H_6} c_{C_2H_5} \quad (5.119)$$

$$c_{C_2H_5} = \left( \frac{k_1}{2k_3} + \sqrt{\left( \frac{k_1}{2k_3} \right)^2 + \frac{k_1 k_4}{k_3 k_5}} \right) c_{C_2H_6}$$

Finally

$$R_{C_2H_4} = k_3 \left( \frac{k_1}{2k_3} + \sqrt{\left( \frac{k_1}{2k_3} \right)^2 + \frac{k_1 k_4}{k_3 k_5}} \right) c_{C_2H_6} \quad (5.120)$$

which can be rewritten as

$$R_{C_2H_4} = k c_{C_2H_6} \quad (5.121)$$

in which

$$k = k_3 \left( \frac{k_1}{2k_3} + \sqrt{\left( \frac{k_1}{2k_3} \right)^2 + \frac{k_1 k_4}{k_3 k_5}} \right)$$

At 925 K,  $k = 0.797 \text{ s}^{-1}$ . The student should prove that

$$c_H = \frac{k_1/k_5}{k_1/2k_3 + \sqrt{(k_1/2k_3)^2 + k_1 k_4/k_3 k_5}} \quad (5.122)$$

$$c_{CH_3} = \frac{2k_1}{k_2} \quad (5.123)$$

The validity of the QSSA is established by solving the set of ODEs

$$\begin{aligned} \frac{dN_{C_2H_6}}{dV} &= -r_1 - r_2 - r_4 + r_5 & \frac{dN_{H_2}}{dV} &= r_4 \\ \frac{dN_{CH_4}}{dV} &= r_2 & \frac{dN_{H_2O}}{dV} &= 0 \\ \frac{dN_{C_2H_4}}{dV} &= r_3 \end{aligned}$$

subject to the initial molar flowrates. During the numerical solution the concentrations needed in the rate equations are computed using

$$c_{C_2H_6} = \left( \frac{N_{C_2H_6}}{N_{C_2H_6} + N_{CH_4} + N_{C_2H_4} + N_{H_2} + N_{H_2O}} \right) \frac{P}{RT} \quad (5.124)$$

and Equations 5.119, 5.122 and 5.123. Note we can neglect  $H$ ,  $CH_3$  and  $C_2H_5$  in the molar flowrate balance (Equation 5.124) because these components are present at levels on the order of  $10^{-6}$  less than  $C_2H_6$ .

Figure 5.14 shows the error in the molar flowrate of ethylene that results from the QSSA. The error is less than 10% after  $3.1 \text{ cm}^3$  of reactor volume; i.e., it takes about 3% of the volume for the error to be less than 10%. Similarly, the error is less than 5% after  $5.9 \text{ cm}^3$  and less than 1% after  $18.6 \text{ cm}^3$ . Figure 5.14 provides evidence that the QSSA is valid for sufficiently large reactors, such as  $100 \text{ cm}^3$ . If the reactor volume were very small (a very short residence time), such as  $20 \text{ cm}^3$ , the QSSA would not be appropriate.

The reader might question what has been gained in this QSSA example. After all, the full model solution requires the simultaneous solution of eight ODEs and the reduced model requires solving five ODEs. If one had experimental data, such as effluent ethane, ethylene, hydrogen and methane concentrations for different residence times ( $V_R/Q_f$ ), one would need to fit these data to find the five rate constants for either the

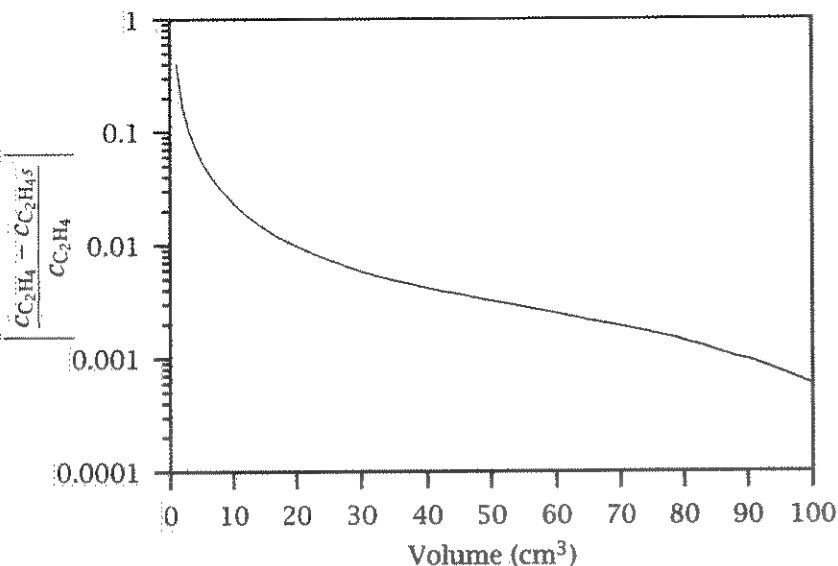


Figure 5.14: Fractional error in the QSSA molar flowrate of  $C_2H_4$  versus reactor volume.

full or the reduced model. However, the results in Figure 5.13 demonstrate that  $CH_4$  is a minor product at this temperature and this also would be found in the experimental data. This suggests we can neglect  $CH_4$  in a species balance when computing mole fractions. Therefore the mass action statement



does a reasonable job of accounting for the changes that occur in the PFR during ethane pyrolysis. QSSA analysis predicts the ethane pyrolysis rate should be first order in ethane concentration and the rate constant is  $k = 0.797 \text{ s}^{-1}$ . Using the mass action statement to describe the reaction stoichiometry and the definition of the rate of production of a component in a single reaction

$$R_j = \nu_j r$$

to relate  $R_{C_2H_6}$  and  $R_{H_2}$  to the result for  $C_2H_4$  found in Equation 5.121, it is possible to solve the problem analytically. Figure 5.15 shows that

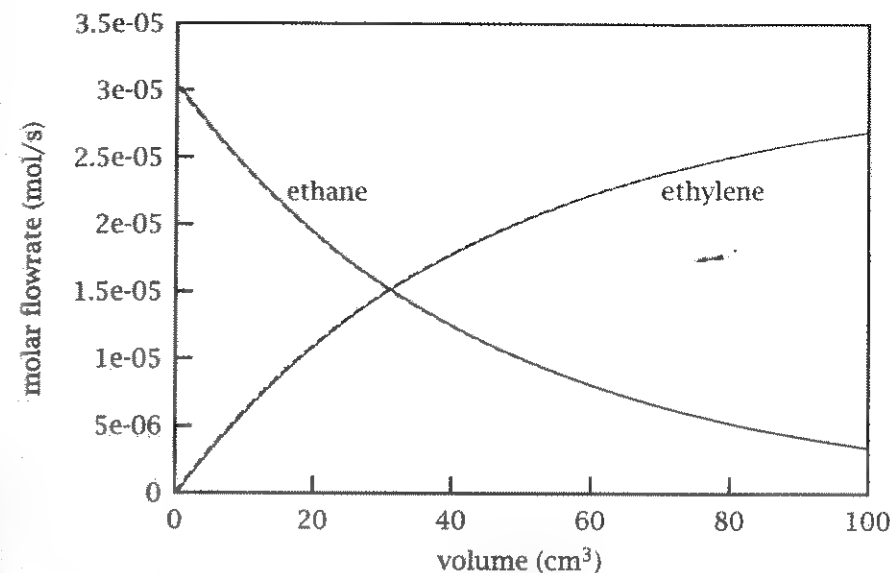


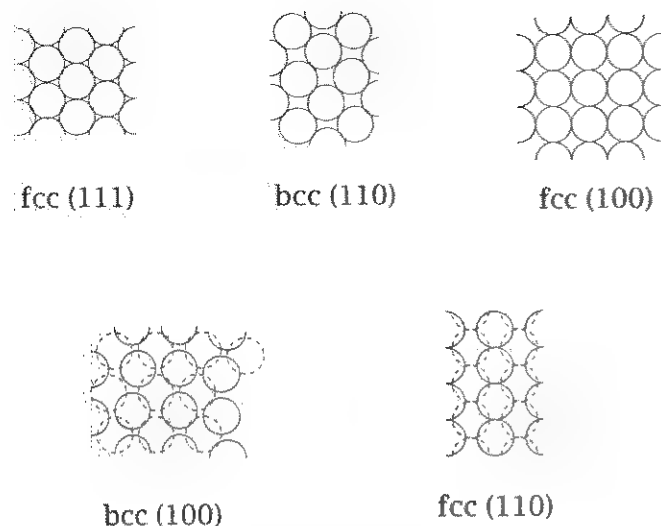
Figure 5.15: Comparison of the molar flowrates of  $C_2H_6$  and  $C_2H_4$  for the exact solution (solid lines) and the simplified kinetic scheme (dashed lines).

nearly equivalent predictions are made for the simplified scheme based on the mass action statement, Reaction 5.125 using Equation 5.121 for the rate expression.

This simple example demonstrates that when one has all the kinetic parameters, they should be used because the QSSA buys very little in this case. The reverse situation of knowing a mechanism but not the rate constants could pose a difficult optimization problem when fitting all the rate constants, and the QSSA analysis can provide a framework for simplifying the kinetic expressions against which the data are tested. Here one would expect a first-order expression to describe the appearance of ethylene adequately.  $\square$

## 5.6 Reactions at Surfaces

Many books have been written on this topic and it continues to be an area of intense research. Our intention in this chapter is to provide the student with a simple picture of surfaces and to provide a basis for



**Figure 5.16:** Surface and second-layer (dashed) atom arrangements for several low-index surfaces.

evaluating heterogeneously catalyzed reaction kinetics and reaction-rate expressions. Keep in mind the goal is to develop a rate expression from a reaction mechanism. This rate expression can be used to direct the regression of rate data when determining the parameters for a heterogeneously catalyzed reaction. To understand how we construct surface reactions and develop rate expressions, it is necessary to discuss the elementary steps that comprise a surface reaction — adsorption, surface reaction and desorption. We also need to have a model of a surface at which the elementary steps occur. To streamline the presentation, the discussion is limited to gas-solid systems and a single chemical isotherm, the Langmuir isotherm. The interested student can find a more complete and thorough discussion of adsorption, surface reaction and desorption in numerous textbooks [5, 36, 1, 13, 24, 3].

A heterogeneously catalyzed reaction takes place at the surface of a catalyst. Catalysts, their properties, and the nature of catalytic surfaces are discussed in Chapter 7. For this discussion, we approximate the surface as a single crystal with a known surface order. The density of atoms at the low-index planes of transition metals is on the order of  $10^{15} \text{ cm}^{-2}$ . Figure 5.16 presents the atomic arrangement of low-index surfaces for various metals. This figure illustrates the packing arrange-

Property	Chemisorption	Physisorption
amount	limited to a monolayer	multilayer possible
specificity	high	none
heat of adsorption	typically > 10 kcal/mol	low (2-5 kcal/mol)
activated	possibly	generally not

**Table 5.7:** Chemisorption and physisorption properties.

ment and different combinations of nearest neighbors that can exist at a surface. For example, an fcc(111) surface atom has six nearest-surface neighbors, an fcc(100) surface atom has four nearest-surface neighbors, and an fcc(110) surface atom has two nearest-surface neighbors. We find that the interaction between an adsorbing molecule and the surface of a single crystal depends on the surface structure.

To facilitate our discussion of heterogeneous reaction kinetics let us consider the oxidation of CO on Pd, an fcc metal. The following mechanism has been proposed for the oxidation reaction over Pd(111) [9]



where the subscript *ads* refers to adsorbed species and S refers to vacant surface sites. This simple example illustrates the steps in a catalytic reaction — adsorption of reactants in Reactions 5.126 and 5.127, reaction of adsorbed components in Reaction 5.128, and desorption of products also shown as part of Reaction 5.128.

Adsorption occurs when an incident atom or molecule sticks to the surface. The adsorbing species can be bound weakly to the surface or it can be held tightly to the surface. The manner by which the adsorbed species is held and the properties it exhibits once adsorbed determine the type of adsorption — physical or chemical. The dynamics of the process by which the incident adsorbate finds the adsorption site is used to construct a rate expression and rate constant for the adsorption step from first principles.

A number of criteria have been applied to distinguish between physical and chemical adsorption. In some cases the distinction is not clear. Table 5.7 lists several properties that can be used to distinguish between physisorption and chemisorption. The most distinguishing characteristics are the degree of coverage and the specificity.

**Chemisorption:** Chemical adsorption occurs when a chemical bond or a partial chemical bond is formed between the surface (adsorbent) and the adsorbate, leading to the specificity of the process. In general, but not always, the formation of a bond limits the coverage to at most one chemisorbed adsorbate for every surface atom. This limit of a single layer or monolayer is exploited later to derive a statement of conservation of sites. The upper limit need not have a one-to-one adsorbate to surface atom stoichiometry; for example, saturation can occur after one-third of all sites are occupied.

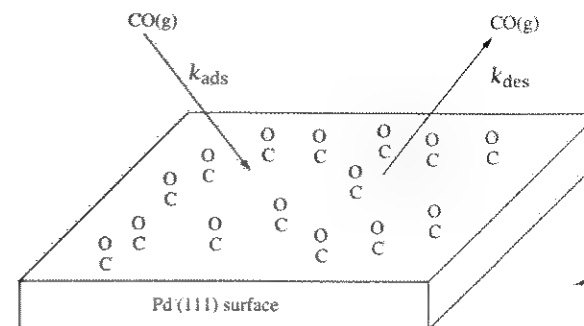
**Physisorption:** Physical adsorption occurs once the partial pressure of the adsorbate is above its saturation vapor pressure. Physisorption is similar to a condensation process and has practically no dependence on the solid surface and the interaction between the adsorbate and adsorbent. Just as water vapor condenses on any cold surface placed in ambient, humid air, a gas such as  $N_2$  condenses on solids maintained at 77 K.

Adsorption is an exothermic process and the magnitude of the heat of adsorption is used to distinguish chemisorption from physisorption. Heats of adsorption greater than 10 kcal/mol are definitely associated with chemisorbed species. Small heats of adsorption (2–5 kcal/mol) do not always indicate physisorption, however. Therefore, it is best to look at more than one property when trying to distinguish between chemisorption and physisorption. Physical adsorption is used to measure the area of high-area oxide catalysts and oxide-supported metal catalysts. Physisorption isotherms and their use are discussed in Chapter 7. The discussion that follows treats only chemisorption.

Figure 5.17 depicts a schematic of a simple adsorption process. Adsorption occurs when an incident CO molecule finds a vacant site. The dynamics of the collision process, accommodation with the surface, and binding to the surface are complex, and the overall rate of this process can change for the same adsorbate on different metals or even different low-index surfaces of the same metal. The flux of CO molecules to the surface (molecules/area·time) is proportional to the number density of incident molecules and the mean kinetic velocity, and is given by

$$F_{CO} = \frac{P_{CO}}{\sqrt{2\pi M_{CO} k_B T}}$$

which, for pressure in torr, equals  $3.83 \times 10^{20} P_{CO}$  molecule/s·cm<sup>2</sup> at



**Figure 5.17:** Schematic representation of the adsorption/desorption process.

300 K. Since most surfaces contain approximately  $10^{15}$  atoms/cm<sup>2</sup>, at  $2.6 \times 10^{-6}$  torr, the number of atoms in a monolayer strikes the surface per second. At one atmosphere, 290 million times the number of atoms in a monolayer strike the surface per second.

Not every collision leads to adsorption. The rate of adsorption can be written as the product of the flux times the probability of sticking

$$r_{ads} = FS(\theta, T)$$

$S$  is known as the sticking coefficient, and is a function of the surface coverage and temperature; the exact functional dependence can be determined once the dynamics of the adsorption process are known [24]. We use the Langmuir adsorption to model the adsorption process.

The Langmuir adsorption model assumes that all surface sites are the same. As long as the incident molecules possess the necessary energy, and entropy considerations are satisfied, adsorption proceeds if a vacant site is available. The Langmuir process also assumes adsorption is a completely random surface process with no adsorbate-adsorbate interactions. Since all sites are equivalent, and adsorbate-adsorbate interactions are neglected, the surface sites can be divided into two groups, vacant and occupied. The surface-site balance is therefore

$$\left\{ \begin{array}{l} \text{Total number of} \\ \text{surface sites} \\ \text{per unit area} \end{array} \right\} = \left\{ \begin{array}{l} \text{Number of} \\ \text{vacant sites} \\ \text{per unit area} \end{array} \right\} + \left\{ \begin{array}{l} \text{Number of} \\ \text{occupied sites} \\ \text{per unit area} \end{array} \right\}$$

or

$$\bar{c}_m = \bar{c}_v + \sum_{j=1}^{n_s} \bar{c}_j \quad (5.129)$$

Dividing by the total number of surface sites leads to the fractional form of the site balance.

$$1 = \frac{\bar{c}_v}{\bar{c}_m} + \sum_{j=1}^{n_s} \frac{\bar{c}_j}{\bar{c}_m} = \theta_v + \sum_{j=1}^{n_s} \theta_j \quad (5.130)$$

Equations 5.129 and 5.130 represent the conservation of sites for non-interacting adsorbates on uniform surfaces. Since active catalysts generally are dispersed on high-surface-area inert carriers, the units for  $\bar{c}_m$  often are number of sites per gram of catalyst.

Figure 5.17 shows the model of a surface consisting of vacant sites and sites covered by chemisorbed CO. The CO in the gas phase is adsorbing on the surface and desorbing from the surface. The surface reaction, and the forward rate of adsorption and the reverse rate of desorption, are given by<sup>6</sup>



$$r_{\text{ads}} = k_1 c_{\text{CO}} \bar{c}_v, \quad r_{\text{des}} = k_{-1} \bar{c}_{\text{CO}}$$

The units of the rate are mol/time-area. The rate expressions follow directly from the reaction mechanism because the reactions are treated as elementary steps. When the adsorption-desorption reactions are in equilibrium, the amount of adsorbed CO is related to the gas-phase partial pressure of CO, an equilibrium constant, and the total number of surface sites. Equating the adsorption and desorption rates at equilibrium gives

$$k_1 c_{\text{CO}} \bar{c}_v = k_{-1} \bar{c}_{\text{CO}}$$

Solving for the surface concentration gives

$$\bar{c}_{\text{CO}} = \frac{k_1}{k_{-1}} c_{\text{CO}} \bar{c}_v = K_1 c_{\text{CO}} \bar{c}_v \quad (5.132)$$

<sup>6</sup>This is an oversimplification of the adsorption process. A more acceptable view is to have CO adsorb into a precursor state and move around on the surface until it finds a suitable adsorption site, S. As long as the precursor state is in equilibrium with the gas phase, the Langmuir description of the adsorption process represented by Reaction 5.131 gives the same result as is obtained by a more accurate description of the adsorption and desorption dynamics. Since experimental results are often consistent with the precursor state in equilibrium with the gas phase, we present the Langmuir model.

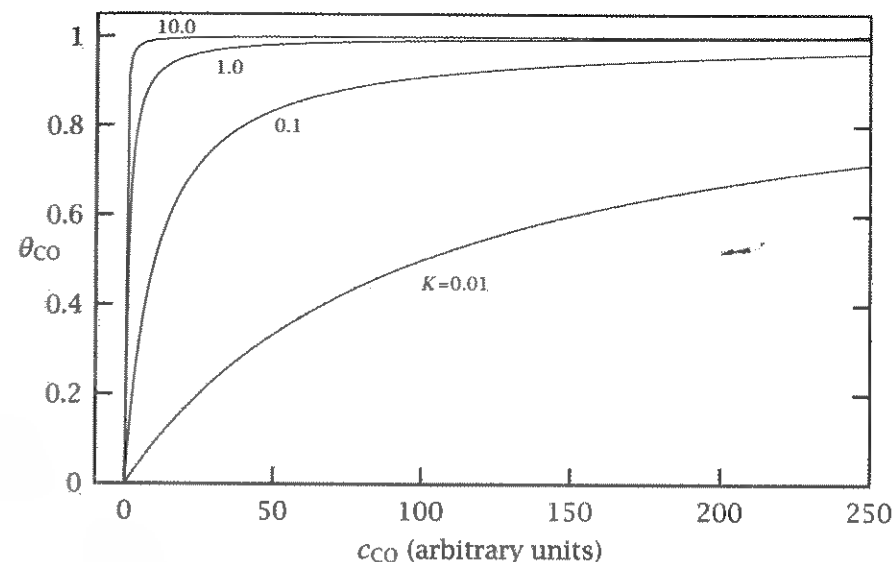


Figure 5.18: Fractional coverage versus adsorbate concentration for different values of the adsorption constant,  $K$ .

The conservation of sites gives

$$\bar{c}_v = \bar{c}_m - \bar{c}_{\text{CO}} \quad (5.133)$$

Substituting Equation 5.133 into Equation 5.132 and rearranging leads to the Langmuir isotherm for single-component, associative adsorption

$$\bar{c}_{\text{CO}} = \frac{\bar{c}_m K_1 c_{\text{CO}}}{1 + K_1 c_{\text{CO}}} \quad (5.134)$$

Dividing both sides of Equation 5.134 by the total concentration of surface sites  $\bar{c}_m$  leads to the fractional form of the single component, associative Langmuir adsorption isotherm.

$$\theta_{\text{CO}} = \frac{K_1 c_{\text{CO}}}{1 + K_1 c_{\text{CO}}} \quad (5.135)$$

Figure 5.18 presents the general shape of the fractional coverage as a function of pressure for different values of  $K$ . In the limit  $K_1 c_{\text{CO}} \gg 1$ , Equations 5.134 and 5.135 asymptotically approach  $\bar{c}_m$  and 1.0, respectively. Since adsorption is exothermic,  $K_1$  decreases with increasing temperature and higher CO partial pressures are required to reach



saturation. Figure 5.18 illustrates the effect of  $K$  on the coverage. In all cases the surface saturates with the adsorbate; however, the concentration at which this saturation occurs is a strong function of  $K$ . Single component adsorption is used to determine the number of active sites,  $\bar{c}_m$ . The adsorption data can be fit to either the Langmuir isotherm (Equation 5.134) or to the "linear" form of the isotherm. The linear form is obtained by taking the inverse of Equation 5.134 and multiplying by  $c_{CO}$

$$\frac{c_{CO}}{\bar{c}_{CO}} = \frac{1}{K_1 \bar{c}_m} + \frac{c_{CO}}{\bar{c}_m} \quad (5.136)$$

A plot of the linear form provides  $1/(K_1 \bar{c}_m)$  for the intercept and  $1/\bar{c}_m$  for the slope.

#### Example 5.6: Fitting Langmuir adsorption constants to CO data

The active area of supported transition metals can be determined by adsorbing carbon monoxide. Carbon monoxide is known to adsorb associatively on ruthenium at 100°C. Use the following uptake data for the adsorption of CO on 10 wt % Ru supported on  $Al_2O_3$  at 100°C to determine the equilibrium adsorption constant and the number of adsorption sites.

$P_{CO}$ (torr)	100	150	200	250	300	400
CO adsorbed ( $\mu\text{mol/g cat}$ )	1.28	1.63	1.77	1.94	2.06	2.21

#### Solution

Figures 5.19 and 5.20 plot the data for Equations 5.134 and 5.136, respectively. From Figure 5.20 we can estimate using least squares the slope,  $0.35 \text{ g}/\mu\text{mol}$ , and the zero concentration intercept,  $1.822 \text{ cm}^3/\text{g}$ , and then calculate

$$K_1 = 0.190 \text{ cm}^3/\mu\text{mol}$$

$$\bar{c}_m = 2.89 \mu\text{mol/g}$$

□

We now consider multiple-component adsorption. If other components, such as B and D, are assumed to be in adsorption-desorption equilibrium along with CO, we add the following two reactions

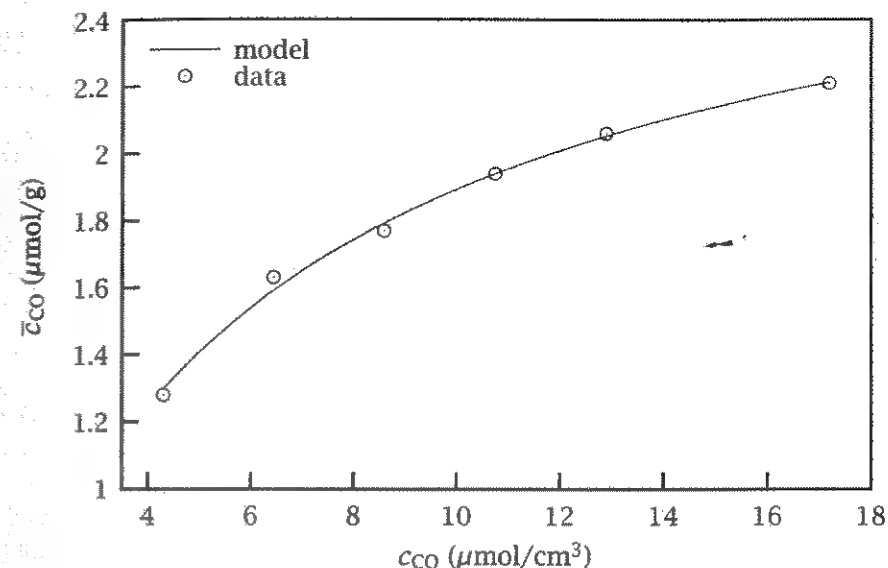
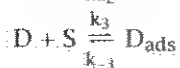
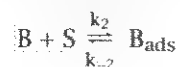


Figure 5.19: Langmuir isotherm for CO uptake on Ru.

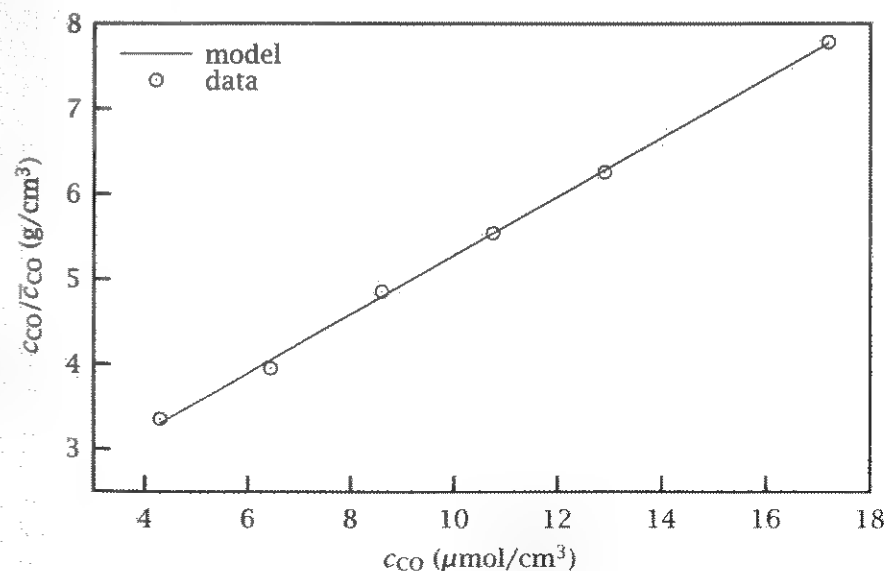


Figure 5.20: Linear form of Langmuir isotherm for CO uptake on Ru.

Following the development that led to Equation 5.132, we can show the equilibrium surface concentrations for B and D are given by

$$\bar{c}_B = K_2 c_B \bar{c}_v \quad (5.137)$$

$$\bar{c}_D = K_3 c_D \bar{c}_v \quad (5.138)$$

The site balance becomes

$$\bar{c}_v = \bar{c}_m - \bar{c}_{CO} - \bar{c}_B - \bar{c}_D \quad (5.139)$$

Substituting Equation 5.139 into Equation 5.132 and rearranging leads to a different isotherm expression for CO.

$$\bar{c}_{CO} = \frac{\bar{c}_m K_{CO} c_{CO}}{1 + K_1 c_{CO} + K_2 c_B + K_3 c_D} \quad (5.140)$$

The terms in the denominator of Equation 5.140 have special significance and the magnitude of the equilibrium adsorption constant, when multiplied by the respective concentration, determines the component or components that occupy most of the sites. Each of the terms  $K_i c_j$  accounts for sites occupied by species  $j$ . From Chapter 3 we know the equilibrium constant is related to the exponential of the heat of adsorption, which is exothermic. Therefore, the more strongly a component is chemisorbed to the surface, the larger the equilibrium constant. Also, if components have very different heats of adsorption, the denominator may be well approximated using only one  $K_i c_j$  term.

The CO oxidation example also contains a dissociative adsorption step



Adsorption requires the  $O_2$  molecule to find a pair of adsorption sites and desorption requires two adsorbed O atoms to be adjacent. Hill [15] and Kisliuk [18, 19] discuss lattice statistics and the probability of finding pairs of sites in two-dimensional arrays presented by the regular arrangement of surface atoms illustrated in Figure 5.16. Boudart and Djega-Mariadassou [3], and Hayward and Trapnell [13], describe how the probability of finding pairs of sites is used to develop rate expressions on surfaces. When a bimolecular surface reaction occurs, such as dissociative adsorption, associative desorption, or a bimolecular surface reaction, the rate in the forward direction depends on the probability of finding pairs of reaction centers. This probability, in turn, depends

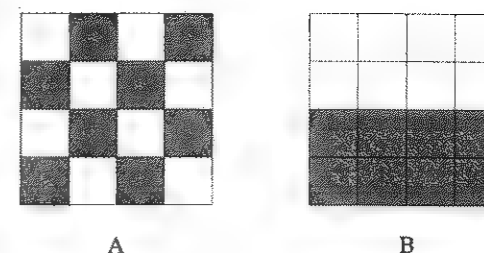


Figure 5.21: Two packing arrangements for a coverage of  $\theta = 1/2$ .

on whether or not the sites are randomly located, and whether or not the adsorbates are mobile on the surface.

The probability of finding pairs of vacant sites can be seen by considering adsorption onto a checkerboard such as Figure 5.21. Let two sites be adjacent if they share a common line segment (i.e., do not count sharing a vertex as being adjacent) and let  $\theta$  equal the fraction of surface covered. The checkerboard has a total 24 adjacent site pairs, which can be found by counting the line segments on adjoining squares. Figure 5.21 presents two possibilities where half the sites (the shaded squares) are covered. For Figure 5.21A the probability an incident gas atom striking this surface in a random location hits a vacant site is  $1 - \theta$  and the probability a gas-phase molecule can dissociate on two adjacent vacant sites is  $p = 0$ . Figure 5.21B has 10 vacant adjacent site pairs. Therefore, the probability of dissociative adsorption is  $p = 10/24$ . The probability of finding vacant adsorption sites is  $0 \leq p \leq 1/2$  for these two examples because the probability depends on the packing arrangement.

For *random* or independent packing of the surface the probability for finding pairs of vacant adjacent sites is  $p = (1 - \theta)^2$ . The probability for random packing can be developed by considering five different arrangements for a site and its four nearest neighbors, with the center site always vacant, as shown in Figure 5.22. The probability of any configuration is  $(1 - \theta)^n \theta^m$ , in which  $n$  is the number of vacant sites and  $m$  is the number of occupied sites. Arrangement 1 can happen only one way. However, arrangement 2 can happen four ways because the covered site can be in four different locations. Similarly, arrangement 3 can happen six different ways, 4 can happen four ways, and 5 can happen only one way. Therefore, the probability of two adjacent

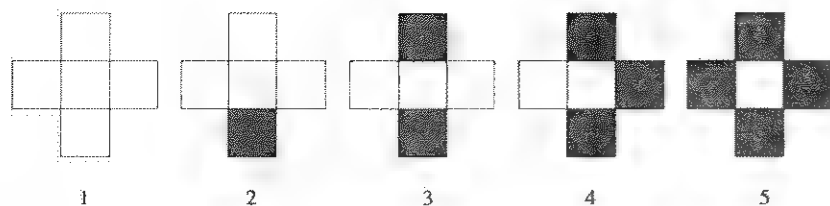


Figure 5.22: Five possible arrangements around a single site.

sites being vacant is

$$p_1 + 4p_2 + 6p_3 + 4p_4 + 0p_5 \quad (5.142)$$

An incoming gas molecule can hit the surface in one of four positions, with one atom in the center and the other pointing either north, south, east or west. For random molecule orientations, the probabilities that a molecule hits the arrangement with a correct orientation are:  $p_1$ ,  $(3/4)4p_2$ ,  $(1/2)6p_3$  and  $(1/4)4p_4$ . Therefore, the probability of a successful collision is

$$\begin{aligned} p &= p_1 + 3p_2 + 3p_3 + p_4 \\ p &= (1 - \theta)^5 + 3(1 - \theta)^4\theta + 3(1 - \theta)^3\theta^2 + (1 - \theta)^2\theta^3 \\ p &= (1 - \theta)^2[(1 - \theta)^3 + 3(1 - \theta)^2\theta + 3(1 - \theta)\theta^2 + \theta^3] \end{aligned}$$

Using a binomial expansion,  $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ , the term in brackets can be written as

$$p = (1 - \theta)^2[(1 - \theta) + \theta]^3$$

or

$$p = (1 - \theta)^2$$

Therefore, we use  $\theta_v \cdot \theta_v$  to represent the probability of finding pairs of vacant sites for dissociative adsorption. The probability of finding other pair combinations is the product of the fractional coverage of the two types of sites.

Returning to Reaction 5.141



and using concentrations of vacant sites instead of fractional coverage, we write the rates of dissociative adsorption and associative desorption of oxygen as

$$r_{\text{ads}} = k_1 \bar{c}_v^2 c_{\text{O}_2} \quad (5.143)$$

$$r_{\text{des}} = k_{-1} \bar{c}_{\text{O}}^2 \quad (5.144)$$

The units on  $k_1$  are such that  $r_{\text{ads}}$  has the units of mol/time·area. At equilibrium, the rate of adsorption equals the rate of desorption leading to

$$\bar{c}_{\text{O}} = \sqrt{K_1 c_{\text{O}_2}} \bar{c}_v \quad (5.145)$$

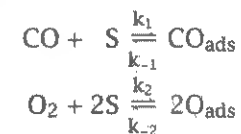
Combining Equation 5.145 with the site balance, Equation 5.129, and rearranging leads to the Langmuir form for single-component dissociative adsorption

$$\bar{c}_{\text{O}} = \frac{\bar{c}_m \sqrt{K_1 c_{\text{O}_2}}}{1 + \sqrt{K_1 c_{\text{O}_2}}} \quad (5.146)$$

The Langmuir isotherms represented in Equations 5.134 and 5.146 are different because each represents a different gas-surface reaction process. Both share the asymptotic approach to saturation at sufficiently large gas concentration, however.

### Example 5.7: Equilibrium CO and O surface concentrations

Determine the equilibrium CO and O surface concentrations when  $\text{O}_2$  and CO adsorb according to



### Solution

At equilibrium the net rates of the two reactions are zero (adsorption rate equals desorption rate)

$$\begin{aligned} r_1 &= 0 = k_1 c_{\text{CO}} \bar{c}_v - k_{-1} \bar{c}_{\text{CO}} \\ r_2 &= 0 = k_2 c_{\text{O}_2} \bar{c}_v^2 - k_{-2} \bar{c}_{\text{O}}^2 \end{aligned}$$

Solving for the surface coverages in terms of the concentration of vacant sites gives

$$\bar{c}_{\text{CO}} = K_1 c_{\text{CO}} \bar{c}_v \quad (5.147)$$

$$\bar{c}_{\text{O}} = \sqrt{K_2 c_{\text{O}_2}} \bar{c}_v \quad (5.148)$$

The remaining unknown is  $\bar{c}_v$ , which can be found using the site balance

$$\bar{c}_m = \bar{c}_v + \bar{c}_{\text{CO}} + \bar{c}_{\text{O}} \quad (5.149)$$

Combining Equations 5.147–5.149 yields

$$\bar{c}_v = \frac{\bar{c}_m}{1 + K_1 c_{\text{CO}} + \sqrt{K_2 c_{\text{O}_2}}} \quad (5.150)$$

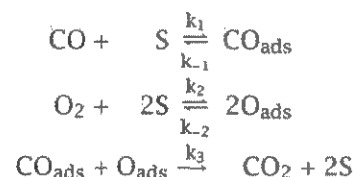
We next substitute the vacant site concentration into Equations 5.147 and 5.148 to give the surface concentrations in terms of gas-phase concentrations and physical constants

$$\bar{c}_{\text{CO}} = \frac{\bar{c}_m K_1 c_{\text{CO}}}{1 + K_1 c_{\text{CO}} + \sqrt{K_2 c_{\text{O}_2}}}$$

$$\bar{c}_{\text{O}} = \frac{\bar{c}_m \sqrt{K_2 c_{\text{O}_2}}}{1 + K_1 c_{\text{CO}} + \sqrt{K_2 c_{\text{O}_2}}}$$

□

The CO oxidation reaction involves associative adsorption of CO, dissociative adsorption of O<sub>2</sub>, and the bimolecular surface reaction



The reaction requires CO<sub>s</sub> and O<sub>s</sub> to occupy adjacent sites. If this reaction step is essentially irreversible, the rate expression is

$$r_3 = k_3 \bar{c}_{\text{CO}} \bar{c}_{\text{O}} \quad (5.151)$$

Note this rate expression is based on the probability of finding two dissimilar paired adsorbates.

#### Example 5.8: Production rate of CO<sub>2</sub>

Find the rate of CO<sub>2</sub> production using the reaction mechanism listed above for CO oxidation. Assume the O<sub>2</sub> and CO adsorption steps are at equilibrium. Make a log-log plot of the production rate of CO<sub>2</sub> versus gas-phase CO concentration at constant gas-phase O<sub>2</sub> concentration. What are the slopes of the production rate at high and low CO concentrations?

#### Solution

The rate of CO<sub>2</sub> production is given by

$$R_{\text{CO}_2} = k_3 \bar{c}_{\text{CO}} \bar{c}_{\text{O}} \quad (5.152)$$

The surface concentrations for CO and O were determined in the preceding example,

$$\bar{c}_{\text{CO}} = \frac{\bar{c}_m K_1 c_{\text{CO}}}{1 + K_1 c_{\text{CO}} + \sqrt{K_2 c_{\text{O}_2}}}$$

$$\bar{c}_{\text{O}} = \frac{\bar{c}_m \sqrt{K_2 c_{\text{O}_2}}}{1 + K_1 c_{\text{CO}} + \sqrt{K_2 c_{\text{O}_2}}}$$

Substituting these concentrations onto Equation 5.152 gives the rate of CO<sub>2</sub> production

$$R_{\text{CO}_2} = \frac{k_3 \bar{c}_m^2 K_1 c_{\text{CO}} \sqrt{K_2 c_{\text{O}_2}}}{(1 + K_1 c_{\text{CO}} + \sqrt{K_2 c_{\text{O}_2}})^2} \quad (5.153)$$

For ease of visualization, we define a dimensionless production rate

$$\bar{R} = R_{\text{CO}_2} / (k_3 \bar{c}_m^2)$$

and plot this production rate versus dimensionless concentrations of CO and O<sub>2</sub> in Figure 5.23. Notice that the production rate goes to zero if either CO or O<sub>2</sub> gas-phase concentration becomes large compared to the other. This feature is typical of competitive adsorption processes and second-order reactions. A large reaction rate in Equation 5.152 requires large surface CO and surface O concentrations. But if one gas-phase concentration is large relative to the other, then that species saturates the surface and the other species surface concentration is small, which causes a small reaction rate. If we hold one of the gas-phase concentrations constant and vary the other, we are taking a slice through the surface in Figure 5.23. Figure 5.24 shows the result when each of the dimensionless gas-phase concentrations is held fixed at 1.0. Notice again that increasing the gas-phase concentration of either reactant increases the rate until a maximum is achieved, and then decreases the rate upon further increase in concentration. High gas-phase concentration of either reactant inhibits the reaction by crowding the other species off the surface. Notice from the plot shown in Figure 5.24 that  $\log \bar{R}$  versus  $\log(K_1 c_{\text{CO}})$  changes in slope from 1.0 to –1.0 as CO concentration increases. As the concentration of O<sub>2</sub> increases, the slope

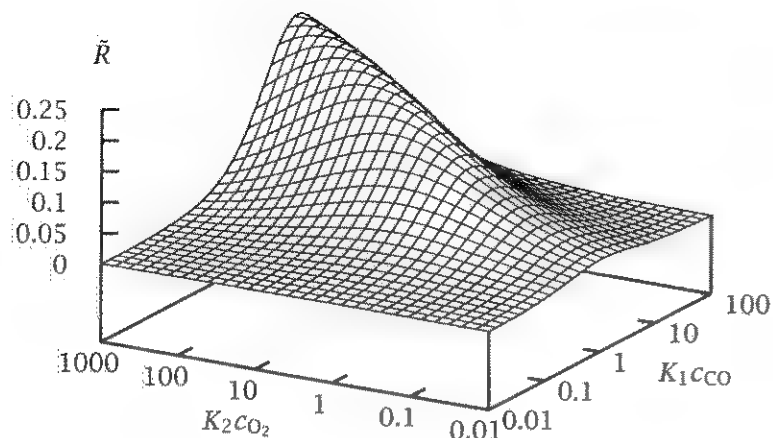


Figure 5.23: Dimensionless CO<sub>2</sub> production rate versus dimensionless gas-phase CO and O<sub>2</sub> concentrations.

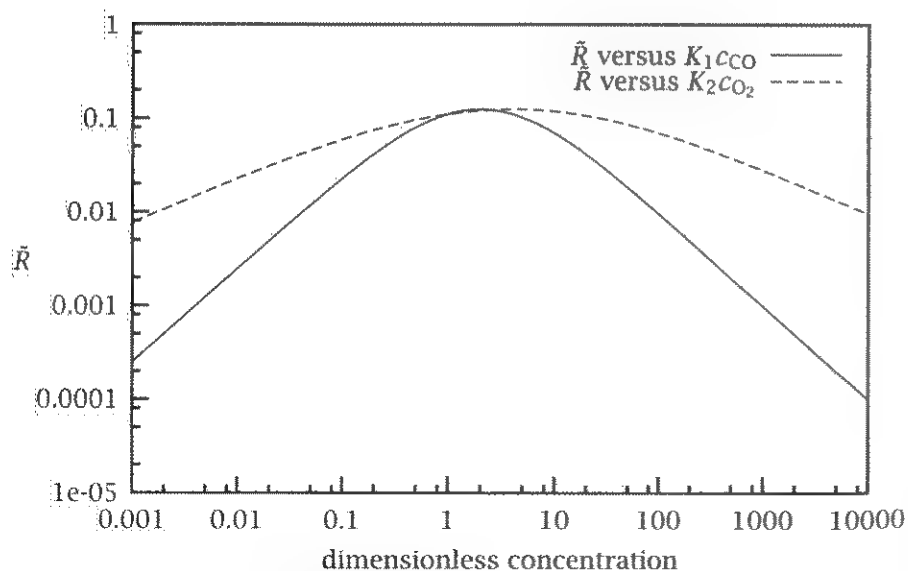


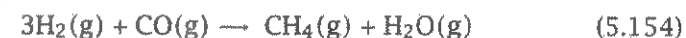
Figure 5.24: Dimensionless CO<sub>2</sub> production rate versus a single dimensionless gas-phase concentration while holding the other fixed at 1.0; slices through Figure 5.23.

changes from 0.5 to -0.5. These values also can be deduced by taking the logarithm of Equation 5.153. □

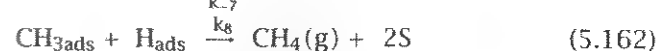
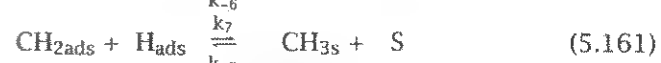
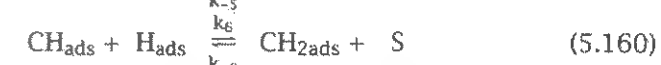
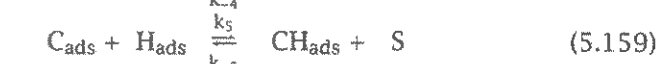
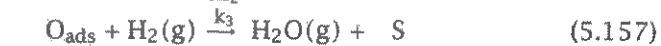
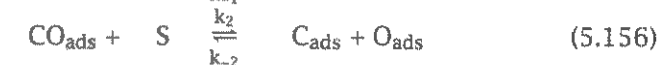
Rate expressions of the form of Equation 5.153 are known as Hougen-Watson or Langmuir-Hinshelwood kinetics [17]. This form of kinetic expression is often used to describe the species production rates for heterogeneously catalyzed reactions. We complete the section on the kinetics of elementary surface reactions by returning to the methane synthesis reaction listed in Section 5.2. The development proceeds exactly as outlined in Section 5.2. But now it is necessary to add a site-balance expression (Equation 5.129) in Step 3.

#### Example 5.9: Production rate of methane

Develop a rate expression for the synthesis of methane. The reaction is proposed to proceed as follows over a ruthenium catalyst [8]. The overall reaction is



and one possible mechanism is



You may assume the reactions shown to be reversible are at equilibrium and that the surface is dominated by adsorbed CO.

**Solution**

Let the species be designated as:

Species	Formula	Name
A <sub>1</sub>	CO(g)	carbon monoxide
A <sub>2</sub>	S	vacant site
A <sub>3</sub>	CO <sub>ads</sub>	adsorbed carbon monoxide
A <sub>4</sub>	C <sub>ads</sub>	adsorbed carbon
A <sub>5</sub>	O <sub>ads</sub>	adsorbed atomic oxygen
A <sub>6</sub>	H <sub>2</sub> (g)	hydrogen
A <sub>7</sub>	H <sub>2</sub> O(g)	water
A <sub>8</sub>	H <sub>ads</sub>	adsorbed atomic hydrogen
A <sub>9</sub>	CH <sub>ads</sub>	adsorbed methyne
A <sub>10</sub>	CH <sub>2ads</sub>	adsorbed methylene
A <sub>11</sub>	CH <sub>3ads</sub>	adsorbed methyl
A <sub>12</sub>	CH <sub>4</sub> (g)	methane

The rate of methane formation is given by Reaction 5.162.

$$R_{12} = k_8 \bar{c}_8 \bar{c}_{11} \quad (5.163)$$

From statements of equilibrium for the reversible reactions

$$\begin{aligned} \bar{c}_3 &= K_1 c_1 \bar{c}_2 & \bar{c}_4 &= K_2 \frac{\bar{c}_3 \bar{c}_2}{\bar{c}_5} & \bar{c}_8 &= \sqrt{K_4 c_6 \bar{c}_2} \\ \bar{c}_9 &= K_5 \frac{\bar{c}_4 \bar{c}_8}{\bar{c}_2} & \bar{c}_{10} &= K_6 \frac{\bar{c}_9 \bar{c}_8}{\bar{c}_2} & \bar{c}_{11} &= K_7 \frac{\bar{c}_{10} \bar{c}_8}{\bar{c}_2} \end{aligned}$$

Substituting for components 4, 9 and 10 into the expression for component 11 gives

$$\bar{c}_{11} = K_2 K_5 K_6 K_7 \frac{\bar{c}_8^3 \bar{c}_3}{\bar{c}_2^2 \bar{c}_5} \quad (5.164)$$

From the reaction stoichiometry,  $R_{12} = R_7$  and<sup>7</sup>

$$\bar{c}_5 = \frac{k_8 \bar{c}_8 \bar{c}_{11}}{k_3 c_6} \quad (5.165)$$

Substituting Equation 5.165 into Equation 5.164 and rearranging gives

$$\bar{c}_{11}^2 = \frac{k_3}{k_8} K_2 K_5 K_6 K_7 c_6 \frac{\bar{c}_8^2 \bar{c}_3}{\bar{c}_2^2}$$

<sup>7</sup>See Exercises 5.9 and 5.10 for a more complete description of when you can legitimately make this assumption.

Substitution of this expression into Equation 5.163 gives

$$R_{12} = \sqrt{k_3 k_8} \sqrt{K_2 K_5 K_6 K_7} \sqrt{c_6} \frac{\bar{c}_8^2 \sqrt{\bar{c}_3}}{\bar{c}_2} \quad (5.166)$$

We are told the surface is saturated with CO, i.e.,  $\theta_{\text{CO}} \cong 1$ . If only CO adsorbs

$$\theta_3 = \frac{\bar{c}_3}{\bar{c}_m} = \frac{K_1 c_1}{1 + K_1 c_1} \cong 1$$

Therefore  $K_1 c_1 \gg 1$  and

$$\bar{c}_2 = \frac{\bar{c}_m}{1 + K_1 c_1} \cong \frac{\bar{c}_m}{K_1 c_1} \quad (5.167)$$

Substituting Equation 5.167 into the expressions for the hydrogen and carbon monoxide surface concentrations, and combining the resulting expressions with Equation 5.166 leads to

$$R_{12} = \sqrt{k_3 k_8 K_2 K_5 K_6 K_7} \bar{c}_m^{3/2} \frac{K_4}{K_1} \frac{(c_6)^{3/2}}{c_1}$$

which simplifies to

$$R_{\text{CH}_4} = k_{\text{eff}} \frac{(c_{\text{H}_2})^{3/2}}{c_{\text{CO}}}$$

□

**5.7 Summary**

We introduced several concepts in this chapter that are the building blocks for reaction kinetics.

- Most reaction processes consist of more than one elementary reaction. The reaction process can be represented by a mass action statement, such as  $\text{C}_2\text{H}_6 \rightarrow \text{C}_2\text{H}_4 + \text{H}_2$  in Example 5.5, but this statement does not describe how the reactants convert into the products. The atomistic description of chemical events is found in the elementary reactions. First principles calculations can be used to predict the order of these elementary reactions and the values of their rate constants.
- The reaction order for elementary reactions is determined by the stoichiometry of the reaction

$$r_i = k_i \prod_{j \in R_i} c_j^{-\nu_{ij}} - k_{-i} \prod_{j \in P_i} c_j^{\nu_{ij}} \quad (5.168)$$

- Two assumptions are generally invoked when developing simplified kinetic rate expressions: (i) some of the elementary reactions are slow relative to the others; the fast reactions can be assumed to be at equilibrium. If only one step is slow, this single reaction determines all production rates, and is called the rate-limiting step; and (ii) the rate of formation of highly reactive intermediate species can be set to zero; the concentration of these intermediates is found from the resulting algebraic relations rather than differential equations.
- The chemical steps involved in heterogeneous reactions (adsorption, desorption, surface reaction) are generally treated as elementary reactions. In many cases, one reaction is the slow step and the remaining steps are at equilibrium. For heterogeneous reactions, we add a site balance to account for the vacant and occupied sites that take part in the reaction steps.

## Notation

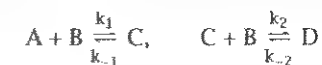
$a_j$	activity of species $j$
$c_j$	concentration of species $j$
$c_{js}$	steady-state concentration of $j$
$\bar{c}_j$	concentration of species $j$ on the catalyst surface
$\bar{c}_m$	total active surface concentration (monolayer coverage concentration)
$\bar{c}_v$	concentration of vacant surface sites
$f_j$	fugacity of species $j$
$f_j^\circ$	standard-state fugacity of pure species $j$
$F_j$	flux of species $j$ from the gas phase to the catalyst surface
$g_{ei}$	degeneracy of the $i$ th electronic energy level
$h$	Planck's constant
$I$	moment of inertia for a rigid rotor
$I_i$	moment of inertia about an axis
$k_B$	Boltzmann constant
$K$	reaction equilibrium constant
$m$	mass of a molecule (atom)
$M_j$	molecular weight of species $j$
$n_j$	moles of species $j$
$N_j$	molar flowrate of species $j$
$p$	probability of finding adjacent pairs of sites

$P$	total pressure
$P_j$	partial pressure of species $j$
$P_i$	product species in reaction $i$
$\left(\frac{q}{V}\right)_j$	molecular partition function of species $j$
$q_{elec}$	electronic partition function
$q_{rot}$	rotational partition function
$q_{vib}$	vibrational partition function
$Q$	partition function
$Q_f$	volumetric flowrate at the reactor inlet
$r_i$	reaction rate for $i$ th reaction
$R$	gas constant
$R_j$	production rate for $j$ th species
$R_i$	reactant species in reaction $i$
$t$	time
$T$	absolute temperature
$V(r)$	intermolecular potential energy
$V_R$	reactor volume
$z$	compressibility factor
$\epsilon_i$	energy of the $i$ th electronic level
$\epsilon_i$	extent of the $i$ th reaction
$\theta_j$	fractional surface coverage of species $j$
$\theta_v$	fractional surface coverage of vacant sites
$\nu_{ij}$	stoichiometric number for the $j$ th species in the $i$ th reaction
$\sigma$	symmetry number, 1 for a heteronuclear molecule, 2 for homonuclear molecule
$\tau$	lifetime of a component
$\phi_j$	fugacity coefficient for species $j$

## 5.8 Exercises

### Exercise 5.1: Equilibrium assumption and nonlinear kinetics

It is common to add groups to difunctional molecules in two consecutive steps:



The difunctional starting material (A) combines with the primary reactant (B) to form the monofunctional intermediate (C), which subsequently combines with B to produce the desired final product (D).



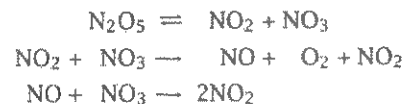
- (a) Write down the material balances for components A, B, C and D if this reaction takes place in a batch reactor.
- (b) Let's assume the first step is very slow compared to the second. Solve for the equilibrium concentration of component D by setting the rate of the second reaction to zero (i.e., the second reaction is at equilibrium compared to the first). Then we can eliminate the ODE for component D in the simplified model and replace it with this algebraic equation.
- (c) What are the differential equations for concentrations of components A, B and C in the simplified model? Hint: don't just set  $r_2 = 0$  in the full model but eliminate  $r_2$  instead.
- (d) A numerical ODE solver like Octave can solve the full set of four ODEs in part 5.1a as easily as the set of three ODEs in part 5.1c. Why might you prefer the simplified model anyway? Be specific.

### Exercise 5.2: Nitrogen dioxide reaction mechanism

The following overall stoichiometry has been observed for the decomposition of gaseous dinitrogen pentoxide to gaseous nitrogen dioxide and oxygen [28].



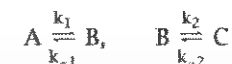
Consider the following proposed mechanism to explain this stoichiometry.



- (a) Write down the production rate of each species in the mechanism.
- (b) After making the quasi-steady-state assumption (QSSA) for the intermediates, write down the production rate of the reactants and products.
- (c) How many rate constants would you need to determine experimentally to use the full model to predict the concentrations of  $\text{N}_2\text{O}_5$ ,  $\text{NO}_2$  and  $\text{O}_2$  in the reactor? How many rate constants would you need to determine experimentally to use the QSSA model to predict these same concentrations?

### Exercise 5.3: Using the QSSA

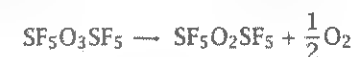
Consider the simple series reaction mechanism taking place in a constant-volume batch reactor



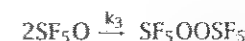
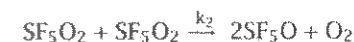
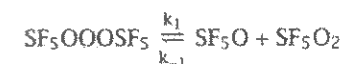
- (a) Write down the mole balance differential equations for species A, B and C. We call this set of differential equations the full model.
- (b) For what values of rate constants would you expect the QSSA to be valid for species B?
- (c) Make the QSSA on species B and write down the resulting differential equations for species A and C.
- (d) Plot  $c_A$ ,  $c_B$  and  $c_C$  versus time for the solution to the full model for initial conditions  $c_A(0) = 1$ ,  $c_B(0) = c_C(0) = 0$ , and rate constants  $k_1 = k_{-2} = 1$ ,  $k_2 = k_{-1} = 20$ . You can solve this model analytically or numerically.
- (e) Now solve the simplified model, either analytically or numerically. Make three plots, one for each species, showing  $c_j(t)$  for the full model and the simplified model. Are you satisfied using the QSSA for this situation? What advantage does the simplified model provide?

### Exercise 5.4: Applying the QSSA to develop a rate expression

The thermal decomposition of bis-pentafluorosulfurtrioxide ( $\text{SF}_5\text{O}_3\text{SF}_5$ )



is postulated to proceed by the following mechanism for oxygen partial pressures greater than 100 torr [7].



The first reaction is not at equilibrium and  $\text{SF}_5\text{O}$  and  $\text{SF}_5\text{O}_2$  are radicals. Develop an expression for the rate of bis-pentafluorosulfurtrioxide decomposition in terms of stable molecules.

**Exercise 5.5: Using QSSA to develop a simplified rate expression**

Example 4.7 presented elementary reactions for ethane pyrolysis in the presence of NO. Use the data provided in Example 4.7 to answer the following questions.

- Apply the quasi-steady state analysis to these reactions, where HNO, H, and C<sub>2</sub>H<sub>5</sub> are reaction intermediates to find the rate of ethylene production in terms of stable molecules.
- Detailed analysis of the products in a PFR reveals that the reactions can be represented with the following mass action statement



where

$$r = k_{\text{eff}} C_{\text{C}_2\text{H}_6}$$

$$k_{\text{eff}} = k_2 \sqrt{\frac{k_1 k_3 k_5}{k_4 k_6}}$$

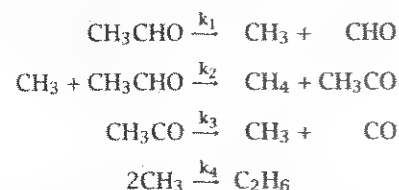
and it is necessary to follow only the concentrations of C<sub>2</sub>H<sub>6</sub>, C<sub>2</sub>H<sub>4</sub>, H<sub>2</sub> and NO.

Assume the reaction takes place in an isothermal PFR operating at constant pressure (1 atm) and constant temperature of 1100 K. The feed to the reactor consists of a mixture of ethane and NO with a molar ratio of 95% ethane and 5% NO. The inlet volumetric flow rate is 600 cm<sup>3</sup>/s. Predict the reactor volume required for 98% of the ethane to react, and determine the activation energy for  $k_{\text{eff}}$ .

- Compare the answer for this simplified model that uses  $k_{\text{eff}}$  with the full solution to this problem in which all components are followed as illustrated in Example 4.7.

**Exercise 5.6: Acetaldehyde decomposition mechanism and kinetics**

The pyrolysis of acetaldehyde, CH<sub>3</sub>CHO, has been proposed to involve the following reactions [33].

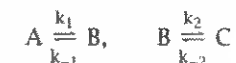


In this mechanism CH<sub>3</sub>, CH<sub>3</sub>CO and CHO are reaction intermediates. The radical CHO undergoes further reactions than are shown, but for simplicity they

are ignored here. Assuming the reaction intermediates (CH<sub>3</sub>, CH<sub>3</sub>CO and CHO) are quasi-steady-state intermediates, determine the rate of methane formation in terms of only stable molecules.

**Exercise 5.7: Two candidate reduced models**

Consider the simple series reaction mechanism taking place in a constant-volume batch reactor



- Write down the mole balance differential equations for species A, B and C. We call this set of differential equations the full model.
- For what range of rate constants would you think it is reasonable to assume the second reaction is at equilibrium compared to the first reaction?
- We write down two candidate simplified models to try to describe this situation. In the first one, set  $r_2 = 0$  and solve for  $c_C$ . Then set  $r_2 = 0$  in the remaining ODEs for  $c_A$  and  $c_B$ .
- In the second one, solve for  $c_C$  as above, but now eliminate  $r_2$  from the differential equations, rather than setting it to zero, and write down differential equations for  $c_A$  and  $c_B$ .
- Solve the three models for the following values (choose any time and concentration units you like). Feel free to solve it analytically or numerically. Plot the concentrations versus time for each model.

$$\begin{array}{llll} k_1 = 1 & k_{-1} = 0 & k_2 = 100 & k_{-2} = 200 \\ c_{A0} = 0.5 & c_{B0} = 0.5 & c_{C0} = 0 & \end{array}$$

Which simplified model correctly describes the full model in the limit of a fast second reaction?

What went wrong in the approach that didn't work?

**Exercise 5.8: Mechanisms and elementary reactions**

- What is the important difference between a reaction with an observed stoichiometry and an elementary reaction?
- List two simple tests a mechanism must pass to be considered a valid description of an overall stoichiometry.

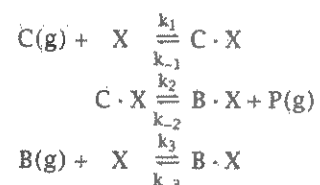
- (c) Name and describe the two major assumptions that are made to simplify the rate expressions for complex reaction networks.
- (d) Describe the transition-state concept and give a chemical example.

### Exercise 5.9: Disproportionation of cumene

The disproportionation of cumene ( $\text{C}_6\text{H}_5\text{CH}(\text{CH}_3)_2$ ) to benzene and propylene



proceeds by the following catalytic mechanism [6]:



in which C, B, P and X represent cumene, benzene, propylene and vacant sites, respectively. The  $\text{A} \cdot \text{X}$  notation means A adsorbed on the catalyst surface. Propylene does not adsorb on the surface. Develop production-rate expressions for the following two situations.

- (a) The surface reaction is rate limiting and irreversible, and the adsorption and desorption of benzene and cumene are at equilibrium. Develop the production-rate expression for propylene.
- (b) The adsorption of cumene is rate limiting and irreversible, and the adsorption and desorption of benzene and the surface reaction are at equilibrium. Develop the production-rate expression for cumene.
- (c) After glancing at the overall stoichiometry,



a colleague claims that the production rate of cumene must also be the negative of the production rates of propylene and benzene. Do you agree? Why or why not?

### Exercise 5.10: Overall stoichiometry and production rates

Consider again the disproportionation of cumene described in Exercise 5.9.

- (a) Assume you are running this reaction in a well-mixed gas-phase CSTR containing a small, solid-catalyst bed. Gas-phase cumene is fed to the reactor, and the effluent gas containing propylene, benzene, and any unreacted cumene is withdrawn. If this reactor achieves steady state, from the CSTR material balances what can you say about the relationships between the steady-state production rates of gas-phase cumene, benzene and propylene?
- (b) Assume that you run the experiment while maintaining low concentrations of adsorbed cumene and benzene compared to the gas-phase concentrations. What can you conclude about the production rates of gas-phase cumene, benzene and propylene under these conditions?
- (c) Corrigan et al. [6] used initial rates to support the mechanism given in Exercise 5.9. What simple experiments can you do to determine which of the previous two sets of assumptions actually describe the chemistry of cumene disproportionation? Explain how you would interpret the results of your proposed experiments to make this determination.

### Exercise 5.11: Fractional coverage for multicomponent adsorption

The conversion of carbon monoxide and hydrogen into synthetic fuels occurs over metals. Carbon monoxide and hydrogen compete for surface sites in the adsorption phase of the synthesis process.

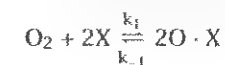
Determine the fraction of surface sites that are covered by CO and atomic hydrogen in the absence of any subsequent reactions. The adsorption reactions are:



The gas-phase conditions are  $P_{\text{CO}} = 3 \text{ atm}$ ,  $P_{\text{H}_2} = 6 \text{ atm}$  and  $T = 398 \text{ K}$ .

### Exercise 5.12: Oxygen adsorption

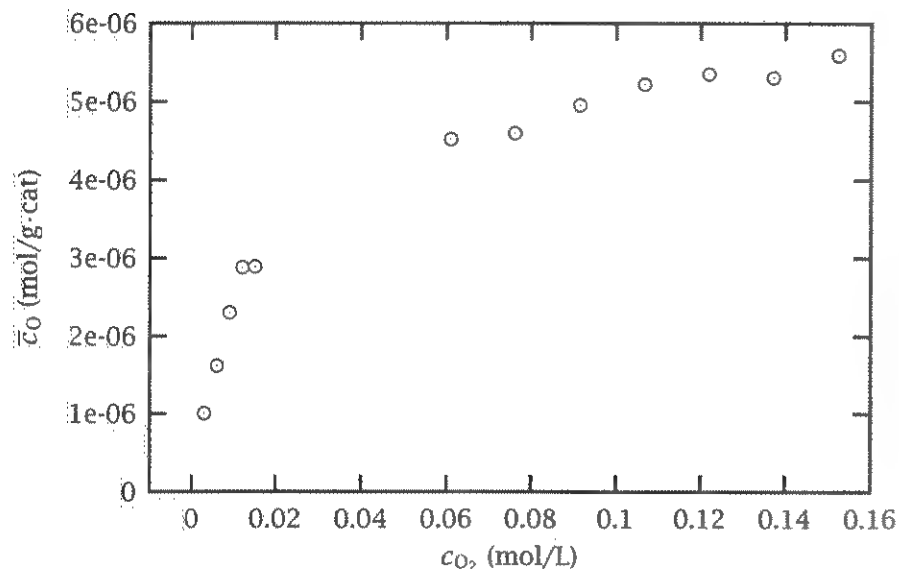
The data in Table 5.8, which are plotted in Figure 5.25, were collected for adsorption of oxygen on a Pd catalyst. The adsorption is dissociative



- (a) Derive an expression for the Langmuir isotherm to model the concentration of adsorbed oxygen in terms of the concentration of gas-phase oxygen. How many unknown parameters does your model contain?

$c_{O_2} \times 10^3$ (mol/L)	3.05	6.10	9.15	12.2	15.2	61.0	76.2	91.5	107	122	137	152
$\bar{c}_O \times 10^6$ (mol/g-cat)	1.00	1.62	2.30	2.88	2.88	4.51	4.59	4.95	5.21	5.35	5.30	5.59

**Table 5.8:** Gas-phase oxygen concentration and adsorbed oxygen concentration.



**Figure 5.25:** Adsorbed oxygen concentration versus gas-phase oxygen concentration.

- (b) Use the experimental data to obtain preliminary estimates of these parameters. You may wish to replot the data as  $1/\bar{c}_O$  versus  $1/\sqrt{c_{O_2}}$ . Don't just plug these numbers into a least-squares calculation without looking at the re-plotted data first. Table 5.8 provides the numerical values of the data plotted in Figure 5.25.

Does the Langmuir model look reasonable for these data? Explain why or why not.

We revisit these data in Exercise 9.7 after we have developed more general parameter estimation methods in Chapter 9.

### Exercise 5.13: Adsorption of propane <sup>8</sup>

Assume we are going to place a large adsorption bed underneath our automobile to store adsorbed propane, which we desorb while driving, to use as a fuel replacement for liquid gasoline. Let's assume that propane adsorbs on the adsorbent material as a Langmuir isotherm,

$$c = \frac{c_m K P}{1 + K P}$$

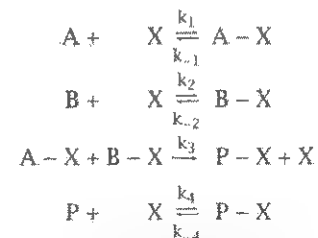
in which  $c$  is the molar concentration of adsorbed propane,  $c_m$  is the monolayer coverage,  $K$  is the adsorption equilibrium constant and  $P$  is the gas-phase propane pressure.

Assume the propane "filling station" can charge the bed with a propane supply pressure of  $P_2 = 10$  atm and the bed is exhausted at a propane partial pressure of  $P_1 = 0.1$  atm. We are interested in selecting an adsorbent material that maximizes the amount of propane that desorbs while changing from  $P_2$  to  $P_1$ .

- Write down an expression for  $\Delta c = c(P_2) - c(P_1)$ . What is  $\Delta c$  for  $K = 0$ , for  $K \rightarrow \infty$ ? Sketch a plot of  $\Delta c$  versus  $K$ .
- What is the optimal adsorbent material ( $K$  value) and what fraction of the adsorbed propane can be taken off as fuel when using this optimal adsorbent material?

### Exercise 5.14: Competitive adsorption and reaction rate

Consider the following heterogeneous reaction in which A and B associatively adsorb on a catalyst surface and undergo reaction to product P, which subsequently desorbs.



At 373 K the equilibrium constants for adsorption are

$$K_1 = 190,000 \quad K_2 = 580,000 \quad K_4 = 75,000 \quad \text{cm}^3/\text{mol}$$

The heats of adsorption are

$$\Delta H_1 = -20,000 \quad \Delta H_2 = -30,000 \quad \Delta H_4 = -12,000 \quad \text{cal/mol}$$

<sup>8</sup>This problem came from a seminar given at UT-Austin by Professor Eduardo Glandt on April 18, 1995 [25].

The rate constant for  $r_3$  is

$$k_3 = 7.6 \times 10^{28} \exp(-15000/T) \frac{(\text{gcat}^2)}{(\text{s})(\text{cm}^3)(\text{mol})}$$

where  $T$  is in Kelvin.

- Develop a Langmuir-Hinshelwood rate expression (i.e., the rate is expressed in terms of gas-phase components) for the surface reaction when you assume the surface reaction is the slow step and the adsorption and desorption processes are at equilibrium.
- Determine the magnitude of rate of the surface reaction ( $r_3$ ) at 373 K if the gas contains a 50:50 mixture of A and B at a pressure of 1.0 atm. What is the value of the rate if a 50:10:40 mixture of A:B:P is present instead?  $C_m = 0.0008 \text{ mol/g cat}$ .
- Examine the effect of changing the composition of the gas in contact with the catalyst on the rate. Plot  $r_3$  for the case where the temperature is 400 K, the total pressure is 1.0 atm, the mole fraction of P is zero, the mole fraction of A is 0.1 and the mole fraction of B is varied between 0 and 0.9. (An inert component that does not adsorb makes up the balance of the gas phase.) Discuss the shape of the curve.
- Now repeat part (c) for the 5 and 10 atm total pressure. Why is there a maximum in the rate versus composition curve at higher pressures?

### Exercise 5.15: Competitive adsorption

Consider two gas-phase components that adsorb associatively on a heterogeneous catalyst



At 373 K the equilibrium constants for adsorption are

$$K_1 = 190,000 \quad K_2 = 580,000 \quad \text{cm}^3/\text{mol}$$

The heats of adsorption are

$$\Delta H_1 = -30,000 \quad \Delta H_2 = -20,000 \quad \text{cal/mol}$$

- Examine the effect of changing the temperature from 300 to 500 K on the fractional coverage of A and B. Plot  $\Theta_A$  and  $\Theta_B$  versus  $T$  when the pressure is 1.0 atm and the gas contains an equimolar mixture of A and B. Comment on the shape of the curves. Why do the curves for the two components have the shapes they do?

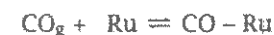
- Examine the effect of pressure on the fractional coverage of A and B. Plot  $\Theta_A$  and  $\Theta_B$  versus pressure between 0 and 3 atm when the temperature is 373 K and the gas contains an equimolar mixture of A and B. Comment on the shape of the curves. Why does one of the components appear to reach an asymptotic saturation level at a lower pressure than the other one?
- Examine the effect of the gas-phase composition on the coverage of A and B. Plot  $\Theta_A$  and  $\Theta_B$  versus the mole fraction of A as it is varied from 0 to 1 when the temperature is 373 K and the pressure is 1.0 atm. Why are the y-axis intercepts different for  $y_A = 0$  and  $y_A = 1$ ?

### Exercise 5.16: Associative versus dissociative adsorption

The active area of supported group-VIII metals can be determined by adsorbing carbon monoxide. Problems arise with the use of CO because it can adsorb associatively or dissociatively. The type of adsorption is a function of the metal type and the adsorption temperature. The following data describe the adsorption of CO on 10% Ru on  $\text{Al}_2\text{O}_3$  at 100°C.

$P_{\text{CO}}$ (torr)	CO ads ( $\mu\text{mol/g cat}$ )
100	1.28
150	1.63
200	1.77
250	1.94
300	2.06
400	2.21

- Test the data to determine if the adsorption is associative or dissociative.



- What is the concentration of total surface ruthenium atoms in terms of mol of sites/gm of catalyst?

### Exercise 5.17: Hougen-Watson kinetics

The irreversible heterogeneous catalytic reaction



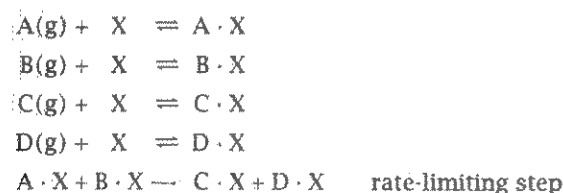
was studied over a wide range of partial pressures of reactants A and B, and products C and D. In the experimental observations reported below the partial

pressure of one component was varied while holding the partial pressures of the other three components fixed. Use the observations to establish the reaction mechanism. Specifically, which candidate reaction mechanism is consistent with all of the data. Explain your reasoning and justify your answer.

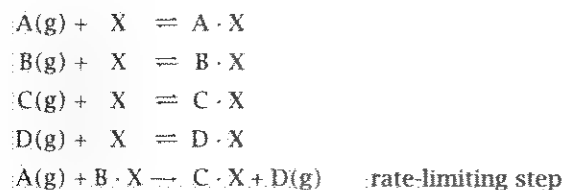
#### Observations:

Partial pressure varied	Partial pressures fixed	Characteristics of the plot of log rate versus log $P$ varied
A	B, C, D	1. At low $P_A$ the curve has a slope of 1. 2. At intermediate $P_A$ the curve has a positive slope less than 1.
B	A, C, D	1. At low $P_B$ the curve has a positive slope slightly less than 1. 2. At intermediate $P_B$ the curve has a maximum. 3. At high $P_B$ the curve has a slope of -1.
C	A, B, D	1. At low $P_C$ the curve has zero slope. 2. At high $P_C$ the curve has slope -2.

#### Mechanism 1:



#### Mechanism 2:

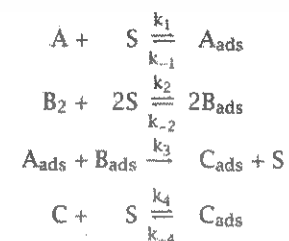


#### Exercise 5.18: Power-law approximation of Hougen-Watson kinetics

The following catalytic reaction is conducted in a 0.25 cm radius spherical pellet.



The mechanism for this reaction is shown below. A, B<sub>2</sub> and C are in adsorption-desorption equilibrium with the surface. The bimolecular surface reaction is irreversible and rate limiting.



Additional data for this problem follow.

Item	Value	Units
$K_1$	130,100	cm <sup>3</sup> /mol
$K_2$	6,500	cm <sup>3</sup> /mol
$K_4$	6,440	cm <sup>3</sup> /mol
$c_A$	$5.83 \times 10^{-5}$	mol/cm <sup>3</sup>
$c_{B_2}$	$1.40 \times 10^{-4}$	mol/cm <sup>3</sup>
$c_C$	$1.17 \times 10^{-5}$	mol/cm <sup>3</sup>
$k_3$	$7.41 \times 10^8$	g cat <sup>2</sup> /(mol cm <sup>3</sup> s)
$\bar{c}_m$	$1.8 \times 10^{-5}$	mol/g cat

- Determine the fractional surface coverages of A, B and C (i.e.,  $\theta_A$ ,  $\theta_B$  and  $\theta_C$ ) at the tabulated conditions.
- Assume you can perform experiments that would enable you to determine the rate of the surface reaction as the concentrations of A, B and C are varied plus and minus 50% from the values in the table. You then fit the rate data to the following power-law rate expression

$$r = k c_A^\alpha c_{B_2}^\beta c_C^\gamma$$

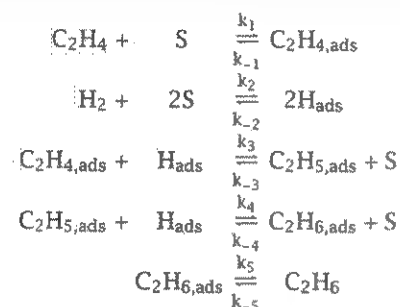
What do you expect the values of  $\alpha$ ,  $\beta$  and  $\gamma$  to be and why?

#### Exercise 5.19: Ethylene hydrogenation

The hydrogenation of ethylene to hydrogen is found experimentally to follow the reaction orders given in the following table [34].

Catalyst	Order in H <sub>2</sub>	Order in C <sub>2</sub> H <sub>4</sub>
Rh	0.85	-0.74
Ru	0.95	-0.59
Co	0.55	-0.19

The following steps have been suggested for the catalytic reaction mechanism [39, p.53]. In this mechanism ethylene adsorbs associatively and H<sub>2</sub> adsorbs dissociatively. There is a stepwise addition of adsorbed hydrogen to form an adsorbed ethyl group, C<sub>2</sub>H<sub>5,ads</sub>, and then adsorbed ethane, C<sub>2</sub>H<sub>6,ads</sub>.



where S is a vacant surface site. The rate of the formation of ethane is that of the rate-limiting surface hydrogenation step. There are two choices. For Scheme I, the rate-limiting step is Reaction 3. For Scheme II, the rate-limiting step is Reaction 4.

1. Develop a rate expression for the rate of ethane formation for Scheme I. Let Reaction 3 be the rate-limiting step and assume all other reactions are at equilibrium. Assume that adsorption from the gas phase follows a Langmuir adsorption isotherm. Neglect the reverse of Reaction 3. When performing a site balance, assume the surface is either vacant or covered with adsorbed ethylene, i.e., the coverage of ethyl, atomic hydrogen and adsorbed ethane are negligibly small.
2. Develop a rate expression for the rate of ethane formation for Scheme II. Let Reaction 4 be the rate-limiting step and assume all other reactions are at equilibrium. Assume that adsorption from the gas phase follows a Langmuir adsorption isotherm. Neglect the reverse of Reaction 4. When performing a site balance, assume the surface is either vacant or covered with adsorbed ethylene and ethyl, i.e., the coverage of atomic hydrogen and adsorbed ethane are negligibly small.
3. Based on the experimental data listed in the table, is Scheme I or II the more probable mechanism and why?

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# 6

## The Energy Balance for Chemical Reactors

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### 6.1 General Energy Balance

To specify the rates of reactions in a nonisothermal reactor, we require a model to determine the temperature of the reactor. The temperature is determined by the energy balance for the reactor. We derive the energy balance by considering an arbitrary reactor volume element, shown in Figure 6.1, as we did in deriving the material balance in Chapter 4. The volume element has inlet and outlet streams with mass flowrates  $m_0$  and  $m_1$ , respectively. In this chapter, we are again neglecting flux of mass through the surface of the volume element except at stream locations 0 and 1. The molar concentrations of component  $j$  in the two streams are given by  $c_{j0}$  and  $c_{j1}$ , and the total energies of

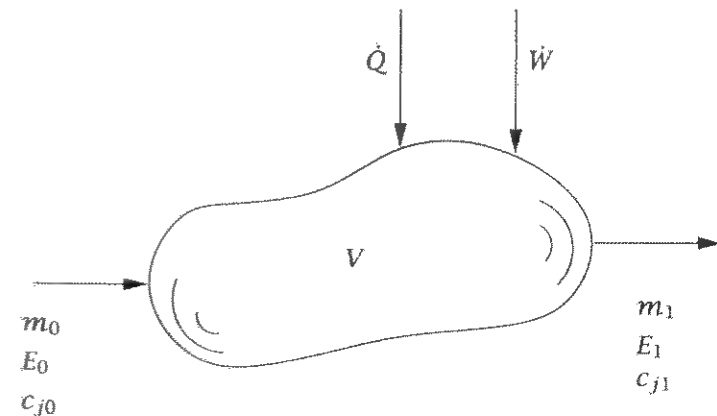


Figure 6.1: Reactor volume element.

the streams are denoted by  $E_0$  and  $E_1$ . The rate of heat added to the system is denoted by  $\dot{Q}$ , and  $\dot{W}$  is the rate of work being done on the system. Be aware that the opposite convention in which  $\dot{W}$  points out of the system in Figure 6.1 is also in use. The statement of conservation of energy for this system takes the form,

$$\left\{ \begin{array}{l} \text{rate of energy} \\ \text{accumulation} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate of energy} \\ \text{entering system} \\ \text{by inflow} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate of energy} \\ \text{leaving system} \\ \text{by outflow} \end{array} \right\} + \left\{ \begin{array}{l} \text{rate of heat} \\ \text{added to system} \end{array} \right\} + \left\{ \begin{array}{l} \text{rate of work} \\ \text{done on system} \end{array} \right\} \quad (6.1)$$

In terms of the defined variables, we can write Equation 6.1 as,

$$\frac{dE}{dt} = m_0 \hat{E}_0 - m_1 \hat{E}_1 + \dot{Q} + \dot{W} \quad (6.2)$$

in which the hat indicates an energy per unit mass.

### 6.1.1 Work Term

It is convenient to split the work term into three parts:  $\dot{W}_f$ , the work done by the flow streams while moving material into and out of the reactor,  $\dot{W}_s$ , the shaft work being done by stirrers, compressors, etc., and  $\dot{W}_b$ , the work done when moving the system boundary.

$$\underbrace{\dot{W}}_{\text{total work}} = \underbrace{\dot{W}_f}_{\text{flow streams}} + \underbrace{\dot{W}_s}_{\text{shaft work}} + \underbrace{\dot{W}_b}_{\text{boundary work}} \quad (6.3)$$

To calculate the work done by the flow streams, we assume the entering fluid has pressure  $P_0$  and a uniform velocity  $v_0$  normal to the bounding surface, and the exiting stream has pressure  $P_1$  and velocity  $v_1$ , and let  $A_0$  and  $A_1$  be the areas on the bounding surface where the streams intersect the boundary as shown in Figure 6.2. In this case  $\dot{W}_f$  is <sup>1</sup>

$$\dot{W}_f = v_0 A_0 P_0 - v_1 A_1 P_1 = Q_0 P_0 - Q_1 P_1$$

We also can express the volumetric flowrate as a mass flowrate divided by the density,  $Q = m/\rho$

$$\dot{W}_f = m_0 \frac{P_0}{\rho_0} - m_1 \frac{P_1}{\rho_1}$$

<sup>1</sup>We are neglecting any normal viscous forces, which are usually much smaller than the pressure force. Consult Bird for a more detailed development from microscopic considerations [5, 6].

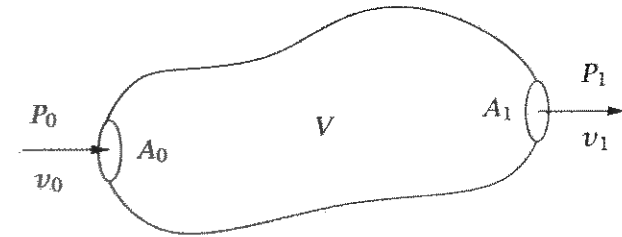


Figure 6.2: Flow streams entering and leaving the volume element.

Note that the units in the product  $mP/\rho$  are  $(m/t)(m/lt^2)/(m/l^3)$ , which gives  $(ml^2/t^2)/t$ , a unit of energy per time as we expect. The overall rate of work can then be expressed as

$$\dot{W} = \dot{W}_f + \dot{W}_s + \dot{W}_b = m_0 \frac{P_0}{\rho_0} - m_1 \frac{P_1}{\rho_1} + \dot{W}_s + \dot{W}_b \quad (6.4)$$

### 6.1.2 Energy Terms

The total energy may be regarded as composed of many forms. Obvious contributions to the total energy arise from the internal, kinetic and potential energies.<sup>2</sup>

$$\hat{E} = \hat{U} + \hat{K} + \hat{\Phi} + \dots$$

For our purposes in this chapter, we consider only these forms of energy. Recalling the definition of enthalpy,  $H = U + PV$ , or expressed on a per-unit mass basis,  $\hat{H} = \hat{U} + P/\rho$ , allows us to rewrite Equation 6.2 as

$$\frac{d}{dt} (U + K + \Phi) = m_0 (\hat{H} + \hat{K} + \hat{\Phi})_0 - m_1 (\hat{H} + \hat{K} + \hat{\Phi})_1 + \dot{Q} + \dot{W}_s + \dot{W}_b \quad (6.5)$$

## 6.2 The Batch Reactor

Since the batch reactor has no flow streams Equation 6.5 reduces to

$$\frac{d}{dt} (U + K + \Phi) = \dot{Q} + \dot{W}_s + \dot{W}_b \quad (6.6)$$

<sup>2</sup>In some cases one might need to consider also electrical and magnetic energies. For example, we might consider the motion of charged ionic species between the plates in a battery cell.

In chemical reactors, we normally assume the internal energy is the dominant contribution and neglect the kinetic and potential energies. Normally we neglect the work done by the stirrer, unless the mixture is highly viscous and the stirring operation draws significant power [13]. Neglecting kinetic and potential energies and shaft work yields

$$\frac{dU}{dt} + P \frac{dV_R}{dt} = \dot{Q} \quad (6.7)$$

in which  $\dot{W}_b = -P dV_R/dt$ . It is convenient to use enthalpy rather than internal energy in the subsequent development. Taking the differential of the definition of enthalpy gives for  $V = V_R$

$$dH = dU + P dV_R + V_R dP$$

Forming the time derivatives and substitution into Equation 6.7 gives

$$\frac{dH}{dt} - V_R \frac{dP}{dt} = \dot{Q} \quad (6.8)$$

For *single-phase systems*, we consider the enthalpy as a function of temperature, pressure and number of moles, and express its differential as

$$dH = \left( \frac{\partial H}{\partial T} \right)_{P, n_j} dT + \left( \frac{\partial H}{\partial P} \right)_{T, n_j} dP + \sum_j \left( \frac{\partial H}{\partial n_j} \right)_{T, P, n_k} dn_j \quad (6.9)$$

The first partial derivative is the definition of the heat capacity,  $C_P$ . Notice this heat capacity is the extensive heat capacity of the reactor contents. Normally we express this quantity as an intensive heat capacity times the amount of material in the reactor. We can express the intensive heat capacity on either a molar or mass basis. We choose to use the heat capacity on a mass basis, so the total heat capacity can be expressed as

$$C_P = V_R \rho \hat{C}_P$$

The second partial derivative can be expressed as

$$\left( \frac{\partial H}{\partial P} \right)_{T, n_j} = V - T \left( \frac{\partial V}{\partial T} \right)_{P, n_j} = V(1 - \alpha T)$$

in which  $\alpha = (1/V)(\partial V/\partial T)_{P, n_j}$  is the coefficient of expansion of the mixture. The partial derivatives appearing in the sum in Equation 6.9 are the partial molar enthalpies,  $H_j$

$$\left( \frac{\partial H}{\partial n_j} \right)_{T, P, n_k} = H_j$$

so Equation 6.9 can be written compactly as

$$dH = V_R \rho \hat{C}_P dT + (1 - \alpha T) V_R dP + \sum_j H_j dn_j \quad (6.10)$$

Forming the time derivatives from this expression and substituting into Equation 6.8 gives

$$V_R \rho \hat{C}_P \frac{dT}{dt} - \alpha T V_R \frac{dP}{dt} + \sum_j H_j \frac{dn_j}{dt} = \dot{Q} \quad (6.11)$$

We note that the material balance for the batch reactor is

$$\frac{dn_j}{dt} = R_j V_R = \sum_{i=1}^{n_r} v_{ij} r_i V_R, \quad j = 1, \dots, n_s \quad (6.12)$$

which upon substitution into Equation 6.11 yields

$$V_R \rho \hat{C}_P \frac{dT}{dt} - \alpha T V_R \frac{dP}{dt} = - \sum_i \Delta H_{Ri} r_i V_R + \dot{Q} \quad (6.13)$$

in which  $\Delta H_{Ri}$  is the heat of reaction

$$\Delta H_{Ri} = \sum_j v_{ij} H_j \quad (6.14)$$

We now consider several special cases. If the reactor operates at constant pressure ( $dP/dt = 0$ ) or the fluid is incompressible ( $\alpha = 0$ ), then Equation 6.13 reduces to

**Incompressible-fluid or constant-pressure reactor.**

$$V_R \rho \hat{C}_P \frac{dT}{dt} = - \sum_i \Delta H_{Ri} r_i V_R + \dot{Q} \quad (6.15)$$

To derive the constant-volume case, we consider the pressure as a function of  $T, V (V = V_R), n_j$ , and express the pressure differential as

$$dP = \left( \frac{\partial P}{\partial T} \right)_{V, n_j} dT + \left( \frac{\partial P}{\partial V} \right)_{T, n_j} dV + \sum_j \left( \frac{\partial P}{\partial n_j} \right)_{T, V, n_k} dn_j$$

For reactor operation at constant volume,  $dV = 0$ , and forming time derivatives and substituting into Equation 6.11 gives

$$\left[ V_R \rho \hat{C}_P - \alpha T V_R \left( \frac{\partial P}{\partial T} \right)_{V, n_j} \right] \frac{dT}{dt} + \sum_j \left[ H_j - \alpha T V_R \left( \frac{\partial P}{\partial n_j} \right)_{T, V, n_k} \right] \frac{dn_j}{dt} = \dot{Q}$$

We note that the first term in brackets is  $C_V = V_R \rho \hat{C}_V$  [15, p. 43]

$$V_R \rho \hat{C}_V = V_R \rho \hat{C}_P - \alpha T V_R \left( \frac{\partial P}{\partial T} \right)_{V, n_j}$$

Substitution of  $C_V$  and the material balance yields the energy balance for the constant-volume batch reactor

**Constant-volume reactor.**

$$\boxed{V_R \rho \hat{C}_V \frac{dT}{dt} = - \sum_i \left[ \Delta H_{Ri} - \alpha T V_R \sum_j v_{ij} \left( \frac{\partial P}{\partial n_j} \right)_{T, V, n_k} \right] r_i V_R + \dot{Q}} \quad (6.16)$$

If we consider an ideal gas, it is straightforward to calculate  $\alpha T = 1$  and  $(\partial P / \partial n_j)_{T, V, n_k} = RT / V$ . Substitution into the constant-volume energy balance gives

**Constant-volume reactor, ideal gas.**

$$\boxed{V_R \rho \hat{C}_V \frac{dT}{dt} = - \sum_i (\Delta H_{Ri} - RT \bar{v}_i) r_i V_R + \dot{Q}} \quad (6.17)$$

where  $\bar{v}_i = \sum_j v_{ij}$ .

If we consider the constant-volume reactor with incompressible fluid ( $\alpha = 0$ ,  $\hat{C}_V = \hat{C}_P$ ), Equation 6.16 reduces to Equation 6.15 as it should because Equation 6.15 is valid for any reactor operation with an incompressible fluid. We also notice that, in the constant-pressure case, the same energy balance applies for any fluid mixture (ideal gas, incompressible fluid, etc.), and that this balance is the same as the balance for an incompressible fluid in a constant-volume reactor. Although the same final balances are obtained for these two cases, the physical situations they describe are completely different.

Liquid-phase reactions often are carried out in batch reactors. Equation 6.15 is therefore a common batch-reactor energy balance. Two cases where Equation 6.15 can be simplified include the case of adiabatic operation ( $\dot{Q} = 0$ ) and the case of heat transfer through a jacket or cooling coil

$$\dot{Q} = U^o A (T_a - T) \quad (6.18)$$

in which  $U^o$  is the overall heat-transfer coefficient. Note that Equation 6.15 must be solved simultaneously, and usually numerically, with the batch-reactor material balance, Equation 6.12.

### Example 6.1: Constant-pressure versus constant-volume reactors

Consider the following two well-mixed, adiabatic, *gas-phase* batch reactors for the elementary and irreversible decomposition of A to B,



**Reactor 1:** The reactor volume is held constant (reactor pressure therefore changes).

**Reactor 2:** The reactor pressure is held constant (reactor volume therefore changes).

Both reactors are charged with pure A at 1.0 atm and  $k$  has the usual Arrhenius activation energy dependence on temperature,

$$k(T) = k_0 \exp(-E/T)$$

The heat of reaction,  $\Delta H_R$ , and heat capacity of the mixture,  $\hat{C}_P$ , may be assumed constant over the composition and temperature range expected.

Write the material and energy balances for these two reactors. Which reactor converts the reactant more quickly?

### Solution

**Material balance.** The material balance for component A in the batch reactor is

$$\frac{d(c_A V_R)}{dt} = R_A V_R$$

Substituting in the reaction-rate expression,  $r = k(T)c_A$ , and using the number of moles of A,  $n_A = c_A V_R$  yields

$$\frac{dn_A}{dt} = -k(T)n_A \quad (6.20)$$

Notice the temperature dependence of  $k(T)$  prevents us from solving this differential equation immediately like we did for the isothermal reactor. We must solve it simultaneously with the energy balance, which provides the information for how the temperature changes. The material balance, Equation 6.20, is the same for *both* reactors.

**Energy balance.** The energy balances for the two reactors are *not* the same. We consider first the constant-volume reactor. For the A → 2B stoichiometry, we substitute the rate expression and  $\bar{v} = 1$  into Equation 6.17 to obtain

$$C_V \frac{dT}{dt} = -(\Delta H_R - RT) kn_A$$

in which  $C_V = V_R \rho \hat{C}_V$  is the total constant-volume heat capacity.

The energy balance for the constant-pressure case follows from Equation 6.15

$$C_P \frac{dT}{dt} = -\Delta H_R kn_A$$

in which  $C_P = V_R \rho \hat{C}_P$  is the total constant-pressure heat capacity. For an ideal gas, we know from thermodynamics that the two total heat capacities are simply related,

$$C_V = C_P - nR \quad (6.21)$$

Comparing the production rates of A and B produces

$$2n_A + n_B = 2n_{A0} + n_{B0}$$

Because there is no B in the reactor initially, subtracting  $n_A$  from both sides yields for the total number of moles

$$n = n_A + n_B = 2n_{A0} - n_A$$

Substitution of the above and Equation 6.21 into the constant-volume case yields

$$\frac{dT}{dt} = -\frac{(\Delta H_R - RT) kn_A}{C_P - (2n_{A0} - n_A)R} \quad \text{constant volume} \quad (6.22)$$

and the temperature differential equation for the constant-pressure case is

$$\frac{dT}{dt} = -\frac{\Delta H_R kn_A}{C_P} \quad \text{constant pressure} \quad (6.23)$$

We see by comparing Equations 6.22 and 6.23 that the numerator in the constant-volume case is larger because  $\Delta H_R$  is negative and the positive  $RT$  is subtracted. We also see the denominator is smaller because  $C_P$  is positive and the positive  $nR$  is subtracted. Therefore the time derivative of the temperature is larger for the constant-volume case. The reaction proceeds more quickly in the constant-volume case. The

constant-pressure reactor is expending work to increase the reactor size, and this work results in a lower temperature and slower reaction rate compared to the constant-volume case. □

### Example 6.2: Liquid-phase batch reactor

The exothermic elementary liquid-phase reaction



is carried out in a batch reactor with a cooling coil to keep the reactor isothermal at 27°C. The reactor is initially charged with equal concentrations of A and B and no C,  $C_{A0} = C_{B0} = 2.0$  mol/L,  $C_{C0} = 0$ .

1. How long does it take to reach 95% conversion?
2. What is the total amount of heat (kcal) that must be removed by the cooling coil when this conversion is reached?
3. What is the maximum *rate* at which heat must be removed by the cooling coil (kcal/min) and at what time does this maximum occur?
4. What is the adiabatic temperature rise for this reactor and what is its significance?

#### Additional data:

Rate constant,  $k = 0.01725$  L/mol·min, at 27°C

Heat of reaction,  $\Delta H_R = -10$  kcal/mol A, at 27°C

Partial molar heat capacities,  $C_{PA} = C_{PB} = 20$  cal/mol·K,

$C_{PC} = 40$  cal/mol K

Reactor volume,  $V_R = 1200$  L

#### Solution

1. Assuming constant density, the material balance for component A is

$$\frac{dc_A}{dt} = -kC_A C_B$$

The stoichiometry of the reaction, and the material balance for B gives

$$C_A - C_B = C_{A0} - C_{B0} = 0$$

or  $c_A = c_B$ . Substitution into the material balance for species A gives

$$\frac{dc_A}{dt} = -kc_A^2$$

Separation of variables and integration gives

$$t = \frac{1}{k} \left[ \frac{1}{c_A} - \frac{1}{c_{A0}} \right]$$

Substituting  $c_A = 0.05c_{A0}$  and the values for  $k$  and  $c_{A0}$  gives

$$t = 551 \text{ min}$$

2. We assume the incompressible-fluid energy balance is accurate for this liquid-phase reactor. If the heat removal is manipulated to maintain constant reactor temperature, the time derivative in Equation 6.15 vanishes leaving

$$\dot{Q} = \Delta H_R r V_R \quad (6.24)$$

Substituting  $dc_A/dt = -r$  and multiplying through by  $dt$  gives

$$dQ = -\Delta H_R V_R dc_A$$

Integrating both sides gives

$$Q = -\Delta H_R V_R (c_A - c_{A0}) = -2.3 \times 10^4 \text{ kcal}$$

3. Substituting  $r = kc_A^2$  into Equation 6.24 yields

$$\dot{Q} = \Delta H_R kc_A^2 V_R$$

The right-hand side is a maximum in absolute value (note it is a negative quantity) when  $c_A$  is a maximum, which occurs for  $c_A = c_{A0}$ , giving

$$\dot{Q}_{\max} = \Delta H_R kc_{A0}^2 V_R = -828 \text{ kcal/min}$$

4. The adiabatic temperature rise is calculated from the energy balance without the heat-transfer term

$$V_R \rho \hat{C}_P \frac{dT}{dt} = -\Delta H_R r V_R$$

Substituting the material balance  $dn_A/dt = -rV_R$  gives

$$V_R \rho \hat{C}_P dT = \Delta H_R dn_A \quad (6.25)$$

Because we are given the partial molar heat capacities (see Equation 3.51 for the definition of partial molar heat capacity), it is convenient to evaluate the total heat capacity as

$$V_R \rho \hat{C}_P = \sum_{j=1}^{n_s} C_{Pj} n_j$$

For a batch reactor, the number of moles can be related to the reaction extent by  $n_j = n_{j0} + \nu_j \epsilon$ , so we can express the right-hand side of the previous equation as

$$\sum_{j=1}^{n_s} C_{Pj} n_j = \sum_j C_{Pj} n_{j0} + \epsilon \Delta C_P$$

in which  $\Delta C_P = \sum_j \nu_j C_{Pj}$ . If we assume the partial molar heat capacities are independent of temperature and composition we have  $\Delta C_P = 0$  and

$$V_R \rho \hat{C}_P = \sum_{j=1}^{n_s} C_{Pj} n_{j0}$$

Integrating Equation 6.25 with constant heat capacity gives

$$\Delta T = \frac{\Delta H_R}{\sum_j C_{Pj} n_{j0}} \Delta n_A$$

The maximum temperature rise corresponds to complete conversion of the reactants and can be computed from the given data

$$\Delta T_{\max} = \frac{-10 \times 10^3 \text{ cal/mol}}{2(2 \text{ mol/L})(20 \text{ cal/mol K})} (0 - 2 \text{ mol/L})$$

$$\Delta T_{\max} = 250 \text{ K}$$

The adiabatic temperature rise indicates the potential danger of a coolant system failure. In this case the reactants contain enough internal energy to raise the reactor temperature by 250 K.



### 6.3 The CSTR

#### Dynamic Operation

For the continuous-stirred-tank reactor (CSTR), we again assume that the internal energy is the dominant contribution to the total energy and take the entire reactor contents as the volume element. Again we denote the feed stream with flowrate  $Q_f$ , density  $\rho_f$ , enthalpy  $\hat{H}_f$ , and component  $j$  concentration  $c_{jf}$ . The outflow stream is flowing out of a well-mixed reactor and its intensive properties are therefore assumed the same as the reactor contents. Its flowrate is denoted  $Q$ . Writing Equation 6.5 for this reactor gives,

$$\frac{dU}{dt} = Q_f \rho_f \hat{H}_f - Q \rho \hat{H} + \dot{Q} + \dot{W}_s + \dot{W}_b \quad (6.26)$$

And if we neglect the shaft work

$$\frac{dU}{dt} + P \frac{dV_R}{dt} = Q_f \rho_f \hat{H}_f - Q \rho \hat{H} + \dot{Q}$$

or if we use the enthalpy rather than internal energy

$$\frac{dH}{dt} - V_R \frac{dP}{dt} = Q_f \rho_f \hat{H}_f - Q \rho \hat{H} + \dot{Q} \quad (6.27)$$

As in the batch reactor, for a single-phase system we consider the change in enthalpy due to changes in temperature, pressure and the number of moles of component  $j$ ,

$$dH = V_R \rho \hat{C}_P dT + (1 - \alpha T) V_R dP + \sum_j H_j dn_j$$

Substitution into Equation 6.27 gives

$$V_R \rho \hat{C}_P \frac{dT}{dt} - \alpha T V_R \frac{dP}{dt} + \sum_j H_j \frac{dn_j}{dt} = Q_f \rho_f \hat{H}_f - Q \rho \hat{H} + \dot{Q} \quad (6.28)$$

The material balance for the CSTR is

$$\frac{dn_j}{dt} = Q_f c_{jf} - Q c_j + \sum_i v_{ij} r_i V_R \quad (6.29)$$

Substitution into Equation 6.28 and rearrangement yields

$$V_R \rho \hat{C}_P \frac{dT}{dt} - \alpha T V_R \frac{dP}{dt} = - \sum_i \Delta H_{Ri} r_i V_R + \sum_j c_{jf} Q_f (H_{jf} - H_j) + \dot{Q} \quad (6.30)$$

Again, a variety of important special cases may be considered. These are listed in Table 6.8 in the chapter summary. A common case is the liquid-phase reaction, which usually is well approximated by the incompressible-fluid equation,

$$V_R \rho \hat{C}_P \frac{dT}{dt} = - \sum_i \Delta H_{Ri} r_i V_R + \sum_j c_{jf} Q_f (H_{jf} - H_j) + \dot{Q} \quad (6.31)$$

In the next section we consider further simplifying assumptions that require less thermodynamic data and yield useful approximations.

#### Steady-State Operation

If the CSTR is at steady state, the time derivatives in Equations 6.29 and 6.30 can be set to zero yielding,

$$Q_f c_{jf} - Q c_j + \sum_i v_{ij} r_i V_R = 0 \quad (6.32)$$

$$- \sum_i \Delta H_{Ri} r_i V_R + \sum_j c_{jf} Q_f (H_{jf} - H_j) + \dot{Q} = 0 \quad (6.33)$$

Equations 6.32 and 6.33 provide  $n_s + 1$  algebraic equations that can be solved simultaneously to obtain the steady-state concentrations and temperature in the CSTR. Note that the heats of reaction  $\Delta H_{Ri}$  are evaluated at the reactor temperature and composition.

For a liquid-phase reaction, there is a final approximation that is often useful. If the heat capacity of the liquid phase does not change significantly with composition or temperature, possibly because of the presence of a large excess of a nonreacting solvent, and we neglect the pressure effect on enthalpy, which is normally small for a liquid, we obtain

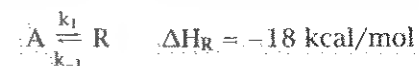
$$H_{jf} - H_j = C_{Pj} (T_f - T)$$

Substitution into Equation 6.33 gives

$$- \sum_i r_i \Delta H_{Ri} V_R + Q_f \rho_f \hat{C}_P (T_f - T) + \dot{Q} = 0 \quad (6.34)$$

#### Example 6.3: Temperature control in a CSTR

An aqueous solution of species A undergoes the following elementary reaction in a 2000 L CSTR as depicted in Figure 6.3.



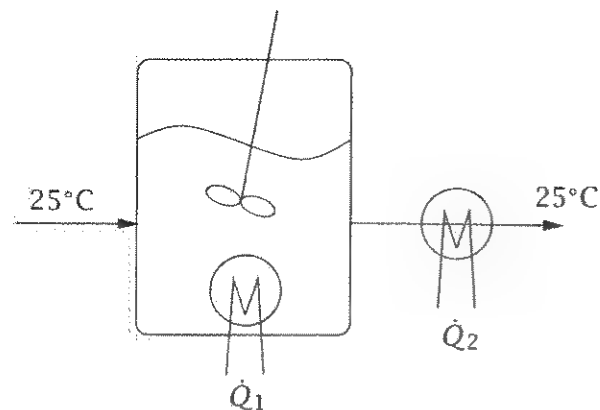


Figure 6.3: CSTR and heat exchangers.

The feed concentration,  $C_{Af}$ , is 4 mol/L and feed flowrate,  $Q_f$ , is 250 L/min. The reaction-rate constants have been determined experimentally

$$k_1 = 3 \times 10^7 e^{-5838/T} \text{ min}^{-1}$$

$$K_1 = 1.9 \times 10^{-11} e^{9059/T}$$

1. At what temperature must the reactor be operated to achieve 80% conversion?
2. What are the heat duties of the two heat exchangers if the feed enters at 25°C and the product is to be withdrawn at this temperature? The heat capacity of feed and product streams can be approximated by the heat capacity of water,  $\hat{C}_P = 1 \text{ cal/g K}$ .

### Solution

1. The steady-state material balances for components A and R in a constant-density CSTR are

$$Q(C_{Af} - C_A) - rV_R = 0$$

$$Q(C_{Rf} - C_R) + rV_R = 0$$

Adding these equations and noting  $C_{Rf} = 0$  gives

$$C_R = C_{Af} - C_A$$

Substituting this result into the rate expression gives

$$r = k_1(C_A - \frac{1}{K_1}(C_{Af} - C_A))$$

Substitution into the material balance for A gives

$$Q(C_{Af} - C_A) - k_1(C_A - \frac{1}{K_1}(C_{Af} - C_A))V_R = 0 \quad (6.35)$$

If we set  $C_A = 0.2C_{Af}$  to achieve 80% conversion, we have one equation and one unknown,  $T$ , because  $k_1$  and  $K_1$  are given functions of temperature. Solving this equation numerically gives

$$T = 334 \text{ K}$$

A word of caution is in order here. Because the reaction is reversible, we do not know if 80% conversion is achievable for any temperature when we attempt to solve Equation 6.35. It may be valuable to first make a plot of the conversion as a function of reactor temperature. If we solve Equation 6.35 for  $C_A$ , we have

$$C_A = \frac{Q/V_R + k_1/K_1}{Q/V_R + k_1(1 + 1/K_1)} C_{Af}$$

or for  $x_A = 1 - C_A/C_{Af}$

$$x_A = \frac{k_1}{Q/V_R + k_1(1 + 1/K_1)} = \frac{k_1\theta}{1 + k_1\theta(1 + 1/K_1)}$$

Figure 6.4 displays  $x_A$  versus  $T$  and we see that the conversion 80% is just reachable at 334 K, and that for any conversion lower than this value, there are two solutions.

2. A simple calculation for the heat-removal rate required to bring the reactor outflow stream from 334 K to 298 K gives

$$\begin{aligned} \dot{Q}_2 &= Q_f \hat{C}_P \Delta T \\ &= (250 \text{ L/min})(1000 \text{ g/L})(1 \text{ cal/g K})(298 - 334 \text{ K}) \\ &= -9 \times 10^3 \text{ kcal/min} \end{aligned}$$

Applying Equation 6.34 to this reactor gives

$$\begin{aligned} \dot{Q}_1 &= k_1(C_A - \frac{1}{K_1}(C_{Af} - C_A))\Delta H_R V_R - Q_f \rho \hat{C}_P (T_f - T) \\ &= -5.33 \times 10^3 \text{ kcal/min} \end{aligned}$$

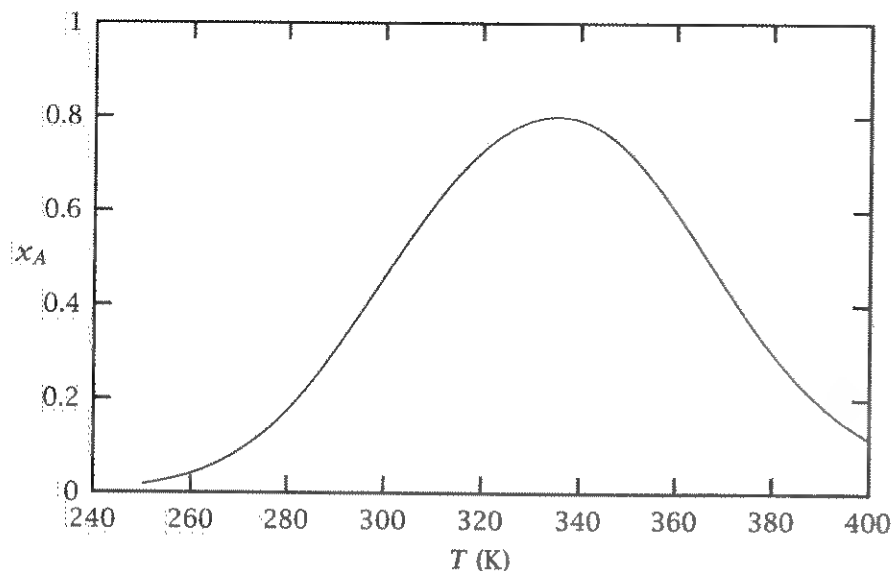
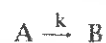


Figure 6.4: Conversion of A versus reactor temperature.

### 6.3.1 Steady-State Multiplicity

The coupling of the material and energy balances for the CSTR can give rise to some surprisingly complex and interesting behavior. Even the steady-state solution of the material and energy balances holds some surprises. In this section we explore the fact that the steady state of the CSTR is not necessarily unique. As many as three steady-state solutions to the material and energy balances may exist for even the simplest kinetic mechanisms. This phenomenon is known as steady-state multiplicity.

We introduce this topic with a simple example [24]. Consider an adiabatic, constant-volume CSTR with the following elementary reaction taking place in the liquid phase



We wish to compute the steady-state reactor conversion and temperature. The data and parameters are listed in Table 6.1.

The material balance for component A is

$$\frac{d(c_A V_R)}{dt} = Q_f c_{Af} - Q c_A + R_A V_R$$

Parameter	Value	Units
$T_f$	298	K
$T_m$	298	K
$\hat{C}_P$	4.0	kJ/kg K
$c_{Af}$	2.0	kmol/m <sup>3</sup>
$k_m$	0.001	min <sup>-1</sup>
$E$	$8.0 \times 10^3$	K
$\rho$	$10^3$	kg/m <sup>3</sup>
$\Delta H_R$	$-3.0 \times 10^5$	kJ/kmol
$U^o$	0	

Table 6.1: Parameter values for multiple steady states.

The production rate is given by

$$R_A = -k(T)c_A$$

For the steady-state reactor with constant-density, liquid-phase streams, the material balance simplifies to

$$0 = c_{Af} - (1 + k\theta)c_A \quad (6.36)$$

Equation 6.36 is one nonlinear algebraic equation in two unknowns:  $c_A$  and  $T$ . The temperature appears in the rate-constant function,

$$k(T) = k_m e^{-E(1/T - 1/T_m)}$$

A second equation is provided by the energy balance, and, in this example, we assume the heat capacity of the mixture is constant and independent of composition and temperature. With these assumptions, the steady-state energy balance reduces to

$$0 = -kc_A \Delta H_R V_R + Q_f \rho_f \hat{C}_P (T_f - T) + U^o A (T_a - T)$$

Dividing through by  $V_R$  and noting  $U^o = 0$  for the adiabatic reactor gives

$$0 = -kc_A \Delta H_R + \frac{C_{Ps}}{\theta} (T_f - T) \quad (6.37)$$

in which  $C_{Ps} = \rho_f \hat{C}_P$ , a heat capacity per volume. The solution of Equations 6.36 and 6.37 for  $c_A$  and  $T$  provide the steady-state CSTR solution. The parameters appearing in the problem are:  $c_{Af}$ ,  $T_f$ ,  $\theta$ ,  $C_{Ps}$ ,  $k_m$ ,  $T_m$ ,  $E$ ,  $\Delta H_R$ . We wish to study this solution as a function of one of these

parameters,  $\theta$ , the reactor residence time. We first examine the isothermal case. Assume the heat of reaction is zero. Equation 6.37 then implies  $T = T_f$  for all residence times. In this case, we already have shown that

$$C_A = \frac{C_{Af}}{1 + k\theta}, \quad x = \frac{k\theta}{1 + k\theta}$$

The  $x$  and  $T$  values for this case are plotted versus residence time as the  $\Delta H_R = 0$  lines in Figures 6.5 and 6.6. This case is much as we expect, the conversion of A increases monotonically with residence time because the reactant molecules have more time to react at longer residence times. Consider the endothermic case,  $\Delta H_R = +5 \times 10^4$  kJ/kmol. Because the heat of reaction is positive, the steady-state temperature decreases with increasing conversion and residence time. The lower temperature decreases the rate constant and one has to operate the reactor at significantly longer residence times to achieve the same conversion as the isothermal case as shown in Figure 6.5. Now consider the solution for exothermic reaction, plotted in Figures 6.5 and 6.6, for  $\Delta H_R = -5, -10, -20, -30 \times 10^4$  kJ/kmol. In these cases, temperature increases with conversion and hence residence time. The rate constant increases significantly with temperature, and higher conversions are achieved at smaller residence times as the exothermicity increases. Note that if the heat of reaction is more exothermic than  $-10$  kJ/kmol, there is a range of residence times in which there is not one but several steady-state solutions, three solutions in this case. The reactor is said to exhibit **steady-state multiplicity** for these values of residence time. For intermediate values of the heat of reaction, one can see the formation of the s-shaped steady-state temperature curves. We also should note that we have assumed the reaction mixture remains in the liquid phase at these temperatures and the reactor pressure.

The points at which the steady-state curves turn are known as **ignition** and **extinction** points. Figures 6.7 and 6.8 show the  $\Delta H_R = -3 \times 10^5$  kJ/kmol case again, without a log scale on the residence-time axis. Consider a reactor at steady state for a small value of residence time, 10 min, at low conversion of A and low temperature. If the feed flowrate were decreased slightly ( $\theta$  increased), there would be a small upset and the reactor would increase in conversion and temperature as it approached the new steady state at the new residence time. Consider the situation at the ignition point, however. Let the reactor be at steady state for  $\theta = 30.9$  min at  $x = 0.09$  and  $T = 311$  K. If there is a small decrease in feed flowrate so the residence time is 32 min, there is no

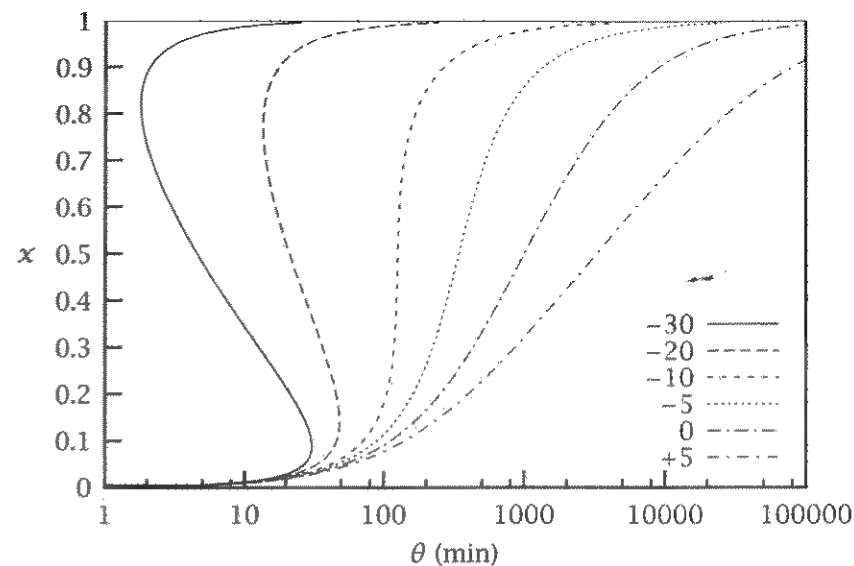


Figure 6.5: Steady-state conversion versus residence time for different values of the heat of reaction ( $\Delta H_R \times 10^{-4}$  kJ/kmol).

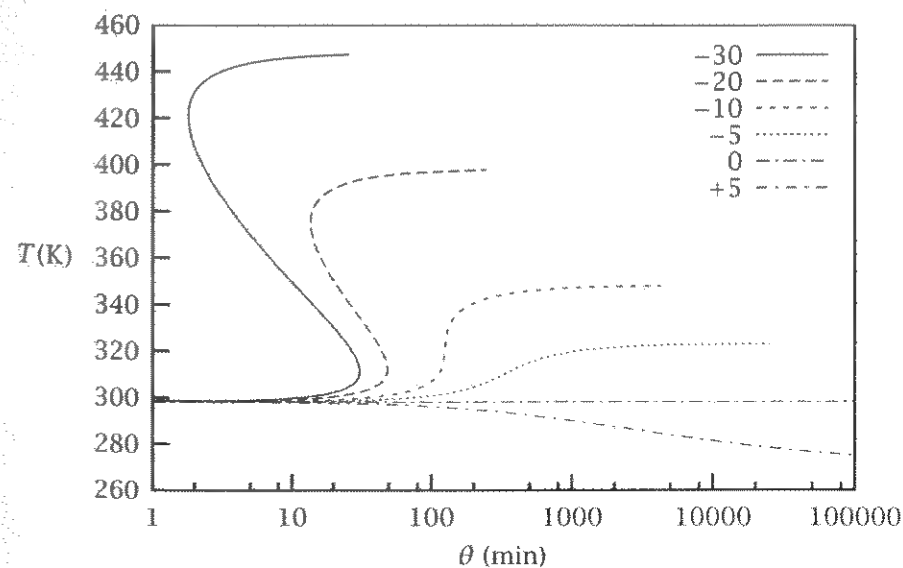


Figure 6.6: Steady-state temperature versus residence time for different values of the heat of reaction ( $\Delta H_R \times 10^{-4}$  kJ/kmol).

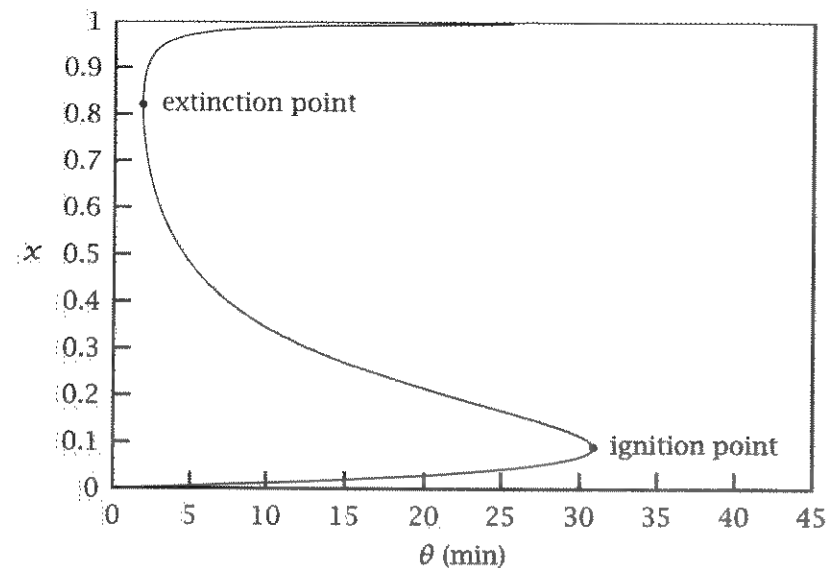


Figure 6.7: Steady-state conversion versus residence time for  $\Delta H_R = -3 \times 10^5$  kJ/kmol; ignition and extinction points.

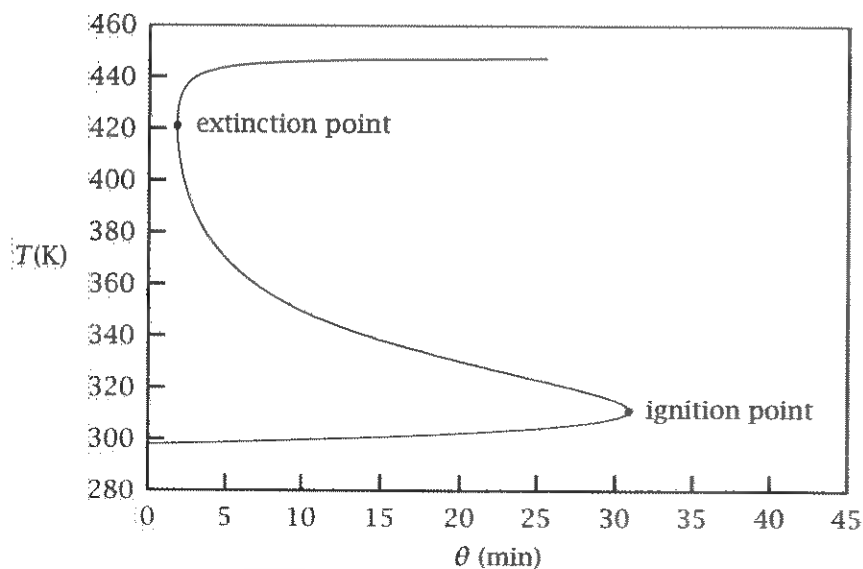


Figure 6.8: Steady-state temperature versus residence time for  $\Delta H_R = -3 \times 10^5$  kJ/kmol; ignition and extinction points.

steady state near the temperature and concentration of the reactor. A large release of heat occurs and the reactor ignites and moves to the steady state near  $x = 1$  and  $T = 448$  K. We can compute this maximal temperature corresponding to complete conversion as follows.

$$T_{\max} - T_f = -\frac{C_{A0}\Delta H_R}{C_{Ps}} = 150 \text{ K}$$

A reactor operating near the extinction point can exhibit the opposite phenomenon. A small increase in feed flowrate causes the residence time to decrease enough so that no steady-state solution exists near the current temperature and concentration. A rapid drop in temperature and increase in concentration of A occurs as the reactor approaches the new steady state. The curves of low temperature and low conversion connecting to the ignition point are known as the **lower branch** of solutions. The curves connecting the extinction point to the high temperature and conversion at large residence time are known as the **upper branch**. The curves connecting the ignition and extinction points together are called the **middle branch**. The points on the middle branch, while satisfying the steady-state equations, are **unstable** solutions. An unstable steady-state solution has the property that small perturbations to a reactor operating at this steady state causes the reactor to leave this solution and approach either the steady state on the upper or lower branch. The choice of which branch the reactor approaches depends on the exact nature of the perturbation the reactor experiences. This situation is much the same as a pencil initially balanced on end on a desk. Which direction the pencil falls is not easily predicted and depends in a sensitive way on the actual perturbation the pencil experiences. The upper and lower branches, on the other hand, are **stable** steady states. The reactor returns to these steady states after small perturbations.

### 6.3.2 Stability of the Steady State

We next discuss why some steady states are stable and others are unstable. This discussion comes in two parts. First we present a plausibility argument and develop some physical intuition by constructing and examining van Heerden diagrams [25]. In the next section we present a rigorous mathematical argument, which has wide applicability in analyzing the stability of any system described by differential equations.

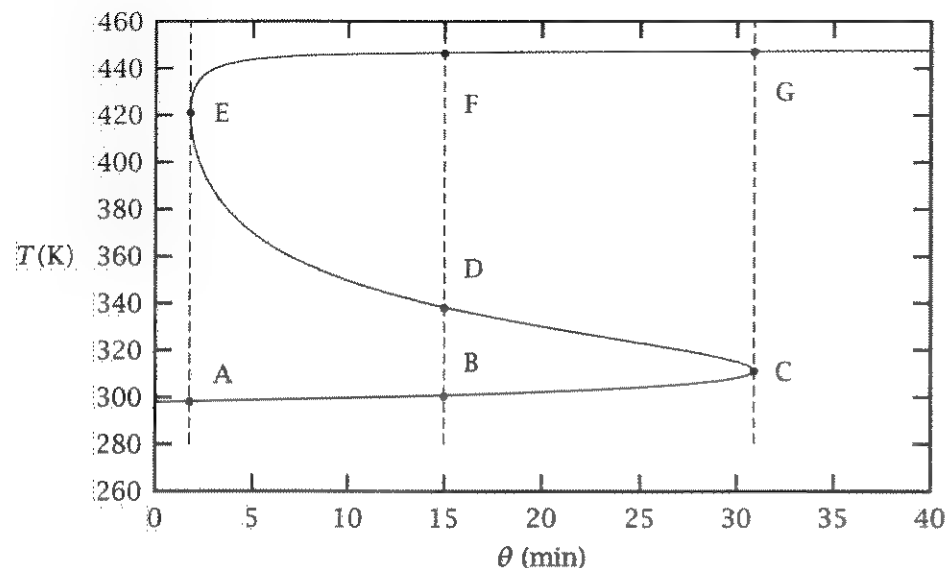


Figure 6.9: Steady-state temperature versus residence time for  $\Delta H_R = -3 \times 10^5$  kJ/kmol.

Point	A	B	C	D	E	F	G
$\theta$ (min)	1.79	15	30.9	15	1.79	15	30.9
$T$ (K)	298.2	300.5	311	338	421	446.5	447

Table 6.2: Selected values of steady-state temperatures and residence times.

### Reactor Stability — Plausibility Argument

Consider again the steady-state temperature versus the residence-time curve shown in Figure 6.9. We have labeled seven points on this curve: A, B and C on the lower branch; D on the middle branch; and E, F and G on the upper branch. The numerical values at these points are listed in Table 6.2. If we first substitute the solution of the mass balance, Equation 6.36, for  $c_A$  into the energy balance, Equation 6.37, we obtain

a single equation for the single unknown  $T$

$$0 = \underbrace{-\frac{k}{1+k\theta} c_{Af} \Delta H_R}_{\dot{Q}_g} + \underbrace{\frac{C_{Ps}}{\theta} (T_f - T)}_{\dot{Q}_r} \quad (6.38)$$

The first term, the reaction rate times the negative of the heat of reaction, we call the heat-generation rate,  $\dot{Q}_g$ . The second term is the enthalpy difference between the inflow and outflow streams, which we call the heat-removal rate,  $\dot{Q}_r$ .

$$\dot{Q}_g = -\frac{k(T)}{1+k(T)\theta} c_{Af} \Delta H_R, \quad \dot{Q}_r = \frac{C_{Ps}}{\theta} (T - T_f)$$

in which we emphasize the temperature dependence of the rate constant,

$$k(T) = k_m e^{-E(1/T - 1/T_m)}$$

Obviously we have a steady-state solution when these two quantities are equal. Consider plotting these two functions as  $T$  varies. The heat-removal rate is simply a straight line with slope  $C_{Ps}/\theta$ . The heat-generation rate is a nonlinear function that is asymptotically constant at low temperatures ( $k(T)$  much less than one) and high temperatures ( $k(T)$  much greater than one). These two functions are plotted for  $\theta = 1.79$  min in Figure 6.10. Notice the two intersections of the heat-generation and heat-removal functions corresponding to steady states A and E. If we decrease the residence time slightly, the slope of the heat-removal line increases and the intersection corresponding to point A shifts slightly. Because the two curves are just tangent at point E, however, the solution at point E disappears, another indicator that point E is an extinction point.

We also can make a plausibility argument on the stability of steady-state A. If we were to increase the reactor temperature slightly, we would be to the right of point A in Figure 6.10. To the right of A we notice that the heat-removal rate is larger than the heat-generation rate. That causes the reactor to cool, which moves the temperature back to the left. In other words, the system responds by resisting our applied perturbation. Similarly, consider a decrease to the reactor temperature. To the left of point A, the heat-generation rate is larger than the heat-removal rate causing the reactor to heat up and move back to the right. Point A is a stable steady state because small perturbations are rejected by the system. Using this argument we see the situation at point E is

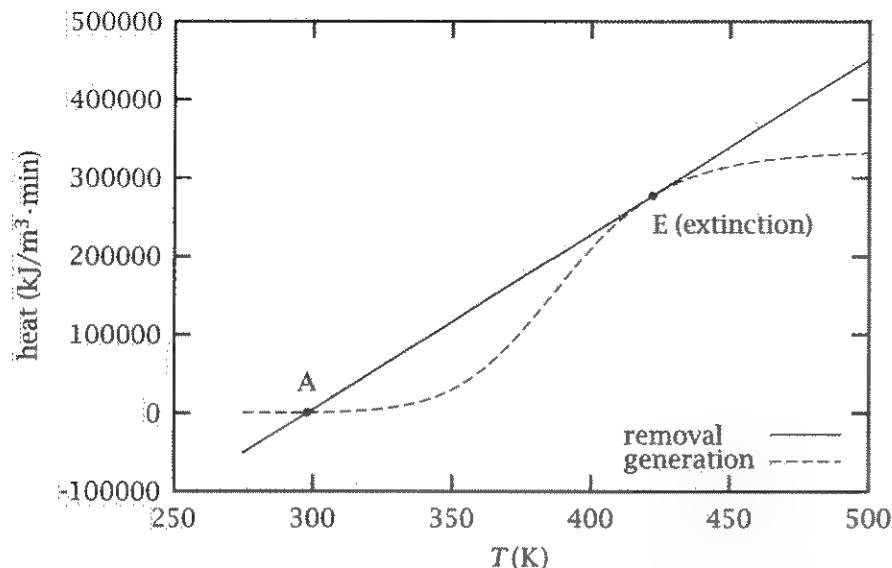


Figure 6.10: Rates of heat generation and removal for  $\theta = 1.79$  min.

quite different from the stability perspective. Increasing the temperature causes the system to cool, but decreasing the temperature also causes the system to cool. The system tends to increase this initial perturbation and does not return to the steady state. That is characteristic of an unstable solution.

Consider next the points on the middle branch. Figure 6.11 displays the heat-generation and heat-removal rates for points B, D and F,  $\theta = 15$  min. Point B on the lower branch is stable as was point A. Point F on the upper branch also is stable because the slope of the heat-generation rate is smaller than the heat-removal rate at point F, just as at point A. At point D, however, the slope of the heat-generation rate is larger than the heat-removal rate. For point D, increasing temperature causes heat generation to be larger than heat removal, and decreasing temperature causes heat generation to be smaller than heat removal. Both of these perturbations are amplified by the system at point D, and this solution is unstable. All points on the middle branch are similar to point D.

Figure 6.12 displays the heat-generation and heat-removal rates for  $\theta = 30.9$  min. Notice that point G on the upper branch is stable and point C, the ignition point, is similar to extinction point E, perturba-

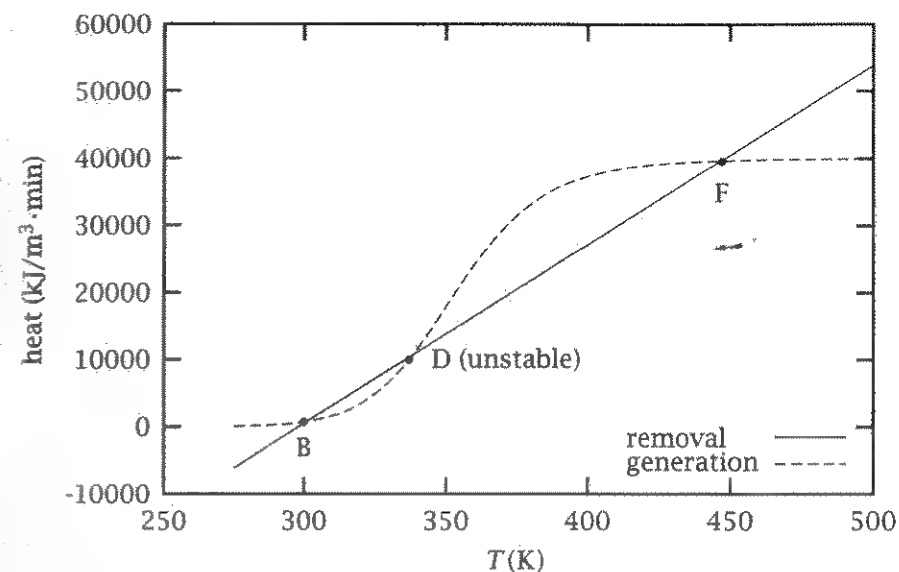


Figure 6.11: Rates of heat generation and removal for  $\theta = 15$  min.

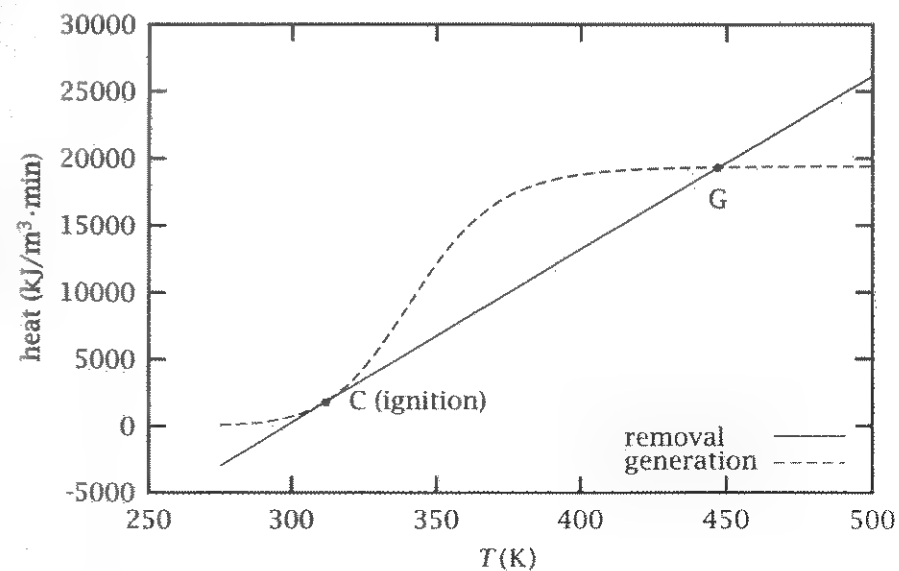


Figure 6.12: Rates of heat generation and removal for  $\theta = 30.9$  min.



tions in one direction are rejected, but in the other direction they are amplified.

### Reactor Stability — Rigorous Argument

The stability discussion of the previous section accompanying the van Heerden diagrams is useful and provides insight, particularly in simple situations. In this section we develop proper arguments that determine stability rigorously in complex situations where physical intuition may not be as readily forthcoming. We also develop analysis tools that tell us more than just whether a steady state is stable or unstable; we also gain insight on the dynamic response of the system to perturbations.

Consider again the dynamic model of the reactor

$$\frac{dc_A}{dt} = \frac{c_{Af} - c_A}{\theta} - kc_A \quad (6.39)$$

$$\frac{dT}{dt} = \frac{U^o}{C_{Ps}}(T_a - T) + \frac{T_f - T}{\theta} - \frac{\Delta H_R}{C_{Ps}}kc_A \quad (6.40)$$

For convenience we summarize these two equations as the following nonlinear differential equations

$$\begin{aligned} \frac{dc_A}{dt} &= f_1(c_A, T) \\ \frac{dT}{dt} &= f_2(c_A, T) \end{aligned} \quad (6.41)$$

Given some values of parameters, we define steady-state solutions to these equations,  $c_{As}, T_s$ , as satisfying

$$0 = f_1(c_{As}, T_s)$$

$$0 = f_2(c_{As}, T_s)$$

Imagine the reactor is initially at this steady state and at  $t = 0$  we perturb the temperature and concentration by small amounts. We would like to know whether or not the system returns to the steady state after this initial condition perturbation. If so, we call the steady-state solution (asymptotically) stable. If not, we call the steady state unstable. Obviously we can solve numerically the nonlinear differential equations to answer this question, but then we answer the question on a case-by-case basis. By linearizing the nonlinear differential equations, we can gain further insight without resorting to full numerical solution. Consider the Taylor series expansion of the nonlinear functions  $f_1, f_2$

around the steady state

$$f_1(c_A, T) = f_1(c_{As}, T_s) + \frac{\partial f_1}{\partial c_A}(c_A - c_{As}) + \frac{\partial f_1}{\partial T}(T - T_s) + \text{higher-order terms}$$

$$f_2(c_A, T) = f_2(c_{As}, T_s) + \frac{\partial f_2}{\partial c_A}(c_A - c_{As}) + \frac{\partial f_2}{\partial T}(T - T_s) + \text{higher-order terms}$$

Define deviation variables,  $\epsilon_1, \epsilon_2$ , that measure how far the system is from the steady state

$$\epsilon_1 = c_A - c_{As}$$

$$\epsilon_2 = T - T_s$$

We can differentiate these deviation variables to see that they also satisfy Equation 6.41. Substituting the Taylor series expansion for  $f_1, f_2$  and neglecting the higher-order terms, which are valid when the perturbations are small, produce the following approximate linear equations for the deviation variables

$$\frac{d\epsilon_1}{dt} = \frac{\partial f_1}{\partial c_A}\epsilon_1 + \frac{\partial f_1}{\partial T}\epsilon_2$$

$$\frac{d\epsilon_2}{dt} = \frac{\partial f_2}{\partial c_A}\epsilon_1 + \frac{\partial f_2}{\partial T}\epsilon_2$$

If we define the Jacobian matrix of  $f_1, f_2$  with respect to  $c_A, T$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial c_A} & \frac{\partial f_1}{\partial T} \\ \frac{\partial f_2}{\partial c_A} & \frac{\partial f_2}{\partial T} \end{bmatrix} \quad (6.42)$$

we can compactly summarize the linear deviation variable equations in the convenient notation

$$\frac{d\epsilon}{dt} = J\epsilon \quad (6.43)$$

For our case, taking partial derivatives shows the Jacobian is

$$J = \begin{bmatrix} -\frac{1}{\theta} - k & -kc_{As}\frac{E}{T_s^2} \\ -\frac{\Delta H_R}{C_{Ps}}k & -\frac{U^o}{C_{Ps}} - \frac{1}{\theta} - \frac{\Delta H_R}{C_{Ps}}kc_{As}\frac{E}{T_s^2} \end{bmatrix} \quad (6.44)$$

If Equation 6.43 governed exactly, rather than approximately, the evolution of the perturbations, we would be essentially finished. Linear

differential equations can be solved analytically. The important feature for us to note is that the solution contains linear combinations of the terms  $e^{\lambda t}$  in which  $\lambda$  are the eigenvalues of  $J$ . In our case the  $2 \times 2$   $J$  matrix has two eigenvalues, which may be real or complex. If complex, they come as complex conjugate pairs. Stability of the linear equations is determined by the real parts of the eigenvalues. In fact,

*The solution to the linear model, Equation 6.43, is asymptotically stable if and only if the eigenvalues of  $J$  have strictly negative real parts.*

We also know that if the eigenvalues are complex, the solution to Equation 6.43 contains oscillatory terms due to the relation

$$e^{(a+bi)t} = e^{at} (\cos(bt) + i \sin(bt))$$

and the period of these oscillatory terms is

$$T = 2\pi/b \quad (6.45)$$

Finally, this linear problem is only important to us because it can be connected to the solution of the full nonlinear problem. Although precise statements of this connection require more mathematics than we can assume, the general result may be summed up for "nice" systems by saying [11]

*For small initial perturbations from the steady state, the local behavior of the nonlinear model, Equation 6.41, is well described by the corresponding behavior of the linear model, Equation 6.43.*

The interested reader should see the first chapter in Guckenheimer and Holmes [11] for an introduction into the mathematical details and important restrictions on this general statement. For the models analyzed in this text, this general statement is valid. This connection forms the basis for all linear stability analysis.

We next use this result to analyze the multiple steady states of the previous section. In Figure 6.13 we plot the eigenvalues of  $J$  versus the steady-state reactor temperature and label the points A-G as indicated in Table 6.2 and shown in Figure 6.9. Both eigenvalues are real for all values of residence time. Notice that the steady state loses stability at point C, the ignition point, as one of the eigenvalues changes sign from negative to positive. Stability is regained on the upper branch at

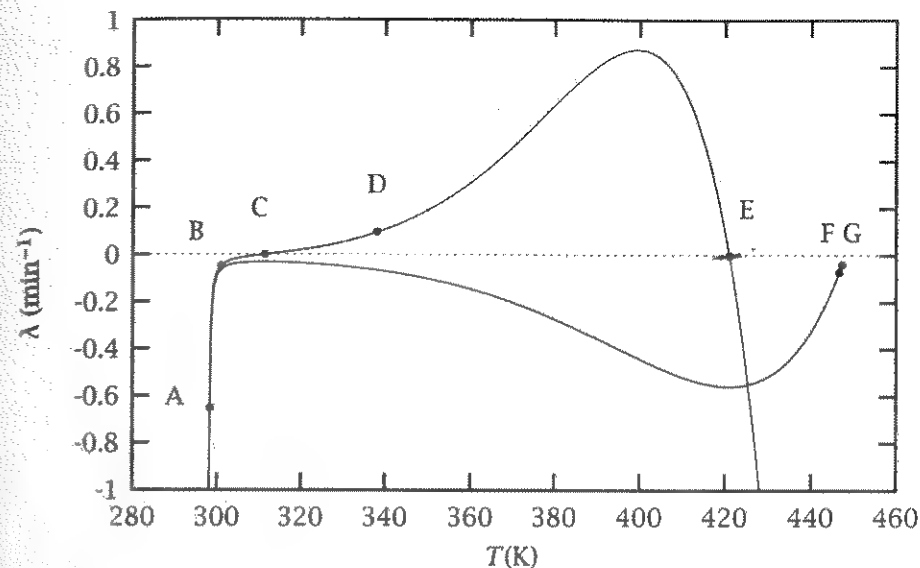


Figure 6.13: Eigenvalues of the Jacobian matrix versus reactor temperature.

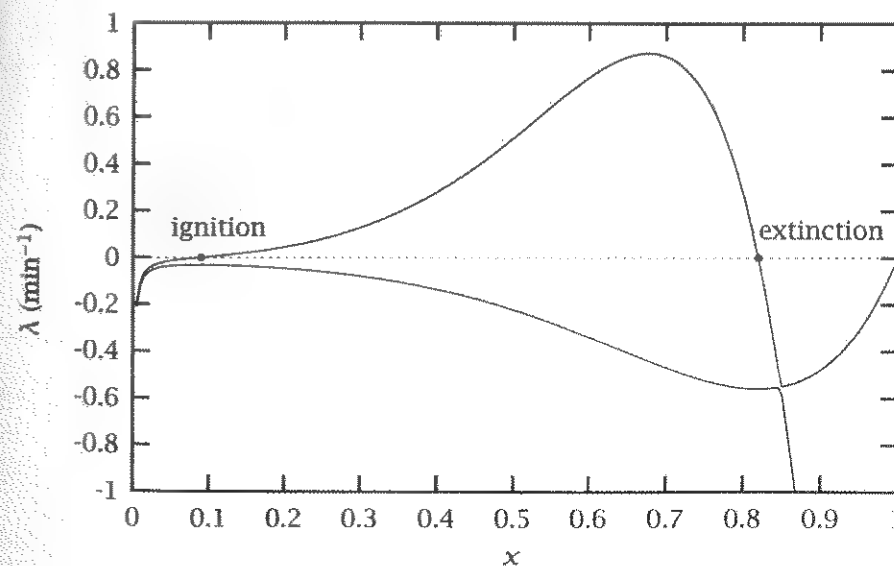
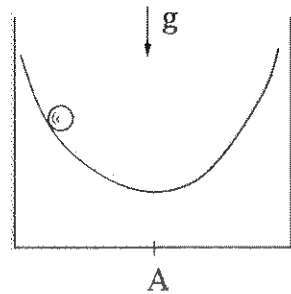
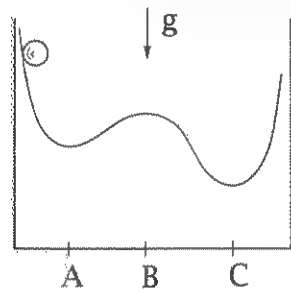


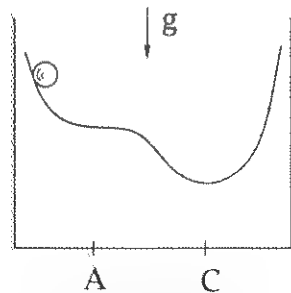
Figure 6.14: Eigenvalues of the Jacobian matrix versus reactor conversion.



**Figure 6.15:** Marble on a track in a gravitational field; point A is the unique, stable steady state.



**Figure 6.16:** Marble on a track with three steady states; points A and C are stable, and point B is unstable.



**Figure 6.17:** Marble on a track with an ignition point (A) and a stable steady state (C).

point E, the extinction point, as the eigenvalue changes sign back from positive to negative. This analysis validates the plausibility argument and shows the entire middle branch is unstable. Figure 6.14 provides another view of the same phenomenon, in which the eigenvalues are plotted versus the conversion instead of temperature; the ignition and extinction points are labeled.

Students may find it helpful to draw an analogy between the chemical reactor with multiple steady states and simple mechanical systems that exhibit the same behavior. Consider a marble on a track in a gravitational field as depicted in Figure 6.15. Based on our physical experience with such systems we conclude immediately that the system has a single steady state, position A, which is asymptotically stable. If we expressed Newton's laws of motion for this system, and linearized the model at point A, we would expect to see eigenvalues with negative real part and nonzero imaginary part because the system exhibits a decaying oscillation back to the steady-state position after a perturbation. The oscillation decays because of the friction between the marble and the track.

Now consider the track depicted in Figure 6.16. Here we have three steady states, the three positions where the tangent curve to

the track has zero slope. This situation is analogous to the chemical reactor with multiple steady states. The steady states A and C are obviously stable and B is unstable. Perturbations from point B to the right are attracted to steady-state C and perturbations to the left are attracted to steady-state A. The significant difference between the reactor and marble systems is that the marble decays to steady state in an oscillatory fashion, and the reactor, with its zero imaginary eigenvalues, returns to the steady state without overshoot or oscillation. Now consider the track depicted in Figure 6.17. We have flattened the track between points A and B in Figure 6.16 so there is just a single point of zero slope, now marked point A. Point A now corresponds to a reactor ignition point as shown in Figures 6.7 and 6.8. Small perturbations push the marble over to the only remaining steady state, point C, which remains stable.

### 6.3.3 Sustained Oscillations, Limit Cycles

We next explore the fact that the dynamic behavior of the CSTR can be more complicated than multiple steady states with ignition, extinction and hysteresis. In fact, at a given operating condition, all steady states may be unstable and the reactor may exhibit sustained oscillations or limit cycles. Consider the same simple kinetic scheme as in the previous section,



but with the following parameter values. Notice that the activation energy in Table 6.3 is significantly larger than in Table 6.1. If we compute the solutions to the steady-state mass and energy balances, Equations 6.36 and 6.37, with these new values of parameters, we obtain the results displayed in Figures 6.18 and 6.19. If we replot these results using a log scaling to stretch out the x-axis, we obtain the results in Figures 6.20 and 6.21. Notice the steady-state solution curve has become a bit deformed and the simple s-shaped multiplicities in Figures 6.7

Param.	Value	Units
$T_f$	298	K
$T_m$	298	K
$\hat{C}_p$	4.0	kJ/kg K
$c_{Af}$	2.0	kmol/m <sup>3</sup>
$k_m(T_m)$	0.004	min <sup>-1</sup>
$E$	$1.5 \times 10^4$	K
$\rho$	$10^3$	kg/m <sup>3</sup>
$\Delta H_R$	$-2.2 \times 10^5$	kJ/kmol

**Table 6.3:** Parameter values for limit cycles.

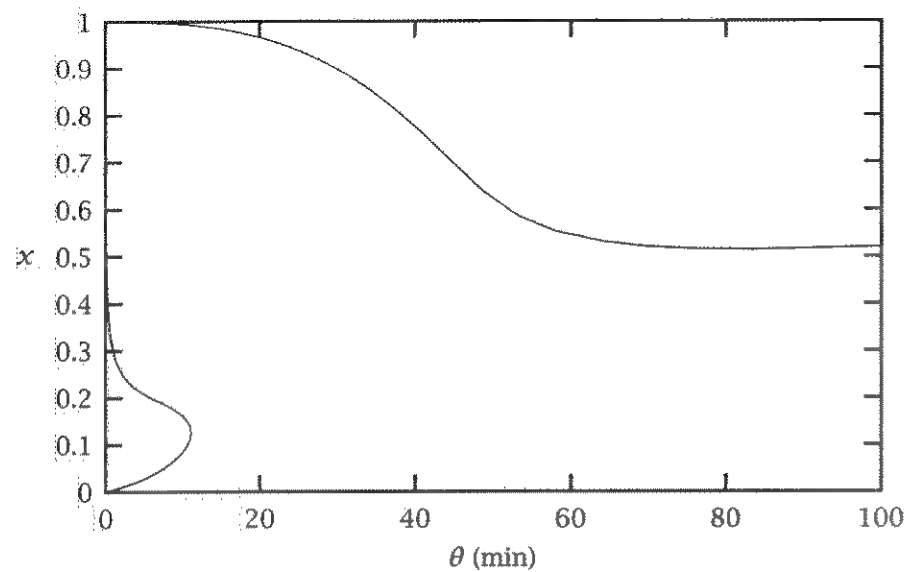


Figure 6.18: Steady-state conversion versus residence time.

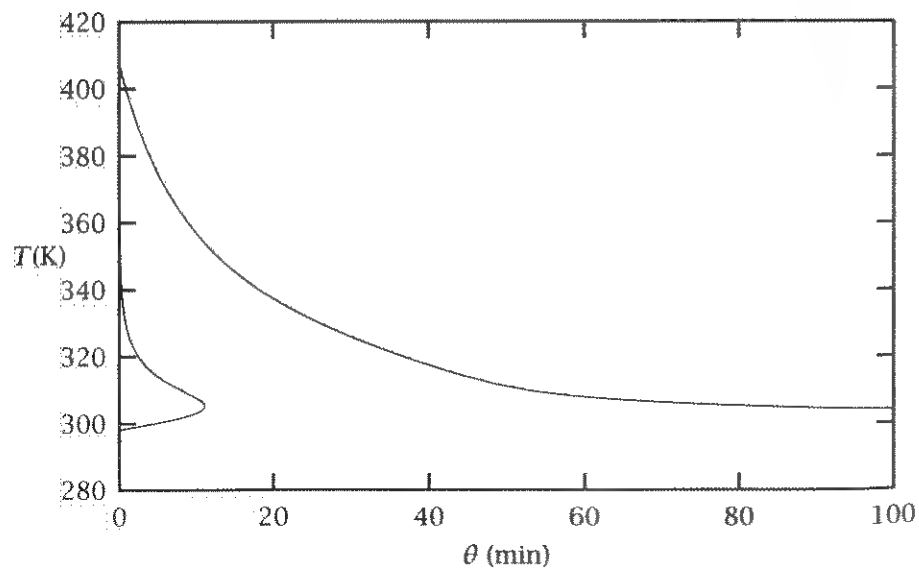


Figure 6.19: Steady-state temperature versus residence time.

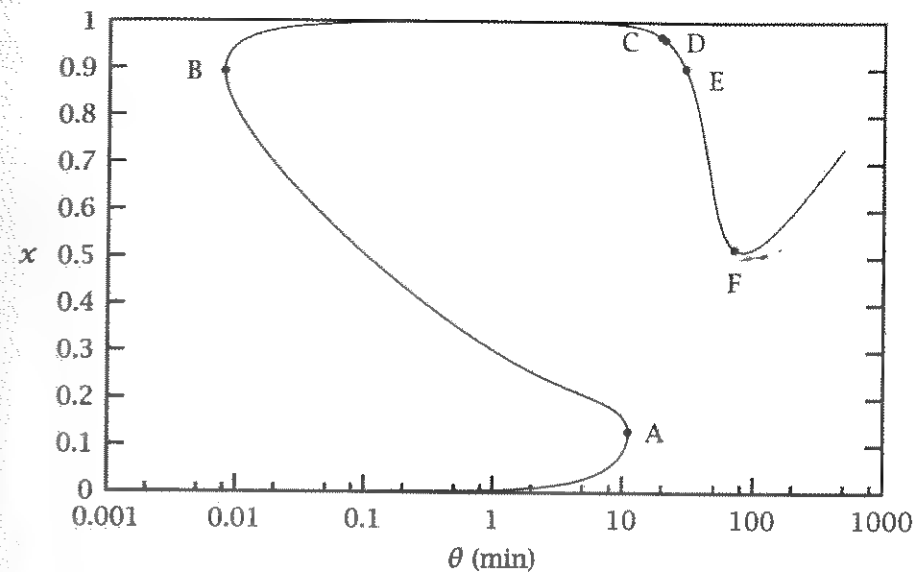


Figure 6.20: Steady-state conversion versus residence time — log scale.

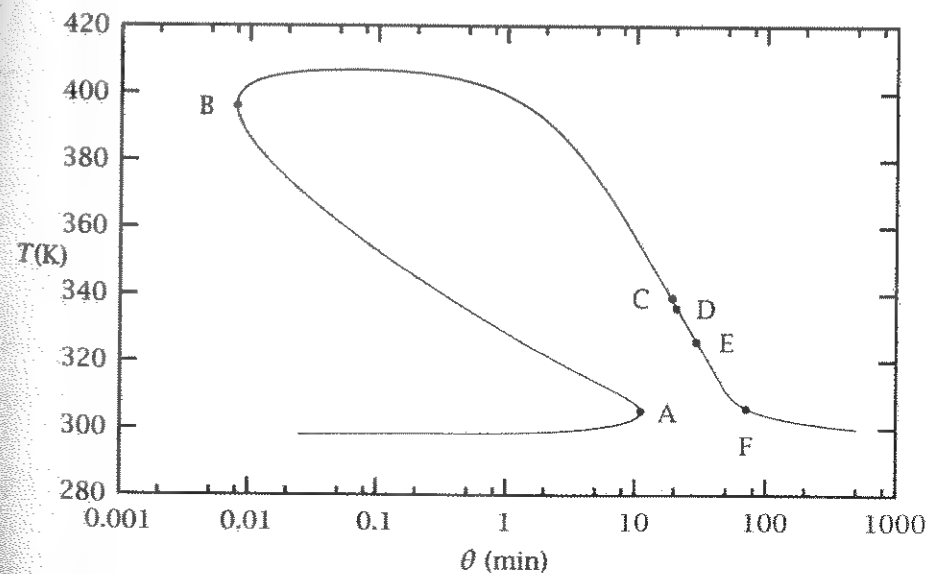


Figure 6.21: Steady-state temperature versus residence time — log scale.

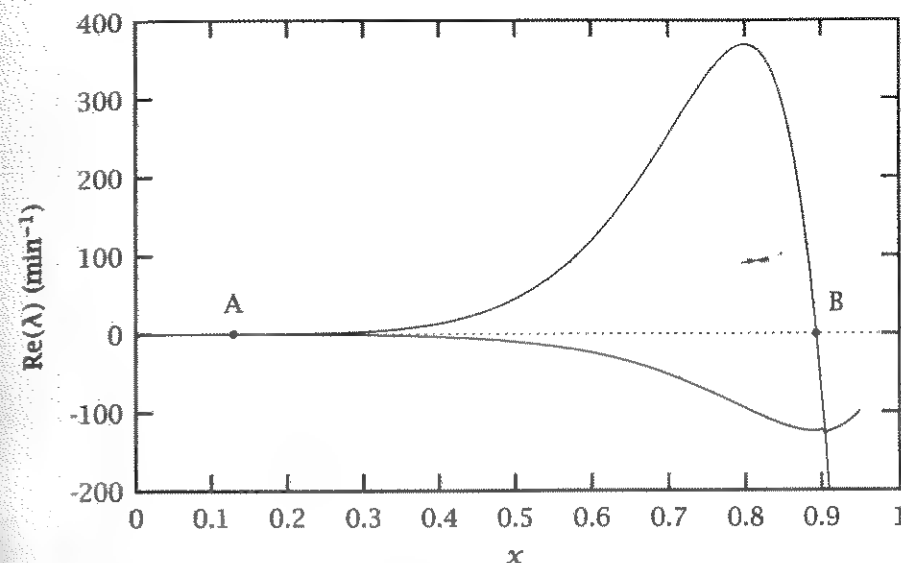
Point	$\theta(\text{min})$	$x$	$T(\text{K})$	$\text{Re}(\lambda)(\text{min}^{-1})$	$\text{Im}(\lambda)(\text{min}^{-1})$
A	11.1	0.125	305	0	0
B	0.008	0.893	396	0	0
C	19.2	0.970	339	-0.218	0
D	20.7	0.962	336	-0.373	0
E	29.3	0.905	327	0	0.159
F	71.2	0.519	306	0	0.0330

**Table 6.4:** Steady state and eigenvalue with largest real part at selected points in Figures 6.20 and 6.21.

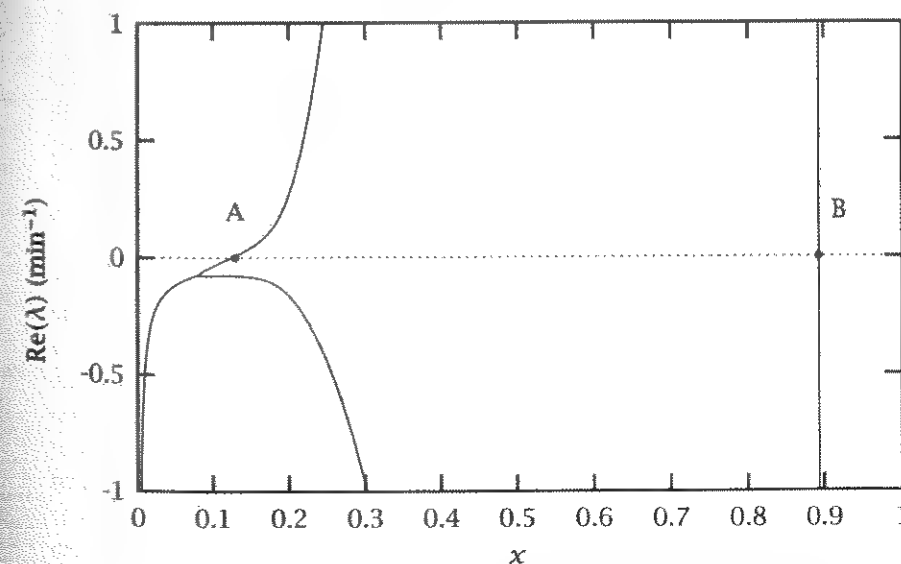
and 6.8 have taken on a mushroom shape with the new parameters. We have labeled points A–F on the steady-state curves in Figures 6.20 and 6.21. Table 6.4 summarizes the locations of these points in terms of the residence times, the steady-state conversions and temperatures, and the eigenvalue of the Jacobian with largest real part. Now we take a walk along the steady-state solution curve and examine the dynamic behavior of the system.

From the beginning of the curve through points A and B, the two eigenvalues are real. Their values are plotted in Figure 6.22 versus the conversion. As in the s-shaped multiplicity of the previous section, we see that an eigenvalue becomes positive at point A, which is again an ignition point. Figure 6.23 shows a magnified view. The middle branch between A and B is unstable and the upper branch after point B is stable again as the positive eigenvalue passes through zero and becomes negative.

As we continue along the upper branch, some surprising things happen. The two eigenvalues remain real and negative as we pass through point C and then merge into the same value at point D. Figure 6.24 shows the real and imaginary parts of the eigenvalues from points C–F. Notice the two eigenvalues approach each other along the real axis until they become identical at point D. After point D the imaginary parts of the eigenvalues are no longer zero, and we have a complex pair of eigenvalues. The two eigenvalues are complex conjugates; they have the same real part and opposite imaginary parts so the eigenvalue curves from D to F are mirror images. But the real parts of the eigenvalues remain negative from point D to E and the steady states along this portion of the upper branch are stable. At point E, however, the pair of eigenvalues cross back into the right half of the complex plane and



**Figure 6.22:** Eigenvalues of the Jacobian matrix versus reactor conversion in the region of steady-state multiplicity.



**Figure 6.23:** Eigenvalues of the Jacobian matrix versus reactor conversion in the region of steady-state multiplicity — magnified y-axis.

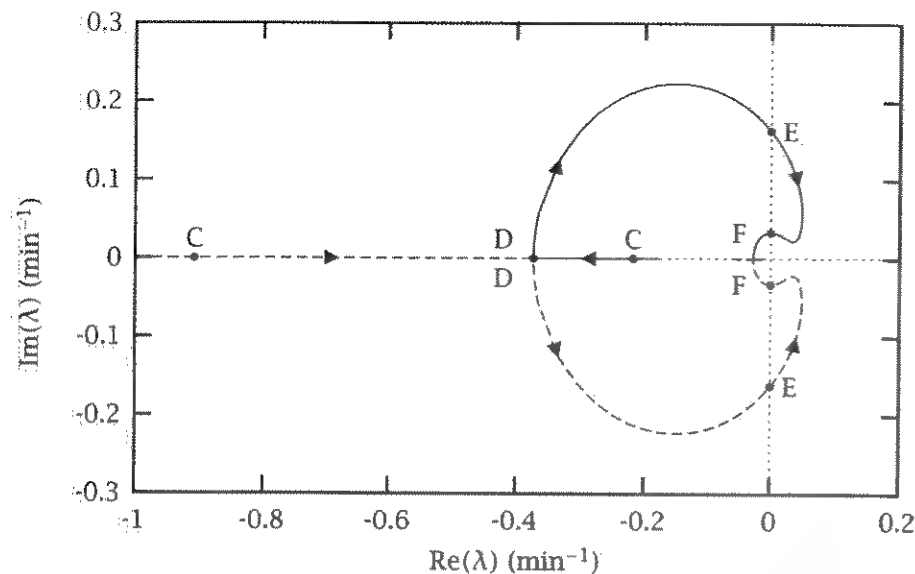


Figure 6.24: Real and imaginary parts of the eigenvalues of the Jacobian matrix near points C-F.

have positive real parts. From point E to F, the eigenvalue analysis indicates the upper branch is unstable. Notice from Figures 6.18 and 6.19, however, that there is no lower-branch steady state for these values of residence time. It appears we have unique steady states and they are unstable. So a pressing issue now is what the reactor does after it is perturbed from these unique and unstable steady states.

Let us see what simulations can show us. A residence time of  $\theta = 35$  min is between points E and F as shown in Table 6.4. Solving the dynamic mass and energy balances with this value of residence time produces Figure 6.25. We see that the solution does not approach the steady state but oscillates continuously. These oscillations are sustained; they do not damp out at large times. Notice also that the amplitude of the oscillation is large, more than 80 K in temperature and 50% in conversion.

We can obtain another nice view of this result if we plot the conversion versus the temperature rather than both of them versus time. This kind of plot is known as a phase plot or phase portrait and is shown in Figure 6.26. Time increases as we walk along the phase plot; the reactor

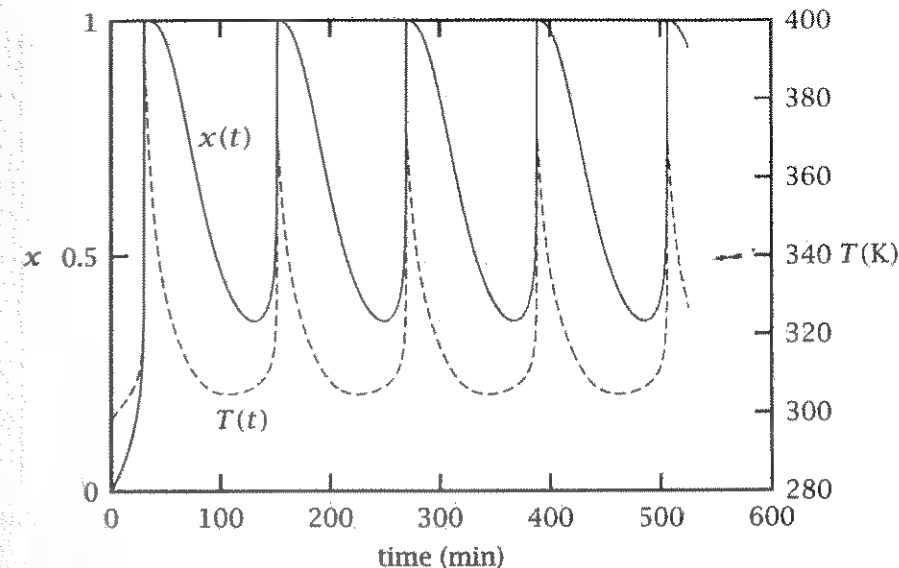


Figure 6.25: Conversion and temperature vs. time for  $\theta = 35$  min.

ignites, then slowly decays, ignites again, and eventually winds onto the steady limit cycle shown in Figure 6.26. Figure 6.27 explores the effect of initial conditions. The trajectory starting with the feed temperature and concentration as in Figure 6.26 is shown again. The trajectory starting in the upper left of the figure has the feed temperature and zero A concentration as its initial condition. Several other initial conditions inside the limit cycle are shown also, including starting the reactor at the unstable steady state. All of these initial conditions wind onto the same final limit cycle. We say that the limit cycle is a global attractor because all initial conditions wind onto this same solution.

If we decrease the residence time to  $\theta = 30$  min, we are close to point E, where the stability of the upper steady state changes. A simulation at this residence time is shown in Figure 6.28. Notice the amplitude of these oscillations is much smaller, and the shape is more like a pure sine wave. Near point E in Figure 6.24, where the transition to instability occurs, the amplitude of the emerging oscillation is small and the period of the oscillation is well predicted by Equation 6.45 ( $T = 2\pi/b$ ). Table 6.4 shows the imaginary part of the eigenvalue at point E is 0.159

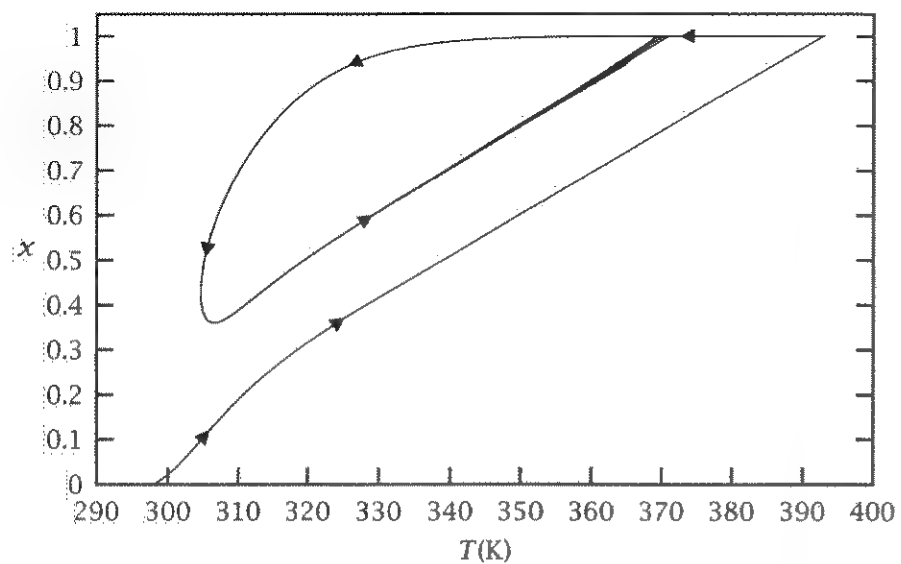


Figure 6.26: Phase portrait of conversion versus temperature for feed initial condition;  $\theta = 35$  min.

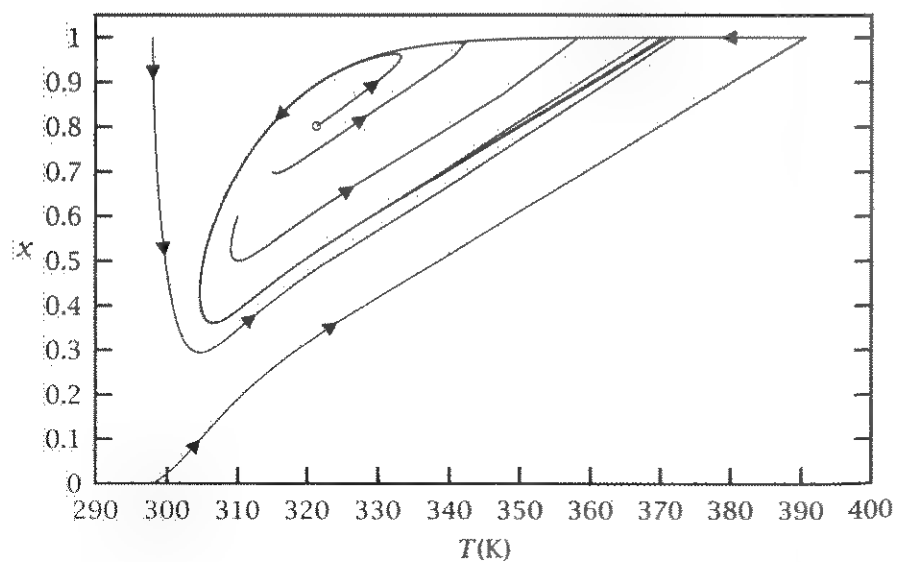


Figure 6.27: Phase portrait of conversion versus temperature for several initial conditions;  $\theta = 35$  min.

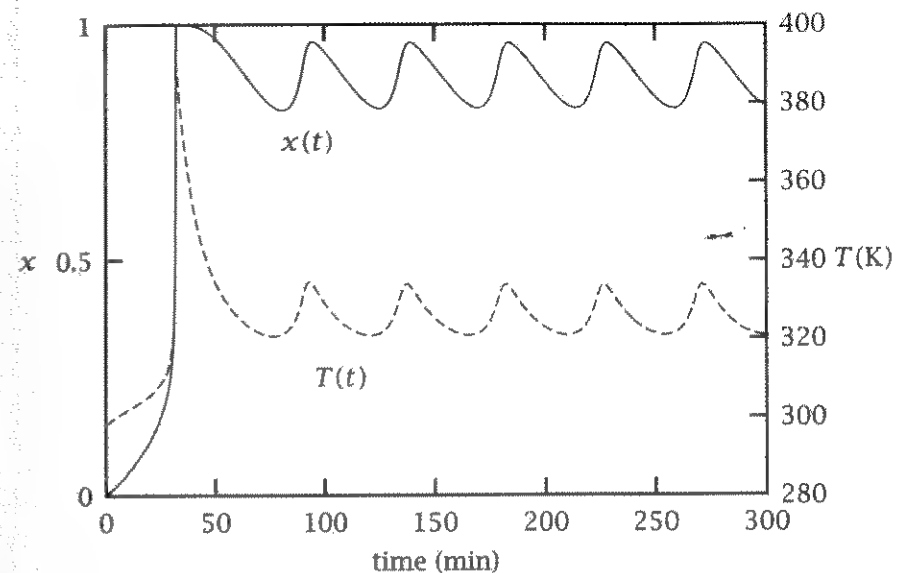


Figure 6.28: Conversion and temperature vs. time for  $\theta = 30$  min.

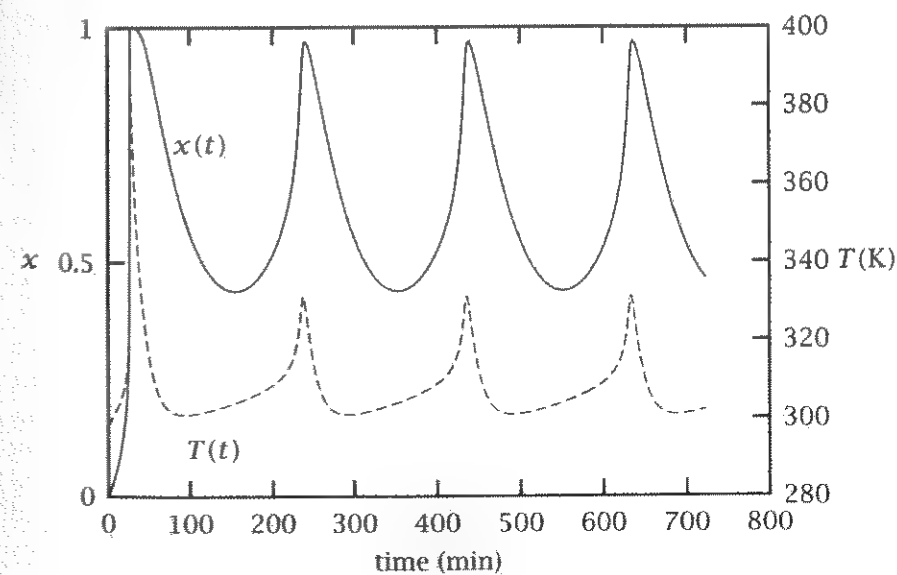
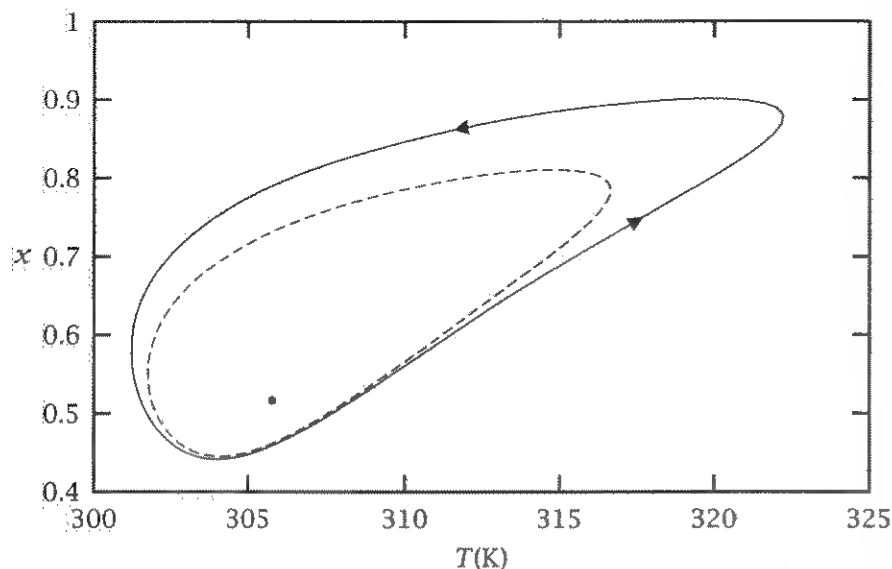


Figure 6.29: Conversion and temperature vs. time for  $\theta = 72.3$  min.





**Figure 6.30:** Phase portrait of conversion versus temperature at  $\theta = 73.1$  min showing stable and unstable limit cycles, and a stable steady state.

so the predicted period is

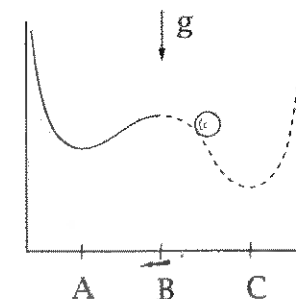
$$T = 2\pi/0.159 \text{ min} = 40 \text{ min}$$

which is in good agreement with the simulation shown in Figure 6.28.

As we pass point F, the eigenvalues in Figure 6.24 cross back into the left-half plane and the steady state is again stable. A simulation near point F is shown in Figure 6.29. Notice, in contrast to point E, the amplitude of the oscillations is not small near point F. Indeed, even though Figure 6.24 indicates the steady state is stable for  $\theta \geq 72.3$  min, that does not mean that limit-cycle solutions are not possible in this part of the parameter space. To see how limit cycles can remain after the steady state regains its stability, consider Figure 6.30, constructed for  $\theta = 73.1$  min. The figure depicts the stable steady state, indicated by a solid circle, surrounded by an unstable limit cycle, indicated by the dashed line.<sup>3</sup> The unstable limit cycle is in turn surrounded by a

<sup>3</sup>This curve was constructed by reversing time in the integration of the mass and energy balances, which stabilizes the unstable limit cycle.

stable limit cycle. Note that all initial conditions outside of the stable limit cycle would converge to the stable limit cycle from the outside. All initial conditions in the region between the unstable and stable limit cycles would converge to the stable limit cycle from the inside. Finally, all initial conditions inside the unstable limit cycle are attracted to the stable steady state. We see that the unstable limit cycle is the boundary between the regions of attraction of the stable steady state and stable limit cycle. We also have a quantitative measure of a perturbation capable of knocking the system from the steady state onto a periodic solution. Exercise 6.12 explores the regions of attraction of the steady state and the stable limit cycle in further detail.



**Figure 6.31:** Marble on a track with three steady states; dashed line indicates frictionless track; point C is not asymptotically stable and is surrounded by limit cycles.

We may modify our simple mechanical system to illustrate somewhat analogous limit-cycle behavior. Consider the marble and track system depicted in Figure 6.31. We have three steady states; steady-state A is again stable and steady-state B is unstable as in Figure 6.16. At this point we cheat thermodynamics a bit to achieve the desired behavior. Imagine the track consists of a frictionless material to the right of point B. Although in violation of the second law, we can either use our imagination or consider extremely low friction material. Without friction in the vicinity of point C, the steady state is not asymptotically stable. Perturbations from point C do not return to the steady state but continually oscillate. The analogy is not perfect because a single limit cycle does not surround the unstable point C as in the chemical reactor shown in Figures 6.27 and 6.30. The amplitude of the oscillation in the marble system depends on the initial perturbation and a continuous family of limit cycles surrounds the unstable steady-state C. But the analogy may prove helpful in demystifying these kinds of reactor behaviors. The reader may also want to consider why we were compelled to violate the second law to achieve sustained oscillations in the simple mechanical system but the reactor can continually oscillate without such violation. Consider what is the essential difference between these thermodynamic systems.

### Further Reading on CSTR Dynamics and Stability

Consideration of the coupled mass and energy balances for the CSTR have led to possible behaviors that may seem surprisingly complex for even the simplest kinetic mechanism, an irreversible first-order reaction. Just because these behaviors are possible does not mean that they are normally observed in reactor operation for something as simple as A goes to B.

Complex kinetics, such as polymerization kinetics [19, 22], lead to complex types of behavior, even under isothermal operation, i.e., *without* the nonlinearity introduced by the rate constant's exponential dependence on temperature. Bailey [4] summarizes some cases of experimentally observed oscillations in CSTRs including chlorination of methyl chloride [7], hydrolysis of acetyl chloride [3], the reaction between methanol and hydrogen chloride [12], and the reaction between sodium thiosulfate and hydrogen peroxide [8].

In addition to the multiple steady states and sustained oscillations, chemical reactors are a rich source of other types of behavior including deterministic chaos [18, 17, 23, 14]. The chemical engineer should be aware of the complex behavior that is possible with simple nonlinear CSTR models, especially if confronted with apparently complex operating data.

## 6.4 The Semi-Batch Reactor

The development of the semi-batch reactor energy balance follows directly from the CSTR energy balance derivation by setting  $Q = 0$ . The main results are summarized in Table 6.9 at the end of this chapter. Note in particular that Equations 6.81–6.83 in the semi-batch reactor Table 6.9 are identical to the corresponding Equations 6.72–6.74 in the CSTR Table 6.8.

It is good practice to enumerate the unknowns and applicable equations when setting up a model for the nonisothermal semi-batch reactor. As discussed in Chapter 4, for a single-phase system, the molar concentrations of the components, and the temperature and the pressure specify all intensive variables of the reactor. If we use the reactor volume as the single extensive variable, then we have  $n_s + 3$  unknowns. As in Chapter 4, we have  $n_s$  equations from the material balances and one equation of state. The energy balance of this chapter provides an additional equation. Finally, we must specify the reactor pressure or some other system constraint that determines the reactor pressure.

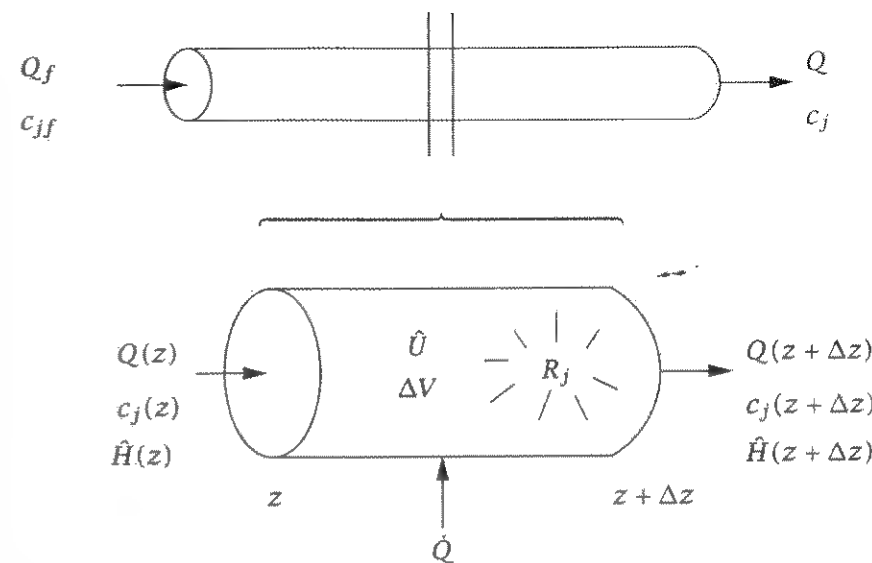


Figure 6.32: Plug-flow reactor volume element.

As in Section 4.5 of Chapter 4, if the reactor outflow term is nonzero as well, then an additional statement of reactor operation, such as constant reactor volume or mass, is required to determine  $Q$ , which then appears in the material and energy balances.

## 6.5 The Plug-Flow Reactor

To derive an energy balance for the plug-flow reactor (PFR), consider the volume element in Figure 6.32. If we write Equation 6.5 for this element and neglect kinetic and potential energies and shaft work, we obtain

$$\frac{\partial}{\partial t}(\rho \hat{U} A_c \Delta z) = m \hat{H}_z - m \hat{H}_{z+\Delta z} + \dot{Q}$$

in which  $A_c$  is the cross-sectional area of the tube,  $R$  is the tube outer radius, and  $\dot{Q}$  is the heat transferred through the wall, normally expressed using an overall heat-transfer coefficient

$$\dot{Q} = U^o 2\pi R \Delta z (T_a - T)$$

Dividing by  $A_c \Delta z$  and taking the limit  $\Delta z \rightarrow 0$ , gives

$$\frac{\partial}{\partial t}(\rho \hat{U}) = -\frac{1}{A_c} \frac{\partial}{\partial z} (Q \rho \hat{H}) + \dot{q}$$

in which  $\dot{q} = (2/R)U^0(T_a - T)$  and we express the mass flowrate as  $m = Q\rho$ . In the steady state, we have

$$\frac{d}{dV}(Q \sum_j c_j H_j) = \dot{q}$$

in which we use the volume instead of length of the reactor,  $V = A_c z$  and partial molar enthalpy to express the enthalpy per volume,

$$\rho \hat{H} = \sum_j c_j H_j$$

The enthalpy term can be broken apart into two terms

$$\frac{d}{dV}(Q \sum_j c_j H_j) = \sum_j \left[ Q c_j \frac{dH_j}{dV} + H_j \frac{d}{dV}(Q c_j) \right] \quad (6.46)$$

We treat the  $\sum_j c_j dH_j$  term by deriving the Gibbs-Duhem equation for enthalpy as follows. We have  $H = \sum_j n_j H_j$  because  $H_j$  is a partial molar enthalpy (see also Exercise 3.6 in Chapter 3). Its differential is therefore

$$dH = \sum_j H_j dn_j + n_j dH_j$$

Subtracting this expression from Equation 6.10, which is valid for single-phase systems,

$$dH = V\rho\hat{C}_P dT + (1 - \alpha T)VdP + \sum_j H_j dn_j$$

yields a Gibbs-Duhem equation for enthalpy

$$0 = V\rho\hat{C}_P dT + (1 - \alpha T)VdP - \sum_j n_j dH_j$$

We express an intensive version of this result by dividing by  $V$  and rearranging to obtain

$$\sum_j c_j dH_j = \rho\hat{C}_P dT + (1 - \alpha T)dP \quad \text{Gibbs-Duhem equation} \quad (6.47)$$

We can form the derivative of Equation 6.47 with reactor volume to obtain

$$\sum_j c_j \frac{dH_j}{dV} = \rho\hat{C}_P \frac{dT}{dV} + (1 - \alpha T) \frac{dP}{dV} \quad (6.48)$$

The second term in Equation 6.46 is expressed using the material balance for the PFR

$$\frac{d}{dV}(Q c_j) = R_j = \sum_i \nu_{ij} r_i$$

Substitution of this result and Equation 6.48 into Equation 6.46 then gives

$$Q\rho\hat{C}_P \frac{dT}{dV} + Q(1 - \alpha T) \frac{dP}{dV} = - \sum_i \Delta H_{Ri} r_i + \dot{q} \quad (6.49)$$

In unpacked tubes, the pressure drop is usually negligible, and for an ideal gas,  $\alpha T = 1$ . For both of these cases, we have

**Ideal gas, or neglect pressure drop.**

$$Q\rho\hat{C}_P \frac{dT}{dV} = - \sum_i \Delta H_{Ri} r_i + \dot{q} \quad (6.50)$$

If the fluid is incompressible,  $\alpha = 0$ , and Equation 6.49 reduces to

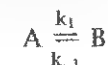
**Incompressible fluid.**

$$Q\rho\hat{C}_P \frac{dT}{dV} + Q \frac{dP}{dV} = - \sum_i \Delta H_{Ri} r_i + \dot{q} \quad (6.51)$$

Equation 6.50 is the usual energy balance for PFRs in this chapter. The next chapter considers packed-bed reactors in which the pressure drop may be significant. Equation 6.49 is more appropriate in that situation unless one can safely make the ideal-gas assumption.

#### Example 6.4: PFR and interstage cooling

Consider the reversible, gas-phase reaction



The reaction is carried out in two long, adiabatic, plug-flow reactors with an interstage cooler between them as shown in Figure 6.33. (Refer to the discussion in Section 1.4.2 for more on interstage cooling.) The feed consists of component A diluted in an inert  $N_2$  stream,  $N_{Af}/N_{If} = 0.1$ ,  $N_{Bf} = 0$ , and  $Q_f = 10,000 \text{ ft}^3/\text{hr}$  at  $P_f = 180 \text{ psia}$  and  $T_f = 830^\circ\text{R}$ . Since the inert stream is present in such excess, we assume that the heat

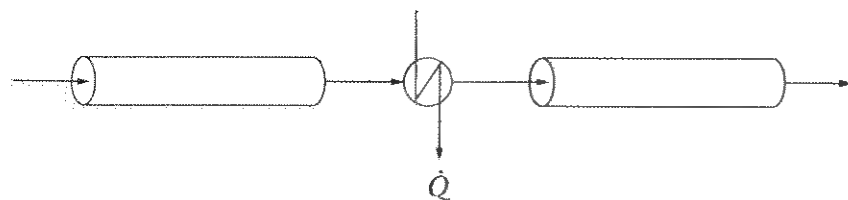


Figure 6.33: Tubular reactors with interstage cooling.

capacity of the mixture is equal to the heat capacity of nitrogen and is independent of temperature for the temperature range we expect. The heat of reaction is  $\Delta H_R = -5850$  BTU/lbmol and can be assumed constant. The value of the equilibrium constant is  $K = k_1/k_{-1} = 1.5$  at the feed temperature.

1. Write down the mole and energy balances that would apply in the reactors. Make sure all variables in your equations are expressed in terms of  $T$  and  $N_A$ . What other assumptions did you make?
2. If the reactors are long, we may assume that the mixture is close to equilibrium at the exit. Using the mole balance, express  $N_A$  at the exit of the first reactor in terms of the feed conditions and the equilibrium constant,  $K$ .
3. Using the energy balance, express  $T$  at the exit of the first reactor in terms of the feed conditions and  $N_A$ .
4. Notice we have two equations and two unknowns because  $K$  is a strong function of  $T$ . Solve these two equations numerically and determine the temperature and conversion at the exit of the first reactor. Alternatively, you can substitute the material balance into the energy balance to obtain one equation for  $T$ . Solve this equation to determine the temperature at the exit of the first reactor. What is the conversion at the exit of the first reactor?
5. Assume that economics dictate that we must run this reaction to 70% conversion to make a profit. How much heat must be removed in the interstage cooler to be able to achieve this conversion at the exit of the second reactor? What are the temperatures at the inlet and outlet of the second reactor?
6. How would you calculate the *actual* conversion achieved for two

PFRs of *specified* sizes (rather than "long" ones) with this value of  $\dot{Q}$ ?

### Solution

1. The steady-state molar flow of A is given by the PFR material balance

$$\frac{dN_A}{dV} = R_A = -r \quad (6.52)$$

and the rate expression for the reversible reaction is given by

$$r = k_1 c_A - k_{-1} c_B = (k_1 N_A - k_{-1} N_B)/Q$$

The molar flow of B is given by  $dN_B/dV = r$ , so we conclude

$$N_B = N_{Af} + N_{Bf} - N_A = N_{Af} - N_A$$

If we assume the mixture behaves as an ideal gas at these conditions,  $c = P/RT$  or

$$Q = \frac{RT}{P} \sum_{j=1}^{n_s} N_j$$

The material balance for inert gives  $dN_I/dV = 0$ , so we have the total molar flow is  $\sum_{j=1}^{n_s} N_j = N_{Af} + N_{If}$  and the volumetric flow-rate is

$$Q = \frac{RT}{P} (N_{Af} + N_{If})$$

and the reaction rate is

$$r = \frac{P}{RT} \left( \frac{k_1 N_A - k_{-1} (N_{Af} - N_A)}{N_{Af} + N_{If}} \right)$$

which is in terms of  $T$  and  $N_A$ . The adiabatic PFR energy balance for an ideal gas is given by

$$\frac{dT}{dV} = -\frac{\Delta H_R}{Q \rho \hat{C}_P} r \quad (6.53)$$

2. For long reactors,  $r = 0$  or

$$k_1 N_A - k_{-1} (N_{Af} - N_A) = 0$$

Dividing by  $k_{-1}$  and solving for  $N_A$  gives

$$N_A = \frac{N_{Af}}{1 + K_1}$$

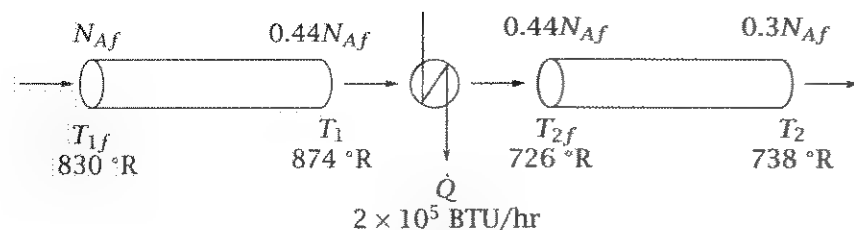


Figure 6.34: Temperatures and molar flows for tubular reactors with interstage cooling.

3. Substituting  $r = -dN_A/dV$  into the energy balance and multiplying through by  $dV$  gives

$$dT = \frac{\Delta H_R}{Q\rho\hat{C}_P} dN_A$$

The term  $Q\rho = m$  in the denominator is the mass flowrate, which is constant and equal to the feed mass flowrate. If we assume the heat of reaction and the heat capacity are weak functions of temperature and composition, we can perform the integral yielding

$$T_1 - T_{1f} = \frac{\Delta H_R}{m\hat{C}_P} (N_A - N_{Af})$$

4.  $T - 830 + 80.1 \left( \frac{1}{1 + 0.0432e^{2944/T}} - 1 \right) = 0,$

$T_1 = 874^\circ\text{R}, \quad x = 0.56$

5.  $\dot{Q} = 200,000 \text{ BTU/hr}, \quad T_{2f} = 726^\circ\text{R}, \quad T_2 = 738^\circ\text{R}$

6. Integrate Equations 6.52 and 6.53.

The results are summarized in Figure 6.34.  $\square$

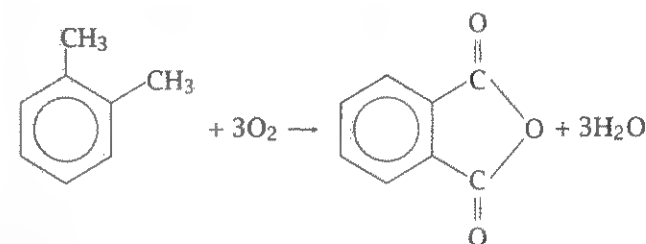
### 6.5.1 Plug-Flow Reactor Hot Spot and Runaway

For exothermic, gas-phase reactions in a PFR, the heat release generally leads to the formation of a reactor hot spot, a point along the reactor length at which the temperature profile achieves a maximum. If the reaction is highly exothermic, the temperature profile can be very sensitive to parameters, and a small increase in the inlet temperature or

reactant feed concentration, for example, can lead to large changes in the temperature profile. A sudden, large increase in the reactor temperature due to a small change in feed conditions is known as reactor runaway. Reactor runaway is highly dangerous, and operating conditions are normally chosen to keep reactors far from the runaway condition. The following example, oxidation of o-xylene to phthalic anhydride, illustrates the PFR hotspot and reactor runaway.

### Example 6.5: Oxidation of o-xylene to phthalic anhydride

The gas-phase oxidation of o-xylene to phthalic anhydride



is highly exothermic. The reaction is carried out in PFR tube bundles with molten salt circulating as the heat transfer fluid [10]. The o-xylene is mixed with air before entering the PFR. The reaction rate is limited by maintaining a low concentration of hydrocarbon in the feed. The mole fraction of o-xylene is less than 2%.

Under these conditions, the large excess of oxygen leads to a pseudo-first-order rate expression

$$r = k_m \exp \left[ -E \left( \frac{1}{T} - \frac{1}{T_m} \right) \right] c_x$$

in which  $c_x$  is the o-xylene concentration. The operating pressure is atmospheric. Calculate the temperature and o-xylene composition profiles. The kinetic parameters are adapted from Van Welsenaere and Froment and given in Table 6.5 [26].

### Solution

If we assume constant thermochemical properties, an ideal gas mixture, and express the mole and energy balances in terms of reactor length,

Parameter	Value	Units
$k_m$	1922.6	$s^{-1}$
$T_a$	625	K
$T_m$	625	K
$P_f$	1.0	atm
$l$	1.5	m
$R$	0.0125	m
$\hat{C}_p$	0.992	kJ/kg K
$U^o$	0.373	kJ/m <sup>2</sup> s K
$\gamma_{xf}$	0.019	
$E/R$	$1.3636 \times 10^4$	K
$\Delta H_R$	$-1.361 \times 10^3$	kJ/kmol
$Q\rho$	$2.6371 \times 10^{-3}$	kg/s

**Table 6.5:** PFR operating conditions and parameters for o-xylene example.

we obtain

$$\frac{dN_x}{dz} = -A_c r$$

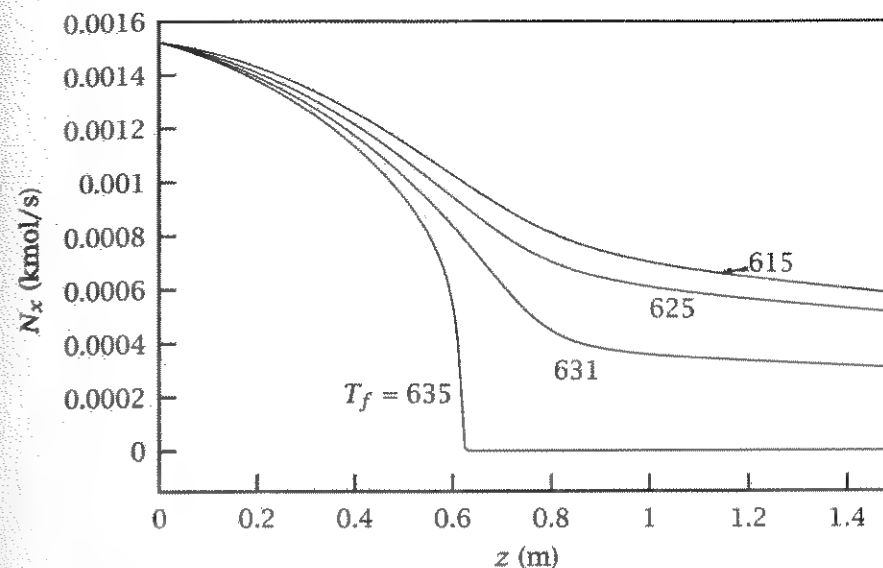
$$\frac{dT}{dz} = -\beta r + \gamma(T_a - T)$$

$$r = k \frac{P}{RT} \frac{N_x}{N}$$

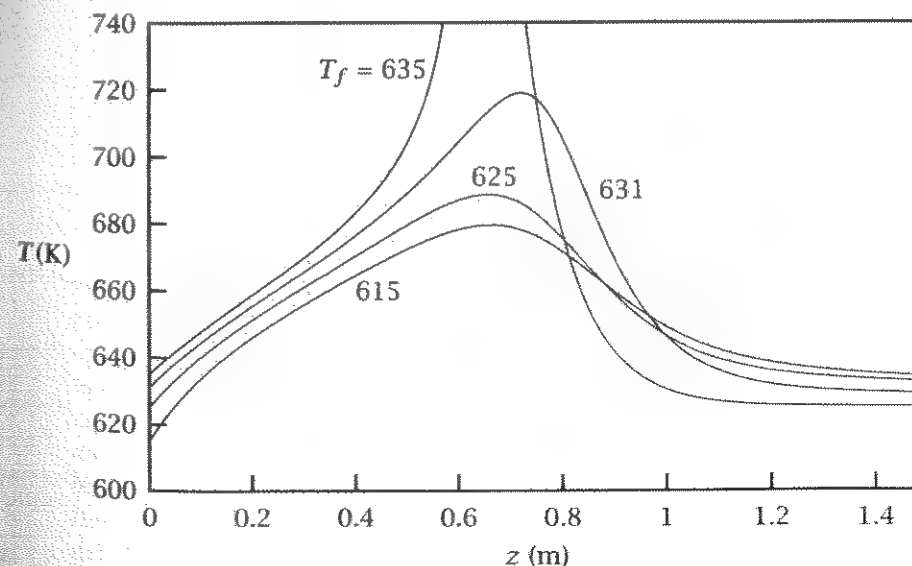
in which

$$\beta = \frac{\Delta H_R A_c}{Q\rho \hat{C}_p}, \quad \gamma = \frac{2\pi R U^o}{Q\rho \hat{C}_p}$$

and the total molar flow is constant and equal to the feed molar flow because of the stoichiometry. Figure 6.35 shows the molar flow of o-xylene versus reactor length for several values of the feed temperature. The corresponding temperature profile is shown in Figure 6.36. We see a hotspot in the reactor for each feed temperature. Notice the hotspot temperature increases and moves down the tube as we increase the feed temperature. Finally, notice if we increase the feed temperature above about 631 K, the temperature spikes quickly to a large value and all of the o-xylene is converted by  $z = 0.6$  m, which is a classic example of reactor runaway. To avoid this reactor runaway, we must maintain the feed temperature below a safe value. This safe value obviously also depends on how well we can control the composition and temperature in the feed stream. Tighter control allows us to operate safely at higher



**Figure 6.35:** Molar flow of o-xylene versus reactor length for different feed temperatures.



**Figure 6.36:** Reactor temperature versus length for different feed temperatures.

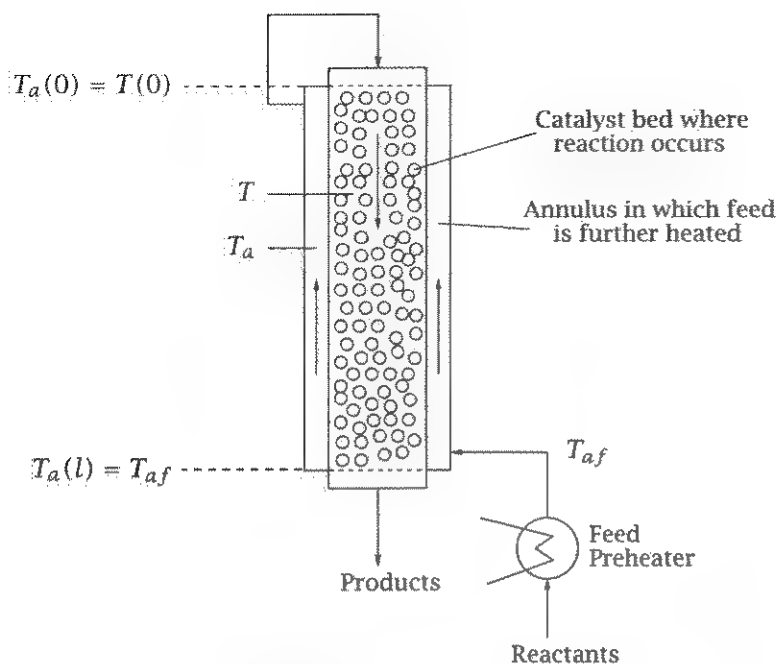


Figure 6.37: Autothermal plug-flow reactor; the heat released by the exothermic reaction is used to preheat the feed.

feed temperatures and feed o-xylene mole fractions, which increases the production rate. □

## 6.5.2 The Autothermal Plug-Flow Reactor

In many applications, it is necessary to heat a feed stream to achieve a reactor inlet temperature having a high reaction rate. If the reaction also is exothermic, we have the possibility to lower the reactor operating cost by heat integration. The essential idea is to use the heat released by the reaction to heat the feed stream. As a simple example of this concept, consider the heat integration scheme depicted in Figure 6.37 [1]. This reactor configuration is known as an autothermal plug-flow reactor. The reactor system is an annular tube. The feed passes through the outer region and is heated through contact with the hot reactor wall. The feed then enters the inner reaction region, which is filled with the catalyst, and flows countercurrently to the feed

stream. The heat released due to reaction in the inner region is used to heat the feed in the outer region. When the reactor is operating at steady state, no external heat is required to preheat the feed. Of course, during the reactor start up, external heat must be supplied to ignite the reactor.

Although recycle of energy can offer greatly lower operating costs, the dynamics and control of these reactors may be complex. We next examine an ammonia synthesis example to show that multiple steady states are possible. Ammonia synthesis is also interesting because of its large impact on the early development of the chemical engineering discipline. Quoting Aftalion [2, p. 101]

While physicists and chemists were linking up to understand the structure of matter and giving birth to *physical chemistry*, another discipline was emerging, particularly in the United States, at the beginning of the twentieth century, that of *chemical engineering* ... it was undoubtedly the synthesis of ammonia by BASF, successfully achieved in 1913 in Oppau, which forged the linking of chemistry with physics and engineering as it required knowledge in areas of analysis, equilibrium reactions, high pressures, catalysis, resistance of materials, and design of large-scale apparatus.

### Example 6.6: Ammonia synthesis

Calculate the steady-state conversion for the synthesis of ammonia using the autothermal process shown in Figure 6.37 [25]. A rate expression for the reaction



over an iron catalyst at 300 atm pressure is suggested by Temkin [21]

$$r = k_{-1}/RT \left[ K^2(T) \frac{P_N P_H^{3/2}}{P_A} - \frac{P_A}{P_H^{3/2}} \right] \quad (6.55)$$

in which  $P_N, P_H, P_A$  are the partial pressures of nitrogen, hydrogen, and ammonia, respectively, and  $K$  is the equilibrium constant for the reaction forming one mole of ammonia. For illustration, we assume the thermochemical properties are constant and the gases form an ideal-gas mixture. More accurate thermochemical properties and a more accurate equation of state do not affect the fundamental behavior predicted by the reactor model.



The steady-state material balance for the ammonia is

$$\frac{dN_A}{dV} = R_A = 2r$$

$$N_A(0) = N_{Af}$$

and the other molar flows are calculated from

$$N_N = N_{Nf} - 1/2(N_A - N_{Af})$$

$$N_H = N_{Hf} - 3/2(N_A - N_{Af})$$

If we assume an ideal gas in this temperature and pressure range, the volumetric flowrate is given by

$$Q = \frac{RT}{P} (N_A + N_N + N_H)$$

The energy balance for the reactor is the usual

$$Q\rho\hat{C}_P \frac{dT}{dV} = -\Delta H_R r + \dot{q} \quad (6.56)$$

in which  $\dot{q}$  is the heat transfer taking place between the reacting fluid and the cold feed

$$\dot{q} = \frac{2}{R} U^o (T_a - T)$$

The material balances for the feed-heating section are simple because reaction does not take place without the catalyst. Without reaction, the molar flow of all species are constant and equal to their feed values and the energy balance for the feed-heating section is

$$Q_a \rho_a \hat{C}_{Pa} \frac{dT_a}{dV_a} = -\dot{q} \quad (6.57)$$

$$T_a(0) = T_{af}$$

in which the subscript  $a$  represents the fluid in the feed-heating section. Notice the heat terms are of opposite signs in Equations 6.57 and 6.56. If we assume the fluid properties do not change significantly over the temperature range of interest, and switch the direction of integration in Equation 6.57 using  $dV_a = -dV$ , we obtain

$$Q\rho\hat{C}_P \frac{dT_a}{dV} = \dot{q} \quad (6.58)$$

$$T_a(V_R) = T_{af} \quad (6.59)$$

Parameter	Value	Units
$P$	300	atm
$Q_0$	0.16	m <sup>3</sup> /s
$A_c$	1	m <sup>2</sup>
$l$	12	m
$T_{af}$	323	K
$\gamma = \frac{2\pi R U^o}{Q\rho\hat{C}_P}$	0.5	1/m
$\beta = \frac{\Delta H_R A_c}{Q\rho\hat{C}_P}$	-2.342	m <sup>2</sup> s K/mol
$\Delta G^\circ$	4250	cal/mol
$\Delta H^\circ$	$-1.2 \times 10^4$	cal/mol
$k_{-10}$	$7.794 \times 10^{11}$	
$E_{-1}/R$	$2 \times 10^4$	K

Table 6.6: Parameter values for Example 6.6; heat of reaction and mixture heat capacity assumed constant.

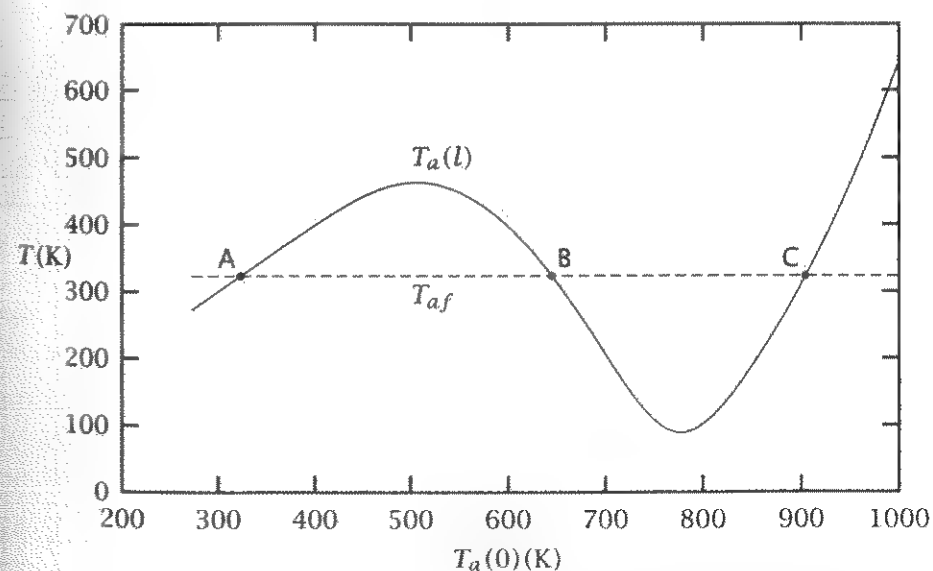


Figure 6.38: Coolant temperature at reactor outlet versus temperature at reactor inlet,  $T_a(l)$  versus  $T_a(0)$ ; intersection with coolant feed temperature  $T_{af}$  indicates three steady-state solutions (A,B,C).

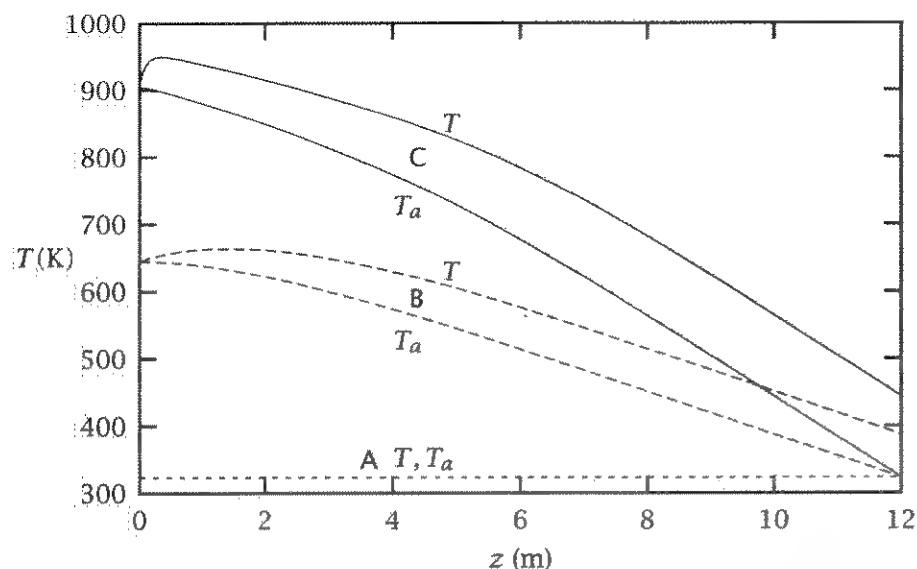


Figure 6.39: Reactor and coolant temperature profiles versus reactor length; lower (A), unstable middle (B), and upper (C) steady states.

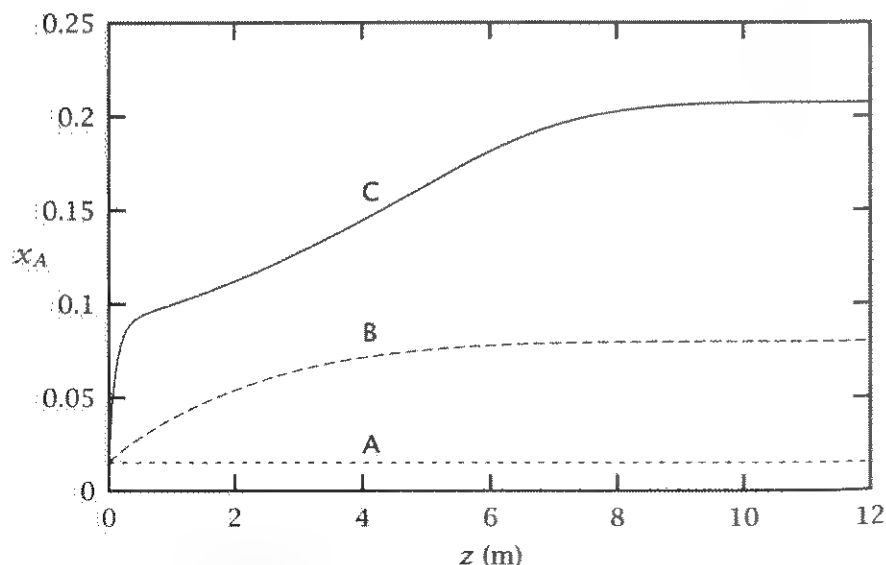


Figure 6.40: Ammonia mole fraction versus reactor length; lower (A), unstable middle (B), and upper (C) steady states.

Finally we require a boundary condition for the reactor energy balance, which we have from the fact that the heating fluid enters the reactor at  $z = 0, T(0) = T_a(0)$ . Combining these balances and boundary conditions and converting to reactor length in place of volume gives the model

$$\begin{aligned} \frac{dN_A}{dz} &= 2A_c r & N_A(0) &= N_{Af} \\ \frac{dT}{dz} &= -\beta r + \gamma(T_a - T) & T(0) &= T_a(0) \\ \frac{dT_a}{dz} &= \gamma(T_a - T) & T_a(l) &= T_{af} \end{aligned} \quad (6.60)$$

in which

$$\beta = \frac{\Delta H_R A_c}{Q \rho \hat{C}_P} \quad \gamma = \frac{2\pi R U^0}{Q \rho \hat{C}_P}$$

Equation 6.60 is a boundary-value problem, rather than an initial-value problem, because  $T_a$  is specified at the exit of the reactor. A simple solution strategy is to guess the reactor inlet temperature, solve the model to the exit of the reactor, and then compare the computed feed preheat temperature to the specified value  $T_{af}$ . This strategy is known as a shooting method. We guess the missing values required to produce an initial-value problem. We solve the initial-value problem, and then iterate on the guessed values until we match the specified boundary conditions. We will see more about boundary-value problems and shooting methods when we treat diffusion in Chapter 7.

### Solution

Figure 6.38 shows the results for the parameter values listed in Table 6.6, which are based on those used by van Heerden [25]. For given values of  $T_a(0)$ , we solve the initial-value problem, Equation 6.60, and plot the resulting  $T_a(V_R)$  as the solid line in Figure 6.38. The intersection of that line with the feed temperature  $T_{af} = 323$  K indicates a steady-state solution. Notice three steady-state solutions are indicated in Figure 6.38 for these values of parameters. The profiles in the reactor for these three steady states are shown in Figures 6.39 and 6.40. It is important to operate at the upper steady state so that a reasonably large production of ammonia is achieved. □

Neglect kinetic and potential energies	
$\frac{dU}{dt} = \dot{Q} + \dot{W}_s + \dot{W}_b$	(6.61)
Neglect shaft work	
$\frac{dU}{dt} + P \frac{dV_R}{dt} = \dot{Q}$	(6.62)
$\frac{dH}{dt} - V_R \frac{dP}{dt} = \dot{Q}$	(6.63)
Single phase	
$V_R \rho \hat{C}_P \frac{dT}{dt} - \alpha T V_R \frac{dP}{dt} + \sum_j H_j \frac{dn_j}{dt} = \dot{Q}$	(6.64)
$V_R \rho \hat{C}_P \frac{dT}{dt} - \alpha T V_R \frac{dP}{dt} = - \sum_i \Delta H_{Ri} \gamma_i V_R + \dot{Q}$	(6.65)
a. Incompressible-fluid or constant-pressure reactor	
$V_R \rho \hat{C}_P \frac{dT}{dt} = - \sum_i \Delta H_{Ri} \gamma_i V_R + \dot{Q}$	(6.66)
b. Constant-volume reactor	
$V_R \rho \hat{C}_V \frac{dT}{dt} = - \sum_i \left[ \Delta H_{Ri} - \alpha T V_R \sum_j v_{ij} \left( \frac{\partial P}{\partial n_j} \right)_{T,V,n_k} \right] \gamma_i V_R + \dot{Q}$	(6.67)
b.1 Constant-volume reactor, ideal gas	
$V_R \rho \hat{C}_V \frac{dT}{dt} = - \sum_i (\Delta H_{Ri} - RT \tilde{v}_i) \gamma_i V_R + \dot{Q}$	(6.68)

Table 6.7: Energy balances for the batch reactor.

## 6.6 Summary

Tables 6.7–6.10 summarize the important energy balances for the batch, continuous-stirred-tank, semi-batch, and plug-flow reactors. In contrast to the material balance, which is reasonably straightforward, choosing the proper energy balance requires some care. It is unwise to select an energy balance from a book without carefully considering the assumptions that have been made in the derivation of that particular

Neglect kinetic and potential energies	
$\frac{dU}{dt} = Q_f \rho_f \hat{H}_f - Q_p \hat{H} + \dot{Q} + \dot{W}_s + \dot{W}_b$	(6.69)
Neglect shaft work	
$\frac{dU}{dt} + P \frac{dV_R}{dt} = Q_f \rho_f \hat{H}_f - Q_p \hat{H} + \dot{Q}$	(6.70)
$\frac{dH}{dt} - V_R \frac{dP}{dt} = Q_f \rho_f \hat{H}_f - Q_p \hat{H} + \dot{Q}$	(6.71)
Single phase	
$V_R \rho \hat{C}_P \frac{dT}{dt} - \alpha T V_R \frac{dP}{dt} + \sum_j H_j \frac{dn_j}{dt} = Q_f \rho_f \hat{H}_f - Q_p \hat{H} + \dot{Q}$	(6.72)
$V_R \rho \hat{C}_P \frac{dT}{dt} - \alpha T V_R \frac{dP}{dt} = - \sum_i \Delta H_{Ri} \gamma_i V_R + \sum_j c_{jf} Q_f (H_{jf} - H_j) + \dot{Q}$	(6.73)
a. Incompressible-fluid or constant-pressure reactor	
$V_R \rho \hat{C}_P \frac{dT}{dt} = - \sum_i \Delta H_{Ri} \gamma_i V_R + \sum_j c_{jf} Q_f (H_{jf} - H_j) + \dot{Q}$	(6.74)
b. Constant-volume reactor	
$V_R \rho \hat{C}_V \frac{dT}{dt} = - \sum_i \left[ \Delta H_{Ri} - \alpha V_R \sum_j v_{ij} \left( \frac{\partial P}{\partial n_j} \right)_{T,V,n_k} \right] \gamma_i V_R + \sum_j c_{jf} Q_f (H_{jf} - H_j) + \dot{Q}$	(6.75)
b.1 Constant-volume reactor, ideal gas	
$V_R \rho \hat{C}_V \frac{dT}{dt} = - \sum_i (\Delta H_{Ri} - RT \tilde{v}_i) \gamma_i V_R + \sum_j c_{jf} Q_f (H_{jf} - H_j) + \dot{Q}$	(6.76)
c. Steady state, constant $\hat{C}_P$ , $P = P_f$	
$- \sum_i \Delta H_{Ri} \gamma_i V_R + Q_f \rho_f \hat{C}_P (T_f - T) + \dot{Q} = 0$	(6.77)

Table 6.8: Energy balances for the CSTR.

<b>Neglect kinetic and potential energies</b>	
$\frac{dU}{dt} = Q_f \rho_f \hat{H}_f + \dot{Q} + \dot{W}_s + \dot{W}_b$	(6.78)
<b>Neglect shaft work</b>	
$\frac{dU}{dt} + P \frac{dV_R}{dt} = Q_f \rho_f \hat{H}_f + \dot{Q}$	(6.79)
$\frac{dH}{dt} - V_R \frac{dP}{dt} = Q_f \rho_f \hat{H}_f + \dot{Q}$	(6.80)
<b>Single phase[t]</b>	
$V_R \rho \hat{C}_P \frac{dT}{dt} - \alpha T V_R \frac{dP}{dt} + \sum_j H_j \frac{dn_j}{dt} = Q_f \rho_f \hat{H}_f + \dot{Q}$	(6.81)
$V_R \rho \hat{C}_P \frac{dT}{dt} - \alpha T V_R \frac{dP}{dt} = - \sum_i \Delta H_{Ri} r_i V_R + \sum_j c_{jf} Q_f (H_{jf} - H_j) + \dot{Q}$	(6.82)
<b>a. Incompressible-fluid or constant-pressure reactor</b>	
$V_R \rho \hat{C}_P \frac{dT}{dt} = - \sum_i \Delta H_{Ri} r_i V_R + \sum_j c_{jf} Q_f (H_{jf} - H_j) + \dot{Q}$	(6.83)
<b>a.1 Constant <math>\hat{C}_P</math></b>	
$V_R \rho \hat{C}_P \frac{dT}{dt} = - \sum_i \Delta H_{Ri} r_i V_R + Q_f \rho_f \hat{C}_P (T_f - T) + \dot{Q}$	(6.84)

Table 6.9: Energy balances for the semi-batch reactor.

energy balance. Adding to the confusion, many books do not state clearly what assumptions have been made in deriving various energy balances, and some books list energy balances that are simply incorrect. See Denn [9] for a thorough discussion of common errors in energy balances and a list of books containing incorrect energy balances.

Nonisothermal reactor design requires the simultaneous solution of the appropriate energy balance and the species material balances. For the batch, semi-batch, and steady-state plug-flow reactors, these balances are sets of initial-value ODEs that must be solved numerically. In very limited situations (constant thermodynamic properties, single

<b>Neglect kinetic and potential energies and shaft work</b>	
$\frac{\partial}{\partial t}(\rho \hat{U}) = - \frac{1}{A_c} \frac{\partial}{\partial z} (Q \rho \hat{H}) + \dot{q}$	(6.85)
<b>Heat transfer with an overall heat-transfer coefficient</b>	
$\dot{q} = \frac{2}{R} U^o (T_a - T)$	(6.86)
<b>Steady state</b>	
$\frac{d}{dV} (Q \rho \hat{H}) = \dot{q}$	(6.87)
<b>Single phase</b>	
$Q \rho \hat{C}_P \frac{dT}{dV} + Q(1 - \alpha T) \frac{dP}{dV} = - \sum_i \Delta H_{Ri} r_i + \dot{q}$	(6.88)
<b>a. Neglect pressure drop, or ideal gas</b>	
$Q \rho \hat{C}_P \frac{dT}{dV} = - \sum_i \Delta H_{Ri} r_i + \dot{q}$	(6.89)
<b>b. Incompressible fluid</b>	
$Q \rho \hat{C}_P \frac{dT}{dV} + Q \frac{dP}{dV} = - \sum_i \Delta H_{Ri} r_i + \dot{q}$	(6.90)

Table 6.10: Energy balances for the plug-flow reactor.

reaction, adiabatic), one can solve the energy balance to get an algebraic relation between temperature and concentration or molar flowrate.

The nonlinear nature of the energy and material balances can lead to multiple steady-state solutions. Steady-state solutions may be unstable, and the reactor can exhibit sustained oscillations. These reactor behaviors were illustrated with exothermic CSTRs and autothermal tubular reactors.

## Notation

$A$	heat transfer area
$A_c$	reactor tube cross-sectional area
$A_i$	preexponential factor for rate constant $i$
$c_j$	concentration of species $j$
$c_{jf}$	feed concentration of species $j$
$c_{js}$	steady-state concentration of species $j$
$c_{j0}$	initial concentration of species $j$
$C_P$	constant-pressure heat capacity
$C_{Pj}$	partial molar heat capacity
$C_{Ps}$	heat capacity per volume
$\hat{C}_P$	constant-pressure heat capacity per mass
$\hat{C}_V$	constant-volume heat capacity per mass
$\Delta C_P$	heat capacity change on reaction, $\Delta C_P = \sum_j \nu_j C_{Pj}$
$E_i$	activation energy for rate constant $i$
$E_k$	total energy of stream $k$
$\hat{E}_k$	total energy per mass of stream $k$
$\Delta G^\circ$	Gibbs energy change on reaction at standard conditions
$H_j$	partial molar enthalpy
$\hat{H}$	enthalpy per unit mass
$\Delta H_{Ri}$	enthalpy change on reaction, $\Delta H_{Ri} = \sum_j \nu_{ij} H_j$
$\Delta H^\circ$	enthalpy change on reaction at standard conditions
$k_i$	reaction rate constant for reaction $i$
$k_m$	reaction rate constant evaluated at mean temperature $T_m$
$K_i$	equilibrium constant for reaction $i$
$\hat{K}$	kinetic energy per unit mass
$l$	tubular reactor length
$m_k$	total mass flow of stream $k$
$n$	reaction order
$n_j$	moles of species $j$ , $V_R c_j$
$n_r$	number of reactions in the reaction network
$n_s$	number of species in the reaction network
$N_j$	molar flow of species $j$ , $Q c_j$
$N_{jf}$	feed molar flow of species $j$ , $Q c_{jf}$
$P$	pressure
$P_j$	partial pressure of component $j$
$\dot{q}$	heat transfer rate per volume for tubular reactor, $\dot{q} = \frac{2}{R} U^\circ (T_m - T)$
$Q$	volumetric flowrate
$Q_f$	feed volumetric flowrate

$\dot{Q}$	heat transfer rate to reactor, usually modeled as $\dot{Q} = U^\circ A (T_a - T)$
$r_i$	reaction rate for $i$ th reaction
$r_{\text{tot}}$	total reaction rate, $\sum_i r_i$
$R$	gas constant
$R_j$	production rate for $j$ th species
$t$	time
$T$	temperature
$T_a$	temperature of heat transfer medium
$T_m$	mean temperature at which $k$ is evaluated
$U^\circ$	overall heat transfer coefficient
$\hat{U}$	internal energy per mass
$v_k$	velocity of stream $k$
$V$	reactor volume variable
$V_j$	partial molar volume of species $j$
$V_j^\circ$	specific molar volume of species $j$
$V_R$	reactor volume
$\Delta V_i$	change in volume upon reaction $i$ , $\sum_j \nu_{ij} V_j$
$\dot{W}$	rate work is done on the system
$x_j$	number of molecules of species $j$ in a stochastic simulation
$x_j$	molar conversion of species $j$
$y_j$	mole fraction of gas-phase species $j$
$z$	reactor length variable
$\alpha$	coefficient of expansion of the mixture, $\alpha = (1/V)(\partial V/\partial T)_{P,n_j}$
$\varepsilon_i$	extent of reaction $i$
$\theta$	reactor residence time, $\theta = V_R/Q_f$
$\nu_{ij}$	stoichiometric coefficient for species $j$ in reaction $i$
$\bar{\nu}_i$	$\sum_j \nu_{ij}$
$\rho$	mass density
$\rho_k$	mass density of stream $k$
$\hat{\Phi}$	potential energy per mass

## 6.7 Exercises

### Exercise 6.1: Batch energy balance with heat transfer

Consider Example 6.2 with the following additional data:

$$\begin{aligned}k_m &= 0.01725 \text{ L/mol} \cdot \text{min} \\T_m &= 300 \text{ K} \\E_a/R &= 2660 \text{ K}\end{aligned}$$

so the temperature dependence of the rate constant is given by

$$k = k_m \exp \left( -\frac{E_a}{R} \left( \frac{1}{T} - \frac{1}{T_m} \right) \right)$$

- What is the adiabatic temperature rise for the reactor?
- How long does it take to reach 95% conversion if the reactor operates isothermally at 27°C?
- How long does it take to reach 95% conversion if the reactor operates adiabatically? Plot  $c_A$  and  $T$  versus time for this case. Put in enough points so we can see a smooth curve.
- Plot  $c_A$  and  $T$  versus time for the nonadiabatic case with heat exchange:

$$U^o A/V_R = 0.01 \text{ kcal}/(\text{min L K})$$

and the temperature of the heat transfer fluid is  $T_a = 27^\circ\text{C}$ .

- Assume the batch is ruined if the temperature exceeds 350 K during the run. What value of heat-transfer coefficient ( $U^o A/V_R$ ) should your design achieve so that this temperature is not exceeded. How long does it take to reach 95% conversion with your design? How should you operate the reactor if you want to speed things up but cannot violate the 350 K limit?

### Exercise 6.2: Batch energy balance with constrained heat-transfer

Consider Example 6.2 with the following additional data:

$$\begin{aligned}k_m &= 0.01725 \text{ L/mol} \cdot \text{min} \\T_m &= 300 \text{ K} \\E_a/R &= 2660 \text{ K}\end{aligned}$$

so the temperature dependence of the rate constant is given by

$$k = k_m \exp \left( -\frac{E_a}{R} \left( \frac{1}{T} - \frac{1}{T_m} \right) \right)$$

The heat-transfer equipment accidentally has been under-designed by using the average rather than the maximum required rate of heat transfer

$$\dot{Q} = \frac{-2.3 \times 10^4}{551} = -41.7 \text{ kcal/min} \quad (6.91)$$

which is much less in absolute value than the required  $-828 \text{ kcal/min}$  calculated in Example 6.2. Assume the control system can only remove heat at this constant rate.

- What is the maximum temperature achieved in this reactor and when does this maximum occur?
- How long does it take for the control system to return the temperature to 27°C?
- When does the reactor achieve 95% conversion?

### Exercise 6.3: CSTR energy balance with multiple reactions

Allyl chloride is to be produced in a 0.83 ft<sup>3</sup> CSTR [20]



The feed is a 4:1 molar ratio of propylene to chlorine and it enters at a feed rate of 0.85 lbmol/hr, 2.0 atm of pressure, and 392°F. The reactor pressure may be assumed constant.

The rate constants have units of lbmol/(hr ft<sup>3</sup> · 0 atm<sup>2</sup>) and are

$$k_1 = 206000 \times \exp \frac{-27200}{RT}$$

$$k_2 = 11.7 \times \exp \frac{-6860}{RT}$$

where  $T$  is in degrees Rankine and  $R$  is in Btu/(lbmol · R). The rate expressions are

$$r_1 = k_1 P_{\text{C}_3\text{H}_6} P_{\text{Cl}_2}$$

$$r_2 = k_2 P_{\text{C}_3\text{H}_6} P_{\text{Cl}_2}$$

The thermodynamic data for this reaction are listed in Table 6.11[16]. The partial molar heat capacities can be calculated using

$$C_{p,j} = A_j + B_j T + C_j T^2 + D_j T^3 \quad \text{cal/mol K}$$

Component	$H_f^{298\text{ K}}$ kcal/mol	A	B ( $\times 10^2$ )	C ( $\times 10^5$ )	D ( $\times 10^9$ )
Cl <sub>2</sub>	0.00	6.432	0.8082	-0.9241	3.695
C <sub>3</sub> H <sub>6</sub>	4.88	0.866	5.602	-2.771	5.266
C <sub>3</sub> H <sub>5</sub> Cl	-0.15	0.604	7.277	-5.442	17.42
HCl	-22.06	7.235	-0.172	0.2976	-0.931
1,2-C <sub>3</sub> H <sub>6</sub> Cl <sub>2</sub>	-39.60	2.496	8.729	-6.219	18.49

Table 6.11: Thermodynamic data for allyl chloride example.

- (a) Compute the molar flowrates of Cl<sub>2</sub>, C<sub>3</sub>H<sub>5</sub>Cl and C<sub>3</sub>H<sub>6</sub>Cl<sub>2</sub>, and the reactor temperature for adiabatic operation. You need to use an algebraic equation solver for this problem, a reasonable first guess on the temperature is 1200°R.
- (b) Now consider an approximate solution along the lines presented by Smith [20] in which the heat of reaction is computed at the inlet temperature and assumed constant. Compare the heat of reaction in Part 6.3a with the value at the inlet temperature, and comment on the differences in the molar flowrates and the temperatures in Parts 6.3a and 6.3b.

#### Exercise 6.4: CSTR stability

An adiabatic CSTR with a first-order, liquid-phase reaction



operates at the conditions shown below

Parameter	Value	Units
$T_f$	298	K
$c_{Af}$	3	kmol/m <sup>3</sup>
$Q_f$	$60 \times 10^{-6}$	m <sup>3</sup> /s
$\Delta H_R$	$-2.09 \times 10^8$	J/kmol
$\hat{C}_p$	$4.19 \times 10^3$	J/kg K
$\rho$	$10^3$	kg/m <sup>3</sup>
$V_R$	$18 \times 10^{-3}$	m <sup>3</sup>
$k$	$4.48 \times 10^6 \exp(-7550/T)$	s <sup>-1</sup> ; T in K

Find the steady-state temperatures and conversions. How many steady states can you find? Which steady states are stable and which are unstable?

#### Exercise 6.5: CSTR energy balance

The liquid-phase reactions



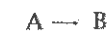
are each first order in the concentration of A. The feed to a nonisothermal CSTR contains pure A at 45 °C and 5 mol/L. Additional information for the reactions and the reactor are provided below. We want to operate this reactor such that the selectivity to B is greater than the selectivity to C, i.e., the rate at which B forms is greater than the rate at which C forms.

Parameter	Value	Units
$M_t$	93,200	g/min
$\Delta H_{r1}$	-12,000	cal/mol
$\Delta H_{r2}$	-15,000	cal/mol
$V$	1000	liter
$Q_f$	100	liter/min
$C_{pf}$	0.22	cal/g K
$k_1$	$3.16 \times 10^{14} \exp(-12,500/T)$	min <sup>-1</sup> ; T in K
$k_2$	$2.52 \times 10^9 \exp(-8,500/T)$	min <sup>-1</sup> ; T in K

- (a) Defining the selectivity as  $S = R_B/R_C$ , determine the rate at which heat is transferred to the surroundings to achieve a selectivity of five.
- (b) Determine the rate at which heat is transferred to the surroundings to achieve a selectivity of four.
- (c) Explain why the answer to Part A or Part B is greater.

#### Exercise 6.6: Finding the CSTR steady state

The first-order reaction



has reaction-rate expression

$$r = k_0 e^{-E/T} c_A$$

in which the units for the rate are kmol/m<sup>3</sup> s,  $c_A$  is in kmol/m<sup>3</sup>, and  $T$  is in K. Determine the effluent temperature and concentration of A for a reactor operating at the conditions listed in the following table.



Parameter	Value	Units
$E$	7550	K
$A_h$	3600	cm <sup>2</sup>
$T_a$	312	K
$T_f$	298	K
$c_{Af}$	3	kmol/m <sup>3</sup>
$U^o$	0.225	J/cm <sup>2</sup> sec K
$\Delta H_R$	$-8.09 \times 10^8$	J/kmol
$k_0$	$4.48 \times 10^6$	1/s
$\rho$	$10^3$	kg/m <sup>3</sup>
$C_p$	$4.19 \times 10^3$	J/kg K
$V_R$	$18 \times 10^{-3}$	m <sup>3</sup>
$Q_f$	$60 \times 10^{-6}$	m <sup>3</sup> /s

### Exercise 6.7: Testing a CSTR operating condition for stability

Consider the reaction of Exercise 6.6.

- (a) Determine if a CSTR operating at 347.6 K is stable or unstable for the following conditions.

Parameter	Value	Units
$E$	7550	K
$T_f$	298	K
$c_{Af}$	3	kmol/m <sup>3</sup>
$\Delta H_R$	$-2.09 \times 10^8$	J/kmol
$k_0$	$4.48 \times 10^6$	1/s
$C_p$	$4.19 \times 10^3$	J/kg-K
$\rho$	$10^3$	kg/m <sup>3</sup>
$V_R$	$18 \times 10^{-3}$	m <sup>3</sup>
$Q_f$	$60 \times 10^{-6}$	m <sup>3</sup> /s

### Exercise 6.8: CSTR ignition

Consider the CSTR parameter values listed in Table 6.1 and the steady-state  $x, T$  versus residence time plots in Figures 6.7 and 6.8.

- (a) Write down the dynamic CSTR equations for the adiabatic reactor. How many parameters are there and what are their values? Choose consistent units.
- (b) Choose a residence time of  $\theta = 10$  min. According to Figures 6.7 and 6.8, how many steady-state solutions are there at this residence time? What are the stable ( $c_A, T$ ) steady-state values and what are the unstable ones? You can just read these off the plots.

Next simulate the dynamic reactor. Choose as initial conditions for  $c_A$  and  $T$ , values near the lower-branch and upper-branch steady states. Simulate the reactor for a time long enough for it to reach steady state. Check that the dynamic simulation converges to a steady state consistent with Figures 6.7 and 6.8 (this is your program debugging check). Once you are sure the program is working correctly simulate the reactor starting from  $c_A = c_{Af}, T = T_f$ . Increase only the initial temperature and simulate again. How warm can the reactor be started before it ignites and reaches the upper branch steady state instead of the lower branch. Hand in plots of  $c_A(t)$  and  $T(t)$  starting one degree below and above this critical temperature.

- (c) If it were advantageous to operate at an unstable steady state, how might you modify the reactor design to stabilize the steady state? Sketch your control scheme. Be sure to show and label clearly your sensor and actuator (if you are interested you can simulate the controlled reactor and see what controller gain is required to achieve stability).
- (d) Change the residence time to  $\theta = 32$  min. How many steady-state solutions are there for this residence time. Start the reactor from the feed conditions and simulate for 1000 min. Does anything unusual happen? Describe what the reactor is doing. Does your simulation point out any safety issues? Can you think of a less violent method to start up the reactor? What values of ( $c_A, T$ ) are you trying to avoid during start up?

### Exercise 6.9: Removing heat in a CSTR

The liquid-phase reaction



$$r = kc_A$$

is to be carried out in a nonisothermal, nonadiabatic CSTR operating at 95 °C. The feed contains pure A at 45 °C and 5 mol/L. Using the additional information provided below determine the temperature of the cooling fluid medium.

Parameter	Value	Units
$E$	12,500	K
$\Delta H_R$	-3700	cal/mol
$V_R$	1000	L
$\rho_f$	932	g/L
$Q_f$	100	L/min
$U^o$	0.54	cal/ min cm <sup>2</sup> K
$C_p$	0.22	cal/g K
$k_0$	$3.16 \times 10^{14}$	min <sup>-1</sup>
$A$	$1.8 \times 10^4$	cm <sup>2</sup>

**Exercise 6.10: CSTR and energy balance**

Consider the reaction



in a liquid-phase, constant-volume CSTR with heating/cooling system. The heat capacity and density of the mixture can be assumed independent of temperature and composition. The following data and parameters are known.

Parameter	Value	Units
$E$	7554	K
$T_f$	298	K
$\hat{C}_{Ps}$	4.19	kJ/kg K
$c_{Af}$	3.0	kmol/m <sup>3</sup>
$V_R$	0.018	m <sup>3</sup>
$k_0$	$4.48 \times 10^6$	s <sup>-1</sup>
$\Delta H_R$	$-2.09 \times 10^5$	kJ/kmol
$\rho$	$10^3$	kg/m <sup>3</sup>
$Q_f$	$6.0 \times 10^{-5}$	m <sup>3</sup> /s

- At what temperature must the reactor be operated to achieve 80% conversion?
- What is the heat duty on the heating/cooling system when operating at this steady state and are you heating or cooling the reactor, i.e., what is the sign of the heat duty?

**Exercise 6.11: CSTR stability**

- For the reaction of A to products, sketch the conversion of A versus mean reactor residence time for an isothermal CSTR.
- Draw a separate sketch of conversion of A and reactor temperature versus residence time for an exothermic reaction in an adiabatic CSTR that exhibits steady-state multiplicity.
- Describe an ignition and extinction point and locate them on the sketch of the CSTR with steady-state multiplicity. What is special about these points? What does hysteresis mean? What are the practical operational problems associated with trying to operate a CSTR near one of these points. Is it possible to operate near one of these points? If not, why not? If so, how do you overcome the operational problems described above?

**Exercise 6.12: Attractors in the CSTR**

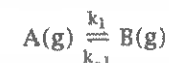
Show which initial conditions are attracted to which solutions in Figure 6.30.

**Exercise 6.13: Semi-batch reactor balance**

Derive the CSTR energy balance given by Equation 6.72 in Table 6.8 by making the assumptions listed in the table. Now derive the semi-batch reactor Equation 6.81 in Table 6.9. Why are these two energy balances identical even though they apply to different reactor types?

**Exercise 6.14: Adiabatic plug-flow reactor**

Consider the elementary, gas-phase isomerization reaction shown below:



This isomerization is to be carried out in an *adiabatic* PFR. Only A and B are present in the reactor feed stream. The feed rate,  $Q_f$ , concentrations,  $c_{Af}$  and  $c_{Bf}$ , temperature,  $T_f$ , and pressure,  $P_f$ , are known at the inlet to the PFR. Pressure drop in the reactor can be neglected.

- What does the stoichiometry tell you about the relationship between the molar flux of B,  $N_B$ , and the molar flux of A,  $N_A$ , at any point in the reactor?
- Write down the mole balance for component A. Remove any reference to  $c_B$  in this equation with the result from 6.14a. Also write down an expression to evaluate the volumetric flowrate,  $Q$ , because it is not constant.
- Starting from the general energy balance equation for a PFR given below, determine the appropriate energy balance equation for the PFR in this problem. Be sure that all variables in both the mole and energy balances can be evaluated in terms of your dependent variables,  $T$  and  $N_A$ .

$$Q\rho\hat{C}_P\frac{dT}{dV} = -\Delta H_R r + \frac{2}{R}U^0(T_a - T)$$

- Compute the equilibrium concentration and conversion of component A that could be achieved if the reactor operated *isothermally* at the inlet temperature given below. The molar ratio of A to B in the feed is 4:1. Assume ideal gas behavior.  
 $T_f = 830^\circ\text{R}$ ;  $P_f = 180\text{ psia}$ ;  $K = k_1/k_{-1} = 1.5$ ;  $R = 10.731\text{ psi}\cdot\text{ft}^3/\text{lbmol}\cdot\text{R}$
- If the equilibrium conversion of component A determined in 6.14d is achieved in the *adiabatic* reactor, calculate the outlet temperature using

the inlet conditions specified in 6.14d. Assume the heat capacities of A and B are the same and the heat capacity and heat of reaction are constant at the values below.

$$\Delta H_R = 5850 \text{ BTU/lbmol} ; C_p = 20.0 \text{ BTU/lbmol}^\circ\text{R} ; Q_f = 10,000 \text{ ft}^3/\text{hr}$$

### Exercise 6.15: PFR energy balance with a single reaction

The gas-phase reaction



is carried out in a PFR. The reaction-rate expression is

$$r = kP_A P_B$$

in which

$$k = 2.06 \times 10^5 \exp(-13,700/T)$$

The units for  $r$  are lbmol/hr ft<sup>3</sup>,  $P$  is in atm, and  $T$  is in °R. The feed is a 4:1 molar ratio of A and B and enters at 500 °F, at a feed rate of 0.85 lbmol/hr and 2 atm pressure.

Assuming constant thermodynamic properties, determine the length of a 2-in inner-diameter (ID) tubular reactor that operates adiabatically and provides 93% conversion of B. The heat of reaction is  $\Delta H_r = -48,000 \text{ Btu/lbmol}$  and the mixture heat capacity is  $C_p = 110 \text{ Btu/lbmol}^\circ\text{R}$ .

### Exercise 6.16: Temperature and the equilibrium limit

The following exothermic, reversible, gas-phase reaction is to be conducted in a PFR



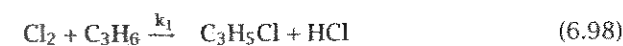
$$r = k \left( c_A - \frac{c_B}{K} \right)$$

You may assume the heat of reaction ( $\Delta H_r = -20,000 \text{ cal/mol}$ ) and the component heat capacities ( $C_{pA} = C_{pB} = 50 \text{ cal/mol K}$ ) are constant. The equilibrium constant at 298 K is  $K = 10^5$ .

Determine the maximum possible conversion of A if the feed contains pure A at a temperature of 300 K.

### Exercise 6.17: PFR energy balance with multiple reactions

Allyl chloride is to be produced in a 25-ft long 2-in ID tube operating as a PFR [20]. The feed is a 4:1 molar ratio of propylene to chlorine and it enters at a feed rate of 0.85 lbmol/hr, 2 atm of pressure, and 392 °F. The reactor pressure may be assumed constant.



The rate constants have units of lbmol/(hr ft<sup>3</sup> atm<sup>2</sup>) and are

$$k_1 = 2.06 \times 10^5 \exp(-27200/(RT))$$

$$k_2 = 11.7 \times \exp \frac{-6860}{RT}$$

in which  $T$  is in °R and  $R$  is in Btu/(lbmol°R). The rate expressions are

$$r_1 = k_1 P_{\text{C}_3\text{H}_6} P_{\text{Cl}_2}$$

$$r_2 = k_2 P_{\text{C}_3\text{H}_6} P_{\text{Cl}_2}$$

The thermodynamic data for this reaction are listed in Table 6.11 [16]. The partial molar heat capacities are expressed in units of cal/mol K and can be calculated using

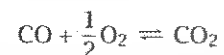
$$C_{pj} = A_j + B_j T + C_j T^2 + D_j T^3$$

Two modes of operation have been considered for this PFR, adiabatic and constant wall temperature. A constant wall temperature of 392 °F can be realized by boiling ethylene glycol on the outer surface of the reactor wall. The inside heat-transfer coefficient is 5 Btu/hr ft<sup>2</sup> °F for a feed rate of 0.85 lbmol/hr and 2 atm total pressure.

- Compute the molar flowrates of Cl<sub>2</sub>, C<sub>3</sub>H<sub>5</sub>Cl and C<sub>3</sub>H<sub>6</sub>Cl<sub>2</sub>, and the reactor temperature as a function of reactor length for adiabatic operation. Plot these dependent variables versus reactor length.
- Compute the molar flowrates of Cl<sub>2</sub>, C<sub>3</sub>H<sub>5</sub>Cl and C<sub>3</sub>H<sub>6</sub>Cl<sub>2</sub>, and the reactor temperature as a function of reactor length for constant wall temperature operation. Plot these dependent variables versus reactor length. What do you notice when you compare the extent of product formation and reactor temperature for both cases? Why do you think this happens?
- Now compare the exact solutions developed above to the approximate solution presented in Smith [20] where the component heat capacities and the heat of reaction are evaluated at the inlet temperature and assumed constant. Plot the molar flowrates of C<sub>3</sub>H<sub>5</sub>Cl and C<sub>3</sub>H<sub>6</sub>Cl<sub>2</sub> for the exact and approximate solutions on different graphs, one for adiabatic and one for nonadiabatic operation. Prepare similar plots for the temperature. Comment on the errors this approximation introduces. Is it sensible that the error is greater for adiabatic operation? Why?

**Exercise 6.18: The adiabatic limit in a PFR**

We wish to convert the CO in a waste stream to CO<sub>2</sub> by reaction in an adiabatic bundle of empty tubes.



The waste stream is a mixture of CO and air, and is mixed with a pure air stream to achieve 100% excess air to CO. This stream enters the reactor tube bundle at 100 °C. You want to calculate the maximum temperature that might be reached in the tube to select appropriate material of construction for the tube bundle.

- What is the maximum outlet temperature that could be achieved? What assumptions do you make to compute this temperature?
- If this temperature is too high for the materials on hand, what simple process design changes can lower the outlet temperature?
- What would you do next if you wanted a more accurate estimate of the temperature of the effluent from the reactor?

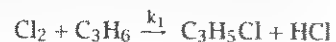
Additional information.

The mean heat capacities and heats of formation at 25 °C are as follows.

Component	$C_{pj}$ cal/mol °C	$-\Delta H_{fj}^\circ$ kcal/mol
CO	6.98	26.42
CO <sub>2</sub>	12.9	94.05
O <sub>2</sub>	8.35	0
N <sub>2</sub>	7.92	0

**Exercise 6.19: Using a simplified PFR energy balance**

Consider the following reaction in an adiabatic, constant-pressure PFR



The rate constant has units of lbmol/(hr ft<sup>3</sup> atm<sup>2</sup>) and is given by

$$k_1 = 2.06 \times 10^5 \exp(-27200/RT)$$

where  $T$  is in °R and  $R$  is in Btu/(lbmol°R). The rate expression is

$$r_1 = k_1 P_{\text{C}_3\text{H}_6} P_{\text{Cl}_2}$$

The feed is a 4:1 molar ratio of propylene to chlorine and it enters at a feed rate of 0.85 lbmol/hr, 2.0 atm pressure, and 392 °F (852°R). The reactor temperature must be maintained below 880°R. The heat of reaction is

$$\Delta H_R = -48,000 \quad \text{Btu/lbmol}$$

and the heat capacity of the reactor feed mixture is

$$C_p = 110 \quad \text{Btu/lbmol}^\circ\text{R}$$

If you assume  $\Delta H_R$  and  $C_p$  are constant and do not change as an inert is added to the feed, determine the minimum inert flowrate (in lbmol/hr) that must be added to the feed to keep the reactor temperature below 880°R when the conversion of chlorine is 95%.

**Exercise 6.20: The effect of heat transfer in an autothermal reactor**

Consider what happens in the ammonia synthesis system described in Example 6.6 if we decrease the heat-transfer parameter,  $\gamma$ . We wish to find the value of  $\gamma$  such that the middle and upper steady states coalesce.

- For a range of  $T_a(0)$  values, solve the initial-value problem, Equation 6.60, and plot the resulting  $T_a(t)$  as in Figure 6.38. Use the parameter values given in Table 6.6.
- Decrease the heat-transfer rate by 20% and repeat this calculation. Continue to decrease  $\gamma$  until you produce Figure 6.41. What is the critical value of  $\gamma$  that corresponds to Figure 6.41?
- What happens if you operate the autothermal reactor with a value of  $\gamma$  less than the critical value found in part 6.20b?

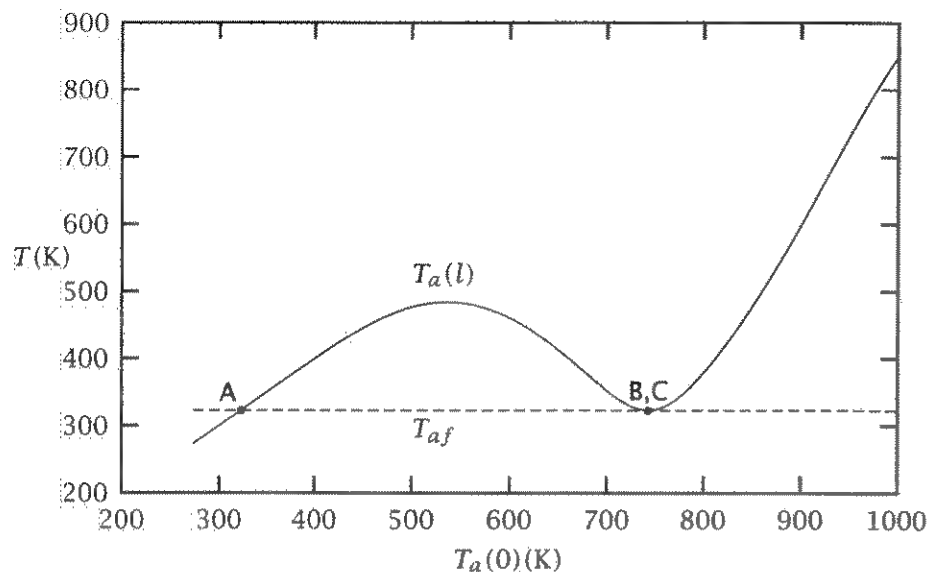
**Exercise 6.21: Moving volume element in PFR**

In Section 6.5 we derived the energy balance for the PFR by considering a fixed volume element. Consider an adiabatic PFR  $\dot{Q} = 0$ , for which we showed

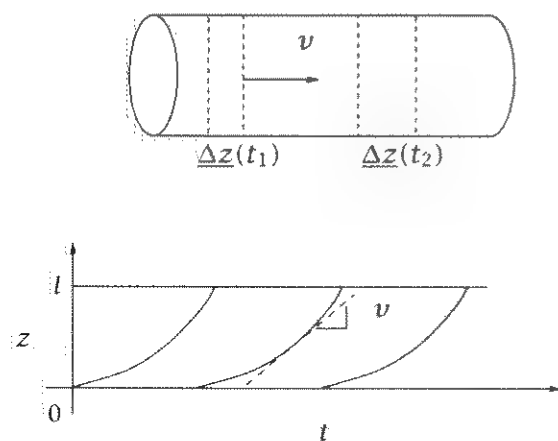
$$\frac{\partial}{\partial t}(\rho \hat{U} A_c \Delta z) = m \hat{H}_z - m \hat{H}_{z+\Delta z} + \underbrace{\dot{W}_b}_{\text{zero}} \quad (6.100)$$

For the fixed volume element we have convection terms because material streams through the boundary, but no work term for moving the boundary because the element is fixed in space. Dividing by  $\Delta z$  and taking the limit leads to the energy balance

$$\frac{\partial(\rho \hat{U})}{\partial t} + \frac{\partial(v \rho \hat{H})}{\partial z} = 0 \quad (6.101)$$



**Figure 6.41:** Coolant temperature at reactor outlet versus temperature at reactor inlet,  $T_a(l)$  versus  $T_a(0)$ , at the critical value of heat-transfer coefficient; upper and middle steady-state solutions have coalesced.



**Figure 6.42:** Volume element moving with the PFR fluid velocity.

in which  $v$  is the PFR fluid velocity.

Consider a volume element that moves with the fluid velocity as depicted in Figure 6.42. The location of the moving element traces out one of the lines shown in the  $(z, t)$  plane. These lines are known as the characteristic lines for Equation 6.101. Notice also the two circular ends of the element move with different velocities if the fluid density is not constant, so this volume element may expand or contract as it moves down the reactor. Applying Equation 6.100 to this moving element produces

$$\frac{\partial}{\partial t} (\underline{\rho \hat{U}} A_c \underline{\Delta z}) = \underbrace{m \hat{H}|_z}_{\text{zero}} - \underbrace{m \hat{H}|_{z+\Delta z}}_{\text{zero}} + \underline{W}_b \quad (6.102)$$

in which the underline represents the quantities in the moving element. Notice for this choice of volume element the convection terms are zero, but the work term for moving the boundary is nonzero. Show that we still arrive at the same energy balance, Equation 6.101, starting with Equation 6.102 for the moving element. It is not difficult to add the heat term to this development as well. The moral of the story is that the choice of volume element is based on convenience; that choice does not affect the final energy balance.

Hints:

1. First show

$$\underline{W}_b = -A_c \frac{\partial(Pv)}{\partial z} \underline{\Delta z}$$

by considering the work done while moving the two ends of the element.

2. Differentiate the left-hand side of Equation 6.102 to obtain

$$\frac{\partial}{\partial t} (\underline{\rho \hat{U}} A_c \underline{\Delta z}) = A_c \left[ \frac{\partial(\underline{\rho \hat{U}})}{\partial t} \underline{\Delta z} + \underline{\rho \hat{U}} \frac{\partial \underline{\Delta z}}{\partial t} \right]$$

3. Compute the change in the volume element size due to the change in the fluid velocity with position, and show

$$\frac{\partial \underline{\Delta z}}{\partial t} = \frac{\partial v}{\partial z} \underline{\Delta z}$$

Use the chain rule to show for  $\underline{\rho \hat{U}} = \rho \hat{U}(z, t)$

$$\frac{\partial(\underline{\rho \hat{U}})}{\partial t} = \frac{\partial(\rho \hat{U})}{\partial t} + \frac{\partial(\rho \hat{U})}{\partial z} \frac{\partial z}{\partial t} \quad (6.103)$$

in which  $\partial z / \partial t = v$  is the fluid velocity because we consider the moving element. The left-hand side of Equation 6.103 is the rate of change of internal energy as we move with the fluid along a characteristic line in Figure 6.42. In the transport literature this derivative also is called the material or substantial derivative. The right-hand side contains the usual partial derivatives in which we hold either  $z$  or  $t$  constant.

4. Combine these results to obtain Equation 6.101.

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### 7.1 Introduction

The analysis and rational design of reactors requires the simultaneous solution of material and energy balances. For a steady-state, tubular, plug-flow reactor, we showed in Chapter 6 that the molar flows and temperature are governed by the following differential equations

$$\frac{d(c_j Q)}{dV} = R_j = \sum_{i=1}^{n_r} \nu_{ij} r_i, \quad j = 1, 2, \dots, n_s \quad (7.1)$$

$$Q \rho \hat{C}_P \frac{dT}{dV} = - \sum_{i=1}^{n_r} \Delta H_{Ri} r_i + \frac{2}{R} U^o (T_a - T) \quad (7.2)$$

A rate expression,  $r_i$ , as a function of concentration and temperature, for each of the  $n_r$  reactions is required to complete the problem description. Equations 7.1 and 7.2 can be used for any reaction within the reactor as long as the plug-flow conditions are satisfied. These balances are applicable to reactions that are homogeneous or heterogeneous; the heterogeneous case is referred to as a fixed-bed reactor.

In a fixed-bed reactor the catalyst pellets are held in place and do not move with respect to a fixed reference frame. Material and energy balances are required for both the fluid, which occupies the interstitial region between catalyst particles, and the catalyst particles, in which the reactions occur. For heterogeneously catalyzed reactions, the effects of intraparticle transport on the rate of reaction must be considered. Catalytic systems operate somewhere between two extremes: kinetic control, in which mass and energy transfer are very rapid; and intraparticle transport control, in which the reaction is very rapid. Separate material and energy balances are needed to describe the concentration and temperature profile inside the catalyst pellet. The concentrations



of each component and the temperature in the pellet are related to the concentrations and temperature in the bulk fluid through the boundary conditions that complete the problem description. Numerical methods are used to solve the resulting coupled differential equations, except in some special cases.

This chapter's goal is to describe the reaction at any location within the reactor. In certain cases, analytical solutions to the transport of mass and energy within the catalyst pellet can be used to develop algebraic relationships that couple the rate of reaction within the pellet to the temperature and concentration that exist in the bulk fluid. Even though the applicability of these relationships is limited, deriving analytical solutions develops intuition and clearly illustrates the coupling between the pellet and the fluid.

Figure 7.1 presents several views of the fixed-bed reactor. Equation 7.1 applies to the bulk fluid in the interstitial volume. The species production rates in the bulk fluid are *essentially zero*. That is the reason we are using a catalyst. Essentially all reaction occurs within the catalyst particles. The fluid in contact with the external surface of the catalyst pellet is denoted with subscript *s*. When we need to discuss both fluid and pellet concentrations and temperatures (see Section 7.7.1), we use a tilde on the variables within the catalyst pellet.

During any catalytic reaction the following steps occur:

1. transport of reactants and energy from the bulk fluid up to the catalyst pellet exterior surface,
2. transport of reactants and energy from the external surface into the porous pellet,
3. adsorption, chemical reaction, and desorption of products at the catalytic sites,
4. transport of products from the catalyst interior to the external surface of the pellet, and
5. transport of products into the bulk fluid.

The coupling of transport processes with chemical reaction can lead to concentration and temperature gradients within the pellet, between the surface and the bulk, or both. For the remainder of this section, the pellet, surface and bulk temperatures are assumed to be equal.

The five steps just listed describe the simultaneous processes that occur in the fixed-bed reactor. Usually one or at most two of these

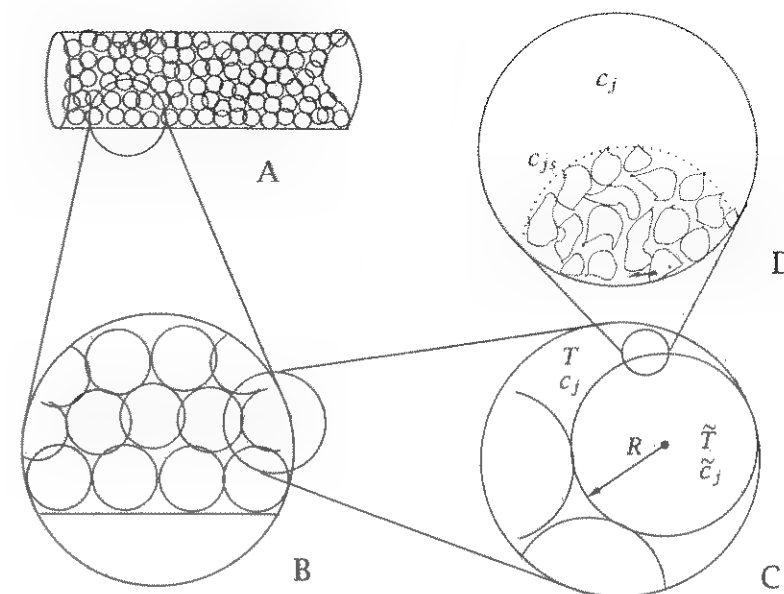


Figure 7.1: Expanded views of a fixed-bed reactor.

steps are rate limiting and act to influence the overall rate of reaction in the pellet. The other steps are inherently faster than the slow step(s) and can accommodate any change in the rate of the slow step. The system is *intraparticle transport controlled* if step 2 is the slow process (sometimes referred to as diffusion limited). For *kinetic or reaction control*, step 3 is the slowest process. Finally, if step 1 is the slowest process, the reaction is said to be *externally transport controlled*.

In this chapter, we model the system on the scale of Figure 7.1C. The problem is solved for one pellet by averaging the microscopic processes that occur on the scale of Figure 7.1D over the volume of the pellet or over a solid surface volume element. This procedure requires an effective diffusion coefficient,  $D_j$ , to be identified that contains information about the physical diffusion process and pore structure.

## 7.2 Catalyst Properties

The chemical steps for catalytic reactions: adsorption, desorption and surface reaction, are covered in Chapter 5. These processes occur at a fluid-solid interface and the rates scale directly with the total number of surface sites,  $\bar{c}_m$ . Table 7.1 lists some of the important commercial

catalysts and their uses [12]. References [12, 13, 14, 15] provide an excellent introduction to catalysis and catalytic processes. To make a catalytic process commercially viable, the number of sites per unit reactor volume should be such that the rate of product formation is on the order of 1 mol/L-hour [21]. In the case of metal catalysts, the metal is generally dispersed onto a high-area oxide such as alumina. Metal oxides also can be dispersed on a second carrier oxide such as vanadia supported on titania, or it can be made into a high-area oxide. These carrier oxides can have surface areas ranging from 0.05 m<sup>2</sup>/g to greater than 100 m<sup>2</sup>/g. The carrier oxides generally are pressed into shapes or extruded into pellets. The following shapes are frequently used in applications: 20–100  $\mu$ m diameter spheres for fluidized-bed reactors, 0.3–0.7 cm diameter spheres for fixed-bed reactors, 0.3–1.3 cm diameter cylinders with a length-to-diameter ratio of 3–4, and up to 2.5 cm diameter hollow cylinders or rings.

Figure 7.1D shows a schematic representation of the cross section of a single pellet. The solid density is denoted  $\rho_s$ . The pellet volume consists of both void and solid. The pellet void fraction (or porosity) is denoted by  $\epsilon$  and

$$\epsilon = \rho_p V_g$$

in which  $\rho_p$  is the effective particle or pellet density and  $V_g$  is the pore volume. The pore structure is a strong function of the preparation method, and catalysts can have pore volumes ( $V_g$ ) ranging from 0.1–1 cm<sup>3</sup>/g pellet. The pores can be the same size or there can be a bimodal distribution with pores of two different sizes, a large size to facilitate transport and a small size to contain the active catalyst sites. Pore sizes can be as small as molecular dimensions (several Ångströms) or as large as several millimeters.

Total catalyst area is generally determined using a physically adsorbed species, such as N<sub>2</sub>. The procedure was developed in the 1930s by Brunauer, Emmett and Teller [7], and the isotherm they developed is referred to as the BET isotherm. In the physisorption process multi-layers are allowed to form. The BET isotherm model treats the first layer differently from all subsequent layers and leads to the following expression

$$v = \frac{v_m c P}{(P_0 - P)(1 + (c - 1)P/P_0)} \quad (7.3)$$

in which  $c$  is a parameter related to the heats of adsorption in the first layer and subsequent layers,  $P_0$  is the normal saturation vapor

Catalyst	Reaction
Metals (e.g., Ni, Pd, Pt, as powders or on supports) or metal oxides (e.g., Cr <sub>2</sub> O <sub>3</sub> )	C=C bond hydrogenation, e.g., olefin + H <sub>2</sub> → paraffin
Metals (e.g., Cu, Ni, Pt)	C=O bond hydrogenation, e.g., acetone + H <sub>2</sub> → isopropanol
Metal (e.g., Pd, Pt)	Complete oxidation of hydrocarbons, oxidation of CO
Fe (supported and promoted with alkali metals)	3H <sub>2</sub> + N <sub>2</sub> → 2NH <sub>3</sub>
Ni	CO + 3H <sub>2</sub> → CH <sub>4</sub> + H <sub>2</sub> O (methanation)
Fe or Co (supported and promoted with alkali metals)	CO + H <sub>2</sub> → paraffins + olefins + H <sub>2</sub> O + CO <sub>2</sub> (+ other oxygen-containing organic compounds) (Fischer-Tropsch reaction)
Cu (supported on ZnO, with other components, e.g., Al <sub>2</sub> O <sub>3</sub> )	CO + 2H <sub>2</sub> → CH <sub>3</sub> OH
Re + Pt (supported on $\eta$ -Al <sub>2</sub> O <sub>3</sub> or $\gamma$ -Al <sub>2</sub> O <sub>3</sub> promoted with chloride)	Paraffin dehydrogenation, isomerization and dehydrocyclization
Solid acids (e.g., SiO <sub>2</sub> -Al <sub>2</sub> O <sub>3</sub> , zeolites)	Paraffin cracking and isomerization
$\gamma$ -Al <sub>2</sub> O <sub>3</sub>	Alcohol → olefin + H <sub>2</sub> O
Pd supported on acidic zeolite	Paraffin hydrocracking
Metal-oxide-supported complexes of Cr, Ti or Zr	Olefin polymerization, e.g., ethylene → polyethylene
Metal-oxide-supported oxides of W or Re	Olefin metathesis, e.g., 2 propylene → ethylene + butene
Ag(on inert support, promoted by alkali metals)	Ethylene + 1/2 O <sub>2</sub> → ethylene oxide (with CO <sub>2</sub> + H <sub>2</sub> O)
V <sub>2</sub> O <sub>5</sub> or Pt	2 SO <sub>2</sub> + O <sub>2</sub> → 2 SO <sub>3</sub>
V <sub>2</sub> O <sub>5</sub> (on metal oxide support)	Naphthalene + 9/2 O <sub>2</sub> → phthalic anhydride + 2 CO <sub>2</sub> + 2 H <sub>2</sub> O
Bismuth molybdate	Propylene + 1/2 O <sub>2</sub> → acrolein
Mixed oxides of Fe and Mo	CH <sub>3</sub> OH + O <sub>2</sub> → formaldehyde (with CO <sub>2</sub> + H <sub>2</sub> O)
Fe <sub>3</sub> O <sub>4</sub> or metal sulfides	H <sub>2</sub> O + CO → H <sub>2</sub> + CO <sub>2</sub>

**Table 7.1:** Industrial reactions over heterogeneous catalysts. This material is used by permission of John Wiley & Sons, Inc., Copyright ©1992 [12].

pressure of the adsorbate at the adsorption temperature, and  $v$  is the volume of gas adsorbed at a particular pressure  $P$ . Uptake data are fit to Equation 7.3 to determine  $c$  and  $v_m$ , in which  $v_m$  is the volume of gas that corresponds to a monolayer of the entire solid. One then assumes each adsorbed molecule in the volume  $v_m$  covers a particular area, i.e., the footprint of the molecule at the adsorption temperature. For  $N_2$  at 77 K the area of each molecule is  $0.162 \text{ nm}^2$ . The total area per gram of catalyst is denoted by  $S_g$ .

### 7.2.1 Effective Diffusivity

We define a diffusion coefficient that takes into account the random distribution of solid and void as one moves from the exterior to the interior of the pellet, so that the balance can be written over the entire pellet. This idea leads to the concept of an effective diffusivity. Starting at any pore opening in Figure 7.1D and moving inward along a straight line to the center, one quickly encounters solid. Inside, any void space transport can be described by the diffusion of a substance under a concentration gradient. The diffusion rate is controlled by the microscopic details of the transport in the voids.

Section 7.2 discussed pore sizes and pore size distributions. Assume here that all pores are straight cylinders and that the pellet has a single pore size distribution. The average radius,  $r_a$ , of the pores is then

$$r_a = \frac{2V_g}{S_g} \quad (7.4)$$

The type of diffusion in the pore, *bulk* or Knudsen, depends on whether the diffusing species collide more often with each other or with the pore wall surface. Liquid-phase diffusion is described by liquid-liquid intermolecular collisions. For a gas-phase molecule the mean free path,  $\lambda$ , is

$$\lambda = \frac{k_B T}{\sqrt{2} \pi \sigma^2 P}$$

and it has a value of approximately  $10^{-5} \text{ cm}$  at 1.0 atm and room temperature for simple gases. When  $\lambda < r_a$  by an order of magnitude, bulk flow dominates and diffusion is described by collisions in the gas phase. When  $\lambda > r_a$  by an order of magnitude, Knudsen flow dominates and diffusion is described by collisions with the pore walls. Finally, when  $r_a$  is approximately equal to the dimensions of the diffusing reactants, transport is described by surface or configurational diffusion. We only consider the first two in this text.

A number of correlations have been developed for bulk flow that describe binary interactions. One such correlation is the Chapman-Enskog relation,

$$D_{AB} = 0.0018583 \frac{\sqrt{T^3 \left( \frac{1}{M_A} + \frac{1}{M_B} \right)}}{P \sigma_{AB}^2 \Omega_{D,AB}}$$

which predicts the diffusivity in  $\text{cm}^2/\text{s}$  for  $T$  in K,  $P$  in atm and  $\sigma_{AB}$  in Ångström units. In multicomponent systems ( $n_s$  species), all the binary interactions must be considered, as well as the contributions relative molar fluxes make to the transport of each component. The correct and complete treatment requires the use of Stefan-Maxwell relations. For example, when only one reaction is present for a multicomponent system, the diffusivity of species  $j$  in the mixture,  $D_{jm}$ , can be found using

$$\frac{1}{D_{jm}} = \left[ \sum_{k \neq j}^{n_s} \frac{1}{D_{jk}} \left( y_k + y_j \frac{v_k}{|v_j|} \right) \right] \frac{1}{1 + \delta_j y_j}$$

in which

$$\delta_j = \frac{\sum_{k=1}^{n_s} v_k}{|v_j|}$$

in which  $v_k$  and  $y_k$  represent the stoichiometric coefficient and mole fraction of component  $k$ , respectively. This one limiting case illustrates that the diffusion coefficient can change with temperature, pressure and composition.

For Knudsen flow the diffusivity of species  $j$  is

$$D_{jK} = \frac{8r_a}{3} \sqrt{\frac{RT}{2\pi M_j}} \quad (7.5)$$

Substitution of the gas constant gives

$$D_{jK} = 9.7 \times 10^3 r_a \sqrt{\frac{T}{M_j}}$$

which predicts the diffusivity in  $\text{cm}^2/\text{s}$  for  $T$  in K and  $M_j$  in g/mol.

Bulk or Knudsen flow dominate when  $r_a$  and  $\lambda$  vary by at least an order of magnitude. A combined diffusivity is needed when  $r_a$  and  $\lambda$  are of the same order of magnitude. Again for the single reaction

$$\frac{1}{D_{jc}} = \sum_{k=1}^{n_s} \frac{1}{D_{jk}} \left( y_k - y_j \frac{N_k}{N_j} \right) + \frac{1}{D_{jK}} \quad (7.6)$$

in which  $N_k$  is the molar flux of component  $k$  in the pore. For binary systems (components A and B) with equimolar counter diffusion, Equation 7.6 reduces to

$$\frac{1}{D_{AC}} = \frac{1}{D_{AB}} + \frac{1}{D_{AK}} \quad (7.7)$$

Equations 7.6 and 7.7 can be used to describe diffusion down a straight cylindrical pore. A porous solid does not consist of straight cylindrical pores, each having the same length and radius. Models for pore structure have been proposed that describe the pore size distribution and orientation as a function of location within the pellet [2]. These microscopic descriptions can be used to predict the porosity, pore size distribution, pore volume and pore area, all of which can be measured experimentally.

We also can use these microscopic models to describe diffusion radially inward, accounting for the changing pore shape and the presence of void or solid as a function of any position ( $r, \theta, \phi$ ) within the pellet. The texts by Aris [2] and Petersen [18] illustrate the approach. In short, the rate of mass transfer is solved rigorously in a pellet at the level depicted in Figure 7.1D to give an equation of the form

$$I_j = -D_j 4\pi r^2 \frac{dc_j}{dr} \xi \quad (7.8)$$

in which  $D_j$  refers to either  $D_{jm}$  or  $D_{jk}$  and  $\xi$  is an integral over the bounding surface of a dimensionless concentration gradient for  $j$  that depends on the microscopic description of the pore surfaces. As the microscopic description changes,  $\xi$  changes; however, this same  $\xi$  would apply to other diffusing species as well. Now if one considers the solid to be an equivalent homogeneous medium, which is the level depicted in Figure 7.1C, then the same rate can be described in terms of an effective diffusion coefficient,  $D_e$ .

$$I_j = -D_{ej} 4\pi r^2 \frac{dc_j}{dr} \quad (7.9)$$

Comparison of Equations 7.8 and 7.9 leads to

$$D_{ej} = D_j \xi$$

The integral  $\xi$  is generally represented as

$$\xi = \frac{\epsilon}{\tau}$$

Catalyst	$\epsilon$	$\tau$
100–110 $\mu\text{m}$ powder packed into a tube	0.416	1.56
pelletized $\text{Cr}_2\text{O}_3$ supported on $\text{Al}_2\text{O}_3$	0.22	2.5
pelletized boehmite alumina	0.34	2.7
Girdler G-58 Pd on alumina	0.39	2.8
Haldor-Topsøe MeOH synthesis catalyst	0.43	3.3
0.5% Pd on alumina	0.59	3.9
1.0% Pd on alumina	0.5	7.5
pelletized Ag/8.5% Ca alloy	0.3	6.0
pelletized Ag	0.3	10.0

Table 7.2: Porosity and tortuosity factors for diffusion in catalysts.

in which  $\tau$  is the tortuosity factor. The use of the ratio  $\epsilon/\tau$  has no real physical significance. It is done to remind us that as a diffusing molecule travels radially inward it encounters solid and void but it diffuses mainly in the void regions. Similarly, the molecule cannot travel in a straight line from the exterior to the center point, rather it must follow a tortuous path that effectively adds to the distance of travel.

Rigorous models such as the parallel, cross-linked, pore model can be used to predict the value of  $\tau$ . If the pores are modeled as straight cylindrical tubes of various lengths, one finds  $\tau = 3$ . Table 7.2 lists experimental values for  $\tau$  [8]. The values of  $\tau$  generally range from about 2 to 7. As more anisotropy in the pore distribution is introduced by the manufacturing process,  $\tau$  increases. Because of this variability in  $\tau$ , it should be determined experimentally for the catalyst of interest.

### 7.3 The General Balances in the Catalyst Particle

In this section we consider the mass and energy balances that arise with diffusion in the solid catalyst particle when considered at the scale of Figure 7.1C. Consider the volume element depicted in Figure 7.2. Assume a fixed laboratory coordinate system in which the velocities are defined and let  $\mathbf{v}_j$  be the velocity of species  $j$  giving rise to molar flux  $N_j$

$$N_j = c_j \mathbf{v}_j, \quad j = 1, 2, \dots, n_s$$

Let  $E$  be the total energy within the volume element and  $\mathbf{e}$  be the flux of total energy through the bounding surface due to all mechanisms of transport. The conservation of mass and energy for the volume element

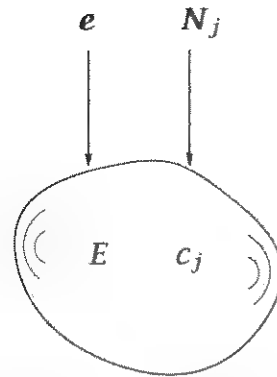


Figure 7.2: Volume element in a stationary solid containing energy density  $E$  and molar concentrations  $c_j$ , with energy flux  $e$  and mass fluxes  $N_j$ .

implies

$$\frac{\partial c_j}{\partial t} = -\nabla \cdot N_j + R_j, \quad j = 1, 2, \dots, n_s \quad (7.10)$$

$$\frac{\partial E}{\partial t} = -\nabla \cdot e \quad (7.11)$$

in which  $R_j$  accounts for the production of species  $j$  due to chemical reaction. As in Chapter 6, we consider the internal energy to be the dominant contribution to total energy, and neglect kinetic, potential and other forms of energy, so  $E = U$ .

Next we consider the fluxes. Since we are considering the diffusion of mass in a stationary, solid particle, we assume the mass flux is well approximated by

$$N_j = -D_j \nabla c_j, \quad j = 1, 2, \dots, n_s$$

in which  $D_j$  is an effective diffusivity for species  $j$ . Notice we are neglecting a variety of other possible mechanisms of mass transport, including the flux of species  $j$  due to gradients in the other species concentrations, and the flux of species  $j$  due to temperature gradients (thermal diffusion), pressure gradients, and gravitational or other external fields (forced diffusion). Bird, Stewart and Lightfoot [5, pp. 767–768] provide further discussion of these contributions to the mass flux.

We approximate the total energy flux by

$$e = -\hat{k} \nabla T + \sum_j N_j H_j$$

This expression accounts for the transfer of heat by conduction, in which  $\hat{k}$  is the effective thermal conductivity of the solid, and transport of energy due to the mass diffusion. We have neglected any viscous heating effects, and the flux of energy due to concentration gradients (Dufour energy) and radiation. So the energy balance becomes

$$\frac{\partial (\rho \hat{U})}{\partial t} = \nabla \cdot \hat{k} \nabla T - \sum_j [(\nabla \cdot N_j) H_j + N_j \cdot \nabla H_j] \quad (7.12)$$

In this chapter, we are concerned mostly with the steady state. Setting the time derivative to zero in Equation 7.10 produces

$$0 = -\nabla \cdot N_j + R_j \quad (7.13)$$

Substituting Equation 7.13 into Equation 7.12, using the definition of the heat of reaction, and setting the time derivative to zero, yields

$$0 = \nabla \cdot \hat{k} \nabla T - \sum_i \Delta H_{Ri} r_i - \sum_j N_j \cdot \nabla H_j$$

Finally, we often assume that the diffusivity, thermal conductivity and partial molar enthalpies are independent of temperature and composition to produce the following coupled mass and energy balances for the steady-state problem

$$0 = D_j \nabla^2 c_j + R_j, \quad j = 1, 2, \dots, n_s \quad (7.14)$$

$$0 = \hat{k} \nabla^2 T - \sum_i \Delta H_{Ri} r_i \quad (7.15)$$

In multiple-reaction, nonisothermal problems, we must solve these equations numerically, so the assumption of constant transport and thermodynamic properties is driven by the lack of data, and not analytical convenience.

## 7.4 Single Reaction in an Isothermal Particle

We start with the simplest cases and steadily remove restrictions and increase the generality. In this section, therefore, we consider a single

reaction taking place in an isothermal particle. We start with the spherical particle, first-order reaction, and neglect the external mass-transfer resistance. Next we consider other catalyst shapes, then other reaction orders, and then other kinetic expressions such as the Hougen-Watson kinetics of Chapter 5. We end this section by considering the effects of finite external mass transfer.

#### 7.4.1 First-Order Reaction in a Spherical Particle

In this section we consider a single, irreversible, first-order reaction occurring in an isothermal spherical pellet



Substituting the production rate into Equation 7.14, expressing the equation in spherical coordinates, and assuming pellet symmetry in  $\theta$  and  $\phi$  coordinates gives

$$D_A \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dc_A}{dr} \right) - kc_A = 0 \quad (7.17)$$

in which  $D_A$  is the effective diffusivity in the pellet for species A. As written here, the first-order rate constant  $k$  has units of inverse time. Be aware that the units for a heterogeneous reaction rate constant are sometimes expressed per mass or per area of catalyst. In these cases, the reaction rate expression includes the conversion factors, catalyst density or catalyst area, as illustrated in Example 7.1. We require two boundary conditions for Equation 7.17. In this section we assume the concentration at the outer boundary of the pellet,  $c_{As}$ , is known, and the symmetry of the spherical pellet implies the vanishing of the derivative at the center of the pellet.<sup>1</sup> Therefore the two boundary conditions for Equation 7.17 are

$$\begin{aligned} c_A &= c_{As}, & r &= R \\ \frac{dc_A}{dr} &= 0 & r &= 0 \end{aligned}$$

At this point we can obtain better insight by converting the problem into dimensionless form. Equation 7.17 has two dimensional quantities, length and concentration. We might naturally choose the sphere

<sup>1</sup> Some may prefer to use other boundary conditions at  $r = 0$ , such as boundedness of  $c_A$  or  $dc_A/dr$ , or a zero source/sink condition. These boundary conditions are equivalent for the problems considered here.

radius  $R$  as the length scale, but we will find that a better choice is to use the pellet's volume-to-surface ratio. For the sphere, this characteristic length is

$$a = \frac{V_p}{S_p} = \frac{\frac{4}{3}\pi R^3}{4\pi R^2} = \frac{R}{3} \quad (7.18)$$

The only concentration appearing in the problem is the surface concentration in the boundary condition, so we use that quantity to nondimensionalize the concentration

$$\bar{r} = \frac{r}{a}, \quad \bar{c} = \frac{c_A}{c_{As}}$$

Dividing through by the various dimensional quantities produces

$$\frac{1}{\bar{r}^2} \frac{d}{d\bar{r}} \left( \bar{r}^2 \frac{d\bar{c}}{d\bar{r}} \right) - \Phi^2 \bar{c} = 0 \quad (7.19)$$

$$\bar{c} = 1 \quad \bar{r} = 3$$

$$\frac{d\bar{c}}{d\bar{r}} = 0 \quad \bar{r} = 0$$

in which  $\Phi$  is given by

$$\Phi = \sqrt{\frac{ka^2}{D_A}} \quad \frac{\text{reaction rate}}{\text{diffusion rate}} \quad \text{Thiele modulus} \quad (7.20)$$

The single dimensionless group appearing in the model is referred to as the Thiele number or Thiele modulus in recognition of Thiele's pioneering contribution in this area [20].<sup>2</sup> The Thiele modulus quantifies the ratio of the reaction rate to the diffusion rate in the pellet.

Equation 7.20 can be considered as a special case of a more general definition

$$\Phi_j = \sqrt{\frac{|R_{js}| a^2}{c_{js} D_j}} = \sqrt{\frac{k c_{As} a^2}{c_{As} D_A}} = \sqrt{\frac{ka^2}{D_A}} \quad (7.21)$$

We now wish to solve Equation 7.19 with the given boundary conditions. Because the reaction is first order, the model is linear and we

<sup>2</sup>In his original paper, Thiele used the term *modulus* to emphasize that this then unnamed dimensionless group was positive. Later when Thiele's name was assigned to this dimensionless group, the term *modulus* was retained. Thiele number would seem a better choice, but the term Thiele modulus has become entrenched.

can derive an analytical solution. It is often convenient in spherical coordinates to consider the variable transformation

$$\bar{c}(\bar{r}) = \frac{u(\bar{r})}{\bar{r}} \quad (7.22)$$

Substituting this relation into Equation 7.19 provides a simpler differential equation for  $u(\bar{r})$ ,

$$\frac{d^2 u}{d\bar{r}^2} - \Phi^2 u = 0 \quad (7.23)$$

with the transformed boundary conditions

$$\begin{aligned} u &= 3 & \bar{r} &= 3 \\ u &= 0 & \bar{r} &= 0 \end{aligned}$$

The boundary condition  $u = 0$  at  $\bar{r} = 0$  ensures that  $\bar{c}$  is finite at the center of the pellet. The student may recall from the differential equations course that the solution to Equation 7.23 is

$$u(\bar{r}) = c_1 \cosh \Phi \bar{r} + c_2 \sinh \Phi \bar{r} \quad (7.24)$$

This solution is analogous to the sine and cosine solutions if one replaces the negative sign with a positive sign in Equation 7.23. These functions are shown in Figure 7.3. Some of the properties of the hyperbolic functions are

$$\begin{aligned} \cosh r &= \frac{e^r + e^{-r}}{2} & \frac{d \cosh r}{dr} &= \sinh r \\ \sinh r &= \frac{e^r - e^{-r}}{2} & \frac{d \sinh r}{dr} &= \cosh r \\ \tanh r &= \frac{\sinh r}{\cosh r} \end{aligned}$$

The constants  $c_1$  and  $c_2$  are determined by the boundary conditions. Substituting Equation 7.24 into the boundary condition at  $\bar{r} = 0$  gives  $c_1 = 0$ , and applying the boundary condition at  $\bar{r} = 3$  gives  $c_2 = 3 / \sinh 3\Phi$ . Substituting these results into Equations 7.24 and 7.22 gives the solution to the model

$$\bar{c}(\bar{r}) = \frac{3 \sinh \Phi \bar{r}}{\bar{r} \sinh 3\Phi} \quad (7.25)$$

Figure 7.4 displays this solution for various values of the Thiele modulus. Note for small values of Thiele modulus, the reaction rate is small

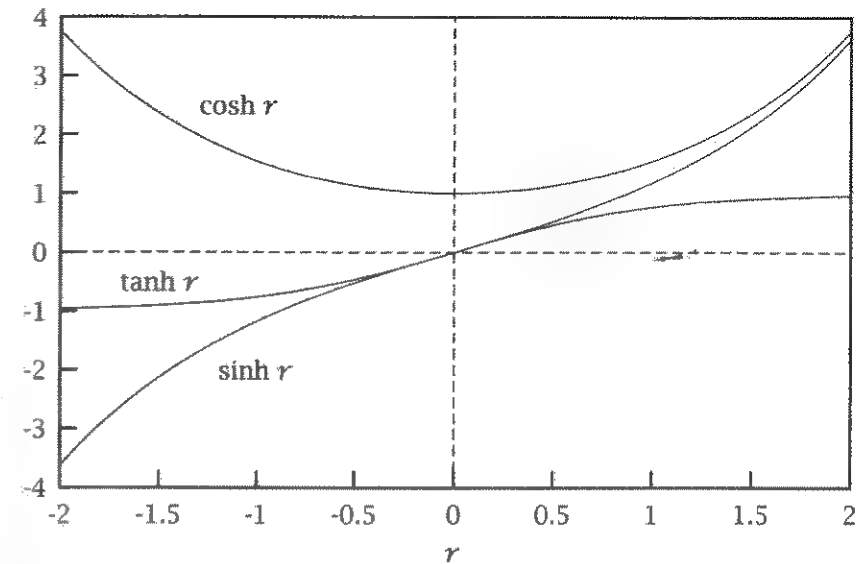


Figure 7.3: Hyperbolic trigonometric functions sinh, cosh and tanh.

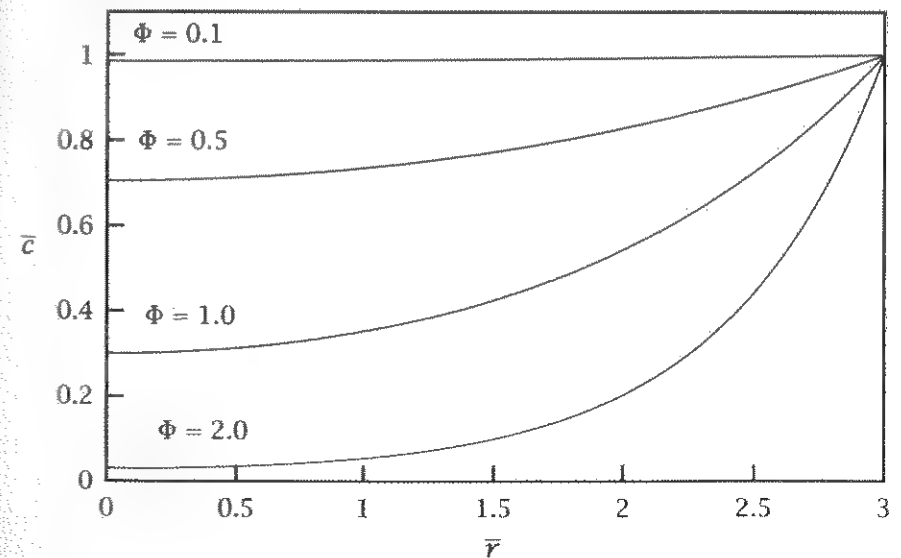


Figure 7.4: Dimensionless concentration versus dimensionless radial position for different values of the Thiele modulus.



compared to the diffusion rate, and the pellet concentration becomes nearly uniform. For large values of Thiele modulus, the reaction rate is large compared to the diffusion rate, and the reactant is converted to product before it can penetrate very far into the pellet.

We now calculate the pellet's overall production rate given this concentration profile. We can perform this calculation in two ways. The first and more direct method is to integrate the local production rate over the pellet volume. The second method is to use the fact that, at steady state, the rate of consumption of reactant within the pellet is equal to the rate at which material fluxes through the pellet's exterior surface. The two expressions are

$$R_{Ap} = \frac{1}{V_p} \int_0^R R_A(r) 4\pi r^2 dr \quad \text{volume integral} \quad (7.26)$$

$$R_{Ap} = -\frac{S_p}{V_p} D_A \left. \frac{dc_A}{dr} \right|_{r=R} \quad \text{surface flux} \quad \text{(assumes steady state)} \quad (7.27)$$

in which the local production rate is given by  $R_A(r) = -kc_A(r)$ . We use the direct method here and leave the other method as an exercise. Substituting the local production rate into Equation 7.26 and converting the integral to dimensionless radius gives

$$R_{Ap} = -\frac{kC_{As}}{9} \int_0^3 \bar{c}(\bar{r}) \bar{r}^2 d\bar{r} \quad (7.28)$$

Substituting the concentration profile, Equation 7.25, and changing the variable of integration to  $x = \Phi \bar{r}$  gives

$$R_{Ap} = -\frac{kC_{As}}{3\Phi^2 \sinh 3\Phi} \int_0^{3\Phi} x \sinh x dx \quad (7.29)$$

The integral can be found in a table or derived by integration by parts to yield finally

$$R_{Ap} = -kC_{As} \frac{1}{\Phi} \left[ \frac{1}{\tanh 3\Phi} - \frac{1}{3\Phi} \right] \quad (7.30)$$

It is instructive to compare this actual pellet production rate to the rate in the absence of diffusional resistance. If the diffusion were arbitrarily fast, the concentration everywhere in the pellet would be equal to the surface concentration, corresponding to the limit  $\Phi = 0$ . The pellet rate for this limiting case is simply

$$R_{As} = -kC_{As} \quad (7.31)$$

We define the effectiveness factor,  $\eta$ , to be the ratio of these two rates

$$\eta \equiv \frac{R_{Ap}}{R_{As}}, \quad \text{effectiveness factor} \quad (7.32)$$

The effectiveness factor is a dimensionless pellet production rate that measures how effectively the catalyst is being used. For  $\eta$  near unity, the entire volume of the pellet is reacting at the same high rate because the reactant is able to diffuse quickly through the pellet. For  $\eta$  near zero, the pellet reacts at a low rate. The reactant is unable to penetrate significantly into the interior of the pellet and the reaction rate is small in a large portion of the pellet volume. The pellet's diffusional resistance is large and this resistance lowers the overall reaction rate. We can substitute Equations 7.30 and 7.31 into the definition of effectiveness factor to obtain for the first-order reaction in the spherical pellet

$$\eta = \frac{1}{\Phi} \left[ \frac{1}{\tanh 3\Phi} - \frac{1}{3\Phi} \right] \quad (7.33)$$

Figures 7.5 and 7.6 display the effectiveness factor versus Thiele modulus relationship given in Equation 7.33. The log-log scale in Figure 7.6 is particularly useful, and we see the two asymptotic limits of Equation 7.33. At small  $\Phi$ ,  $\eta \approx 1$ , and at large  $\Phi$ ,  $\eta \approx 1/\Phi$ . Figure 7.6 shows that the asymptote  $\eta = 1/\Phi$  is an excellent approximation for the spherical pellet for  $\Phi \geq 10$ . For large values of the Thiele modulus, the rate of reaction is much greater than the rate of diffusion, the effectiveness factor is much less than unity, and we say the pellet is *diffusion limited*. Conversely, when the diffusion rate is much larger than the reaction rate, the effectiveness factor is near unity, and we say the pellet is *reaction limited*.

#### Example 7.1: Using the Thiele modulus and effectiveness factor

The first-order, irreversible reaction ( $A \rightarrow B$ ) takes place in a 0.3 cm radius spherical catalyst pellet at  $T = 450$  K. At 0.7 atm partial pressure of A, the pellet's production rate is  $-2.5 \times 10^{-5}$  mol/(g s). Determine the production rate at the same temperature in a 0.15 cm radius spherical pellet. The pellet density is  $\rho_p = 0.85$  g/cm<sup>3</sup>. The effective diffusivity of A in the pellet is  $D_A = 0.007$  cm<sup>2</sup>/s.

#### Solution

We can use the production rate and pellet parameters for the 0.3 cm pellet to find the value for the rate constant  $k$ , and then compute the Thiele

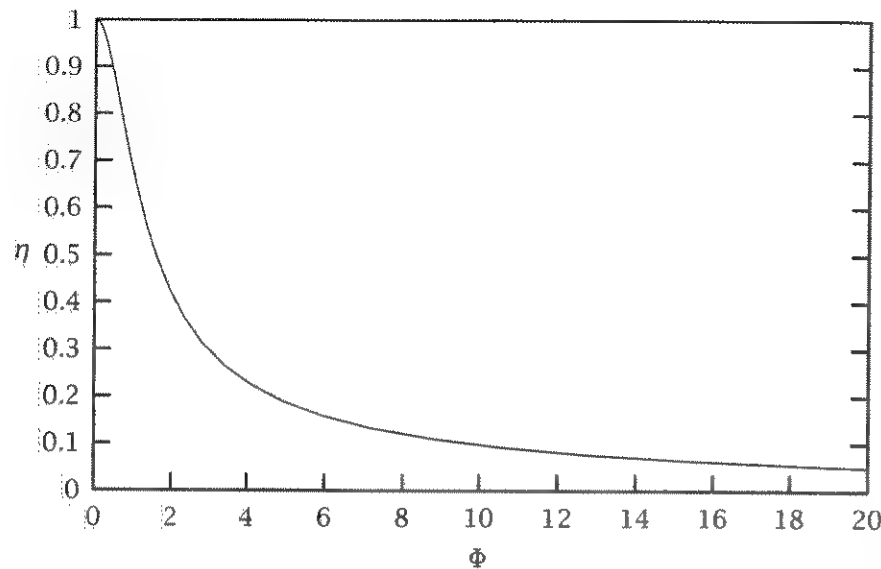


Figure 7.5: Effectiveness factor versus Thiele modulus for a first-order reaction in a sphere.

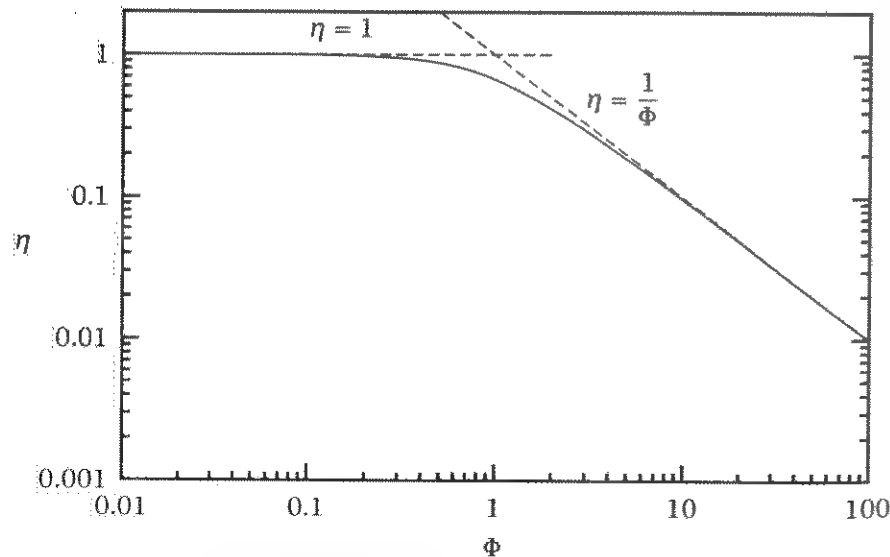


Figure 7.6: Effectiveness factor versus Thiele modulus for a first-order reaction in a sphere (log-log scale).

modulus, effectiveness factor and production rate for the smaller pellet.

We have three unknowns,  $k$ ,  $\Phi$ ,  $\eta$ , and the following three equations

$$R_{Ap} = -\eta k c_{As} \quad (7.34)$$

$$\Phi = \sqrt{\frac{k a^2}{D_A}} \quad (7.35)$$

$$\eta = \frac{1}{\Phi} \left[ \frac{1}{\tanh 3\Phi} - \frac{1}{3\Phi} \right] \quad (7.36)$$

The production rate is given in the problem statement. Solving Equation 7.35 for  $k$ , and substituting that result and Equation 7.36 into 7.34, give one equation in the unknown  $\Phi$

$$\Phi \left[ \frac{1}{\tanh 3\Phi} - \frac{1}{3\Phi} \right] = -\frac{R_{Ap} a^2}{D_A c_{As}} \quad (7.37)$$

The surface concentration and pellet production rates are given by

$$c_{As} = \frac{0.7 \text{ atm}}{\left(82.06 \frac{\text{cm}^3 \text{ atm}}{\text{mol K}}\right) (450 \text{ K})} = 1.90 \times 10^{-5} \text{ mol/cm}^3$$

$$R_{Ap} = \left(-2.5 \times 10^{-5} \frac{\text{mol}}{\text{g s}}\right) \left(0.85 \frac{\text{g}}{\text{cm}^3}\right) = -2.125 \frac{\text{mol}}{\text{cm}^3 \text{ s}}$$

Substituting these values into Equation 7.37 gives

$$\Phi \left[ \frac{1}{\tanh 3\Phi} - \frac{1}{3\Phi} \right] = 1.60$$

This equation can be solved numerically yielding the Thiele modulus

$$\Phi = 1.93$$

Using this result, Equation 7.35 gives the rate constant

$$k = 2.61 \text{ s}^{-1}$$

The smaller pellet is half the radius of the larger pellet, so the Thiele modulus is half as large or  $\Phi = 0.964$ , which gives  $\eta = 0.685$ . The production rate is therefore

$$R_{Ap} = -0.685 (2.6 \text{ s}^{-1}) (1.90 \times 10^{-5} \text{ mol/cm}^3) = -3.38 \times 10^{-5} \frac{\text{mol}}{\text{cm}^3 \text{ s}}$$

We see that decreasing the pellet size increases the production rate by almost 60%. Notice that this type of increase is possible only when the pellet is in the diffusion-limited regime. □

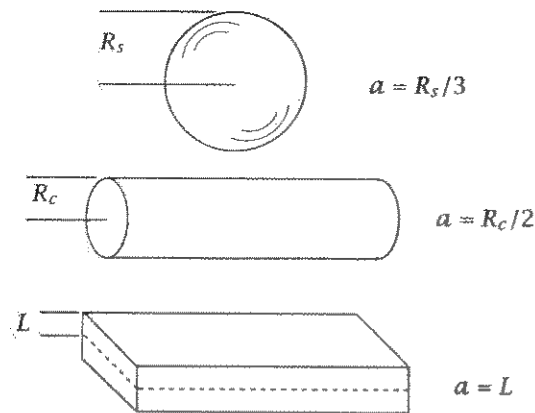


Figure 7.7: Characteristic length  $a$  for sphere, semi-infinite cylinder and semi-infinite slab.

#### 7.4.2 Other Catalyst Shapes: Cylinders and Slabs

Here we consider the cylinder and slab geometries in addition to the sphere covered in the previous section. To have a simple analytical solution, we must neglect the end effects; we therefore consider in addition to the sphere of radius  $R_s$ , the semi-infinite cylinder of radius  $R_c$ , and the semi-infinite slab of thickness  $2L$ , depicted in Figure 7.7. We can summarize the reaction-diffusion mass balance for these three geometries by

$$D_A \frac{1}{r^q} \frac{d}{dr} \left( r^q \frac{dc_A}{dr} \right) - kc_A = 0 \quad (7.38)$$

in which

$$\begin{aligned} q &= 2 && \text{sphere} \\ q &= 1 && \text{cylinder} \\ q &= 0 && \text{slab} \end{aligned}$$

The associated boundary conditions are

$$\begin{aligned} c_A &= c_{As} && \begin{cases} r = R_s & \text{sphere} \\ r = R_c & \text{cylinder} \\ r = L & \text{slab} \end{cases} \\ \frac{dc_A}{dr} &= 0 && r = 0 \text{ all geometries.} \end{aligned}$$

The characteristic length  $a$  is again best defined as the volume-to-surface ratio, which gives for these geometries

$$\begin{aligned} a &= \frac{R_s}{3} && \text{sphere} \\ a &= \frac{R_c}{2} && \text{cylinder} \\ a &= L && \text{slab} \end{aligned}$$

The dimensionless form of Equation 7.38 is

$$\frac{1}{\bar{r}^q} \frac{d}{d\bar{r}} \left( \bar{r}^q \frac{d\bar{c}}{d\bar{r}} \right) - \Phi^2 \bar{c} = 0 \quad (7.39)$$

$$\begin{aligned} \bar{c} &= 1 && \bar{r} = q + 1 \\ \frac{d\bar{c}}{d\bar{r}} &= 0 && \bar{r} = 0 \end{aligned}$$

in which the boundary conditions for all three geometries can be compactly expressed in terms of  $q$ .

The effectiveness factor for the different geometries can be evaluated using the integral and flux approaches, Equations 7.26–7.27, which lead to the two expressions

$$\eta = \frac{1}{(q+1)^q} \int_0^{q+1} \bar{c} \bar{r}^q d\bar{r} \quad (7.40)$$

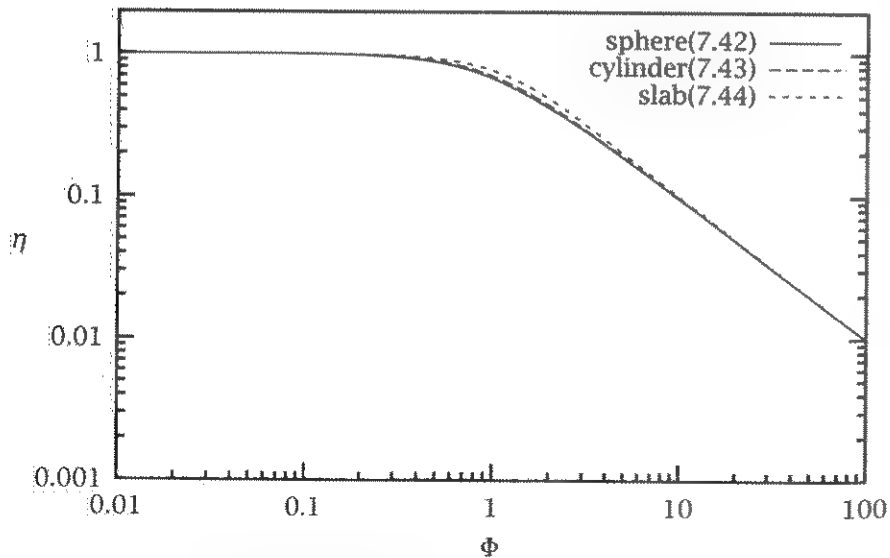
$$\eta = \frac{1}{\Phi^2} \left. \frac{d\bar{c}}{d\bar{r}} \right|_{\bar{r}=q+1} \quad (7.41)$$

We have already solved Equations 7.39 and 7.41 for the sphere,  $q = 2$ . Analytical solutions for the slab and cylinder geometries also can be derived. See Exercise 7.1 for the slab geometry. The results are summarized in Table 7.3. Note that  $I_0$  and  $I_1$  are modified Bessel functions of the first kind of orders zero and one, respectively.

The effectiveness factors versus Thiele modulus for the three geometries are plotted in Figure 7.8. Although the functional forms listed in Table 7.3 appear quite different, we see in Figure 7.8 that these solutions are quite similar. The effectiveness factor for the slab is largest, the cylinder is intermediate, and the sphere is the smallest at all values of Thiele modulus. The three curves have identical small  $\Phi$  and large  $\Phi$  asymptotes. The maximum difference between the effectiveness factors of the sphere and the slab  $\eta$  is about 16%, and occurs at  $\Phi = 1.6$ . For  $\Phi < 0.5$  and  $\Phi > 7$ , the difference between all three effectiveness factors is less than 5%.

Sphere	$\eta = \frac{1}{\Phi} \left[ \frac{1}{\tanh 3\Phi} - \frac{1}{3\Phi} \right]$ (7.42)
Cylinder	$\eta = \frac{1}{\Phi} \frac{I_1(2\Phi)}{I_0(2\Phi)}$ (7.43)
Slab	$\eta = \frac{\tanh \Phi}{\Phi}$ (7.44)

**Table 7.3:** Closed-form solution for the effectiveness factor versus Thiele modulus for the sphere, semi-infinite cylinder, and semi-infinite slab.



**Figure 7.8:** Effectiveness factor versus Thiele modulus for the sphere, cylinder and slab.

### 7.4.3 Other Reaction Orders

For reactions other than first order, the reaction-diffusion equation is nonlinear and numerical solution is required. We will see, however, that many of the conclusions from the analysis of the first-order reaction case still apply for other reaction orders. We consider  $n$ th-order, irreversible reaction kinetics



The reaction-diffusion equation and boundary conditions for this case are

$$D_A \frac{1}{r^q} \frac{d}{dr} \left( r^q \frac{dc_A}{dr} \right) - kc_A^n = 0 \quad (7.46)$$

We might naturally define the Thiele modulus as in Equation 7.21, but the results for various reaction orders have a common asymptote if we instead define

$$\Phi = \sqrt{\frac{n+1}{2} \frac{kc_{As}^{n-1} a^2}{D_A}} \quad \text{Thiele modulus } n\text{th-order reaction} \quad (7.47)$$

$$\frac{1}{\bar{r}^q} \frac{d}{d\bar{r}} \left( \bar{r}^q \frac{d\bar{c}}{d\bar{r}} \right) - \frac{2}{n+1} \Phi^2 \bar{c}^n = 0$$

$$\bar{c} = 1 \quad \bar{r} = q + 1$$

$$\frac{d\bar{c}}{d\bar{r}} = 0 \quad \bar{r} = 0$$

$$\eta = \frac{1}{(q+1)^q} \int_0^{q+1} \bar{c}^n \bar{r}^q d\bar{r}$$

$$\eta = \frac{n+1}{2} \frac{1}{\Phi^2} \left. \frac{d\bar{c}}{d\bar{r}} \right|_{\bar{r}=q+1}$$

Figure 7.9 shows the effect of reaction order for  $n \geq 1$  in a spherical pellet. As the reaction order increases, the effectiveness factor decreases. Notice that the definition of Thiele modulus in Equation 7.47 has achieved the desired goal of giving all reaction orders a common asymptote at high values of  $\Phi$ . Figure 7.10 shows the effectiveness factor versus Thiele modulus for reaction orders less than unity. Notice the discontinuity in slope of the effectiveness factor versus Thiele modulus that occurs when the order is less than unity. Recall from

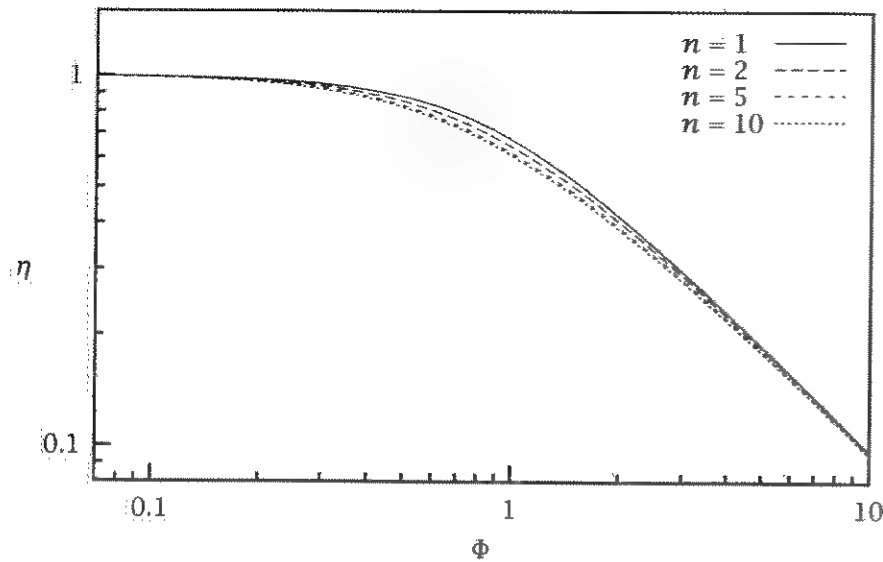


Figure 7.9: Effectiveness factor versus Thiele modulus in a spherical pellet; reaction orders greater than unity.

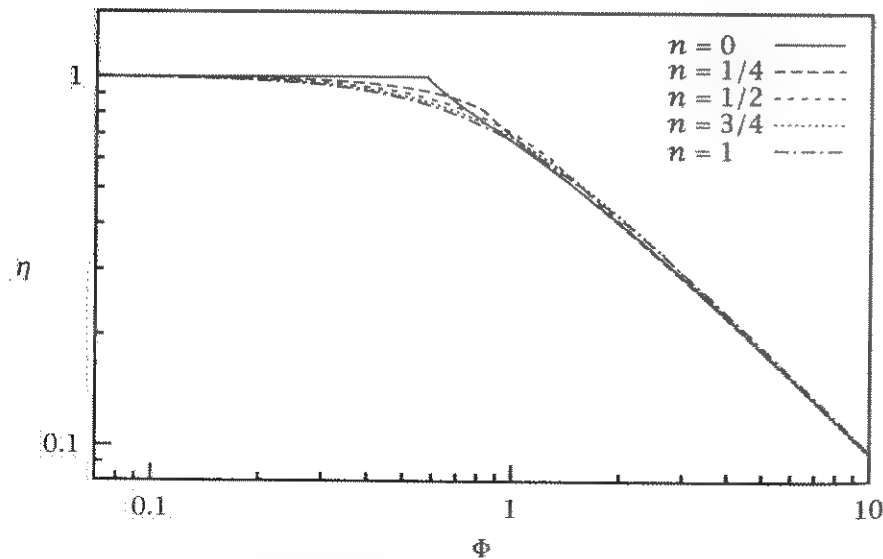


Figure 7.10: Effectiveness factor versus Thiele modulus in a spherical pellet; reaction orders less than unity.

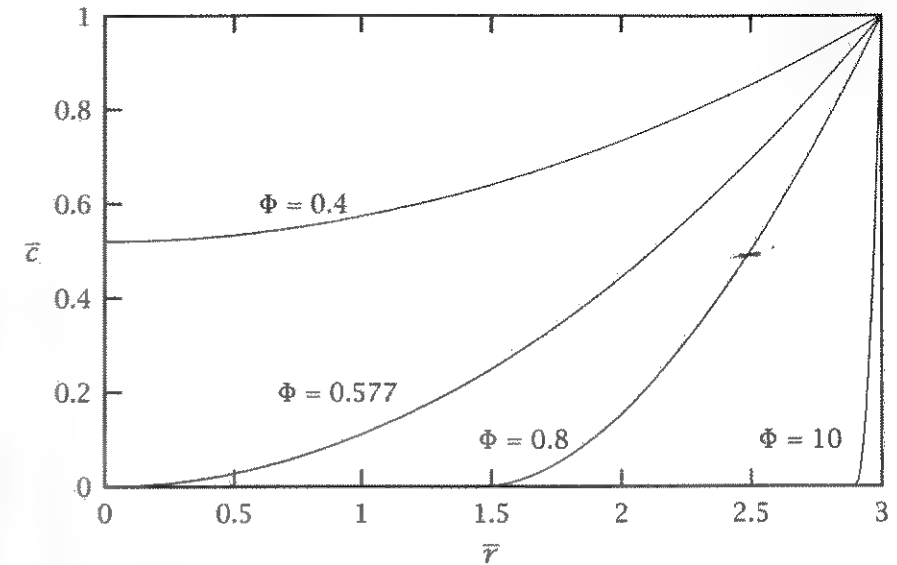


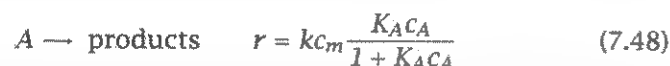
Figure 7.11: Dimensionless concentration versus radius for zero-order reaction ( $n=0$ ) in a spherical pellet ( $q=2$ ); for large  $\Phi$  the inner region of the pellet has zero A concentration.

the discussion in Chapter 4 that if the reaction order is less than unity in a batch reactor, the concentration of A reaches zero in finite time. Equation 4.28 shows the discontinuity that is introduced into the mass balance when the concentration reaches zero, and Figure 4.8 displays the discontinuity in the slope of concentration versus time. In the reaction-diffusion problem in the pellet, the same kinetic effect causes the discontinuity in  $\eta$  versus  $\Phi$ . For large values of Thiele modulus, the diffusion is slow compared to reaction, and the A concentration reaches zero at some nonzero radius inside the pellet. For orders less than unity, an inner region of the pellet has identically zero A concentration. Figure 7.11 shows the reactant concentration versus radius for the zero-order reaction case in a sphere at various values of Thiele modulus. For  $\Phi = 0.577$ , the A concentration reaches zero just at the center of the pellet. Notice that the discontinuity in slope of  $\eta$  versus  $\Phi$  in Figure 7.10 occurs at this same  $\Phi$  value. For larger values of  $\Phi$ , the A concentration reaches zero at finite radius. This radius increases

with increase in Thiele modulus so that for  $\Phi = 10$ , only a small outer shell of the pellet has nonzero A concentration. The same qualitative picture holds for all values of reaction order less than one.

#### 7.4.4 Hougen-Watson Kinetics

Given the discussion in Section 5.6 of adsorption and reactions on catalyst surfaces, it is reasonable to expect our best catalyst rate expressions may be of the Hougen-Watson form. In this section we study an example reaction mechanism of this form. Consider the following reaction and rate expression



This expression arises when gas-phase A adsorbs onto the catalyst surface and the reaction is first order in the adsorbed A concentration. If we consider the slab catalyst geometry, the mass balance is

$$D_A \frac{d^2 c_A}{dr^2} - k \bar{c}_m \frac{K_A c_A}{1 + K_A c_A} = 0$$

and the boundary conditions are

$$\begin{aligned} c_A &= c_{As} & r &= L \\ \frac{dc_A}{dr} &= 0 & r &= 0 \end{aligned}$$

We would like to study the effectiveness factor for these kinetics. First we define dimensionless concentration and length as before to arrive at the dimensionless reaction-diffusion model

$$\frac{d^2 \bar{c}}{d\bar{r}^2} - \bar{\Phi}^2 \frac{\bar{c}}{1 + \phi \bar{c}} = 0 \quad (7.49)$$

$$\begin{aligned} \bar{c} &= 1 & \bar{r} &= 1 \\ \frac{d\bar{c}}{d\bar{r}} &= 0 & \bar{r} &= 0 \end{aligned} \quad (7.50)$$

in which we now have two dimensionless groups

$$\bar{\Phi} = \sqrt{\frac{k \bar{c}_m K_A a^2}{D_A}}, \quad \phi = K_A c_{As} \quad (7.51)$$

We use the tilde to indicate  $\bar{\Phi}$  is a good first guess for a Thiele modulus for this problem, but we will find a better candidate subsequently. The new dimensionless group  $\phi$  represents a dimensionless adsorption constant. The effectiveness factor is calculated from

$$\eta = \frac{R_{Ap}}{R_{As}} = \frac{-(S_p/V_p) D_A \left. \frac{dc_A}{dr} \right|_{r=a}}{-k \bar{c}_m K_A c_{As} / (1 + K_A c_{As})}$$

which becomes upon definition of the dimensionless quantities

$$\eta = \frac{1 + \phi}{\bar{\Phi}^2} \left. \frac{d\bar{c}}{d\bar{r}} \right|_{\bar{r}=1} \quad (7.52)$$

Now we wish to define a Thiele modulus so that  $\eta$  has a common asymptote at large  $\Phi$  for all values of  $\phi$ . This goal was accomplished for the  $n$ th-order reaction as shown in Figures 7.9 and 7.10 by including the factor  $(n+1)/2$  in the definition of  $\Phi$  given in Equation 7.47. We now turn our attention to finding the right Thiele modulus for the Hougen-Watson kinetics problem. First we calculate  $\eta$  for large  $\bar{\Phi}$ .

**Asymptotic behavior for large  $\bar{\Phi}$ .** As a first step we change the independent variable from  $\bar{r}$  to

$$z = \bar{\Phi}(1 - \bar{r}) \quad (7.53)$$

in order to remove  $\bar{\Phi}$  from the differential equation [18]. Then we take the limit  $\bar{\Phi} \rightarrow \infty$  to obtain the model describing the asymptotic case. Changing variables in the reaction-diffusion model, Equations 7.49-7.50, and taking the limit produces

$$\frac{d^2 \bar{c}}{dz^2} - \frac{\bar{c}}{1 + \phi \bar{c}} = 0 \quad (7.54)$$

$$\bar{c} = 1 \quad z = 0$$

$$\frac{d\bar{c}}{dz} = 0 \quad z = \infty$$

The effectiveness factor is obtained by changing variables in Equation 7.52, producing

$$\eta = -\frac{1 + \phi}{\bar{\Phi}} \left. \frac{d\bar{c}}{dz} \right|_{z=0} \quad (7.55)$$

We need to perform one integral of Equation 7.54, without resorting to numerical integration, to evaluate the effectiveness factor.

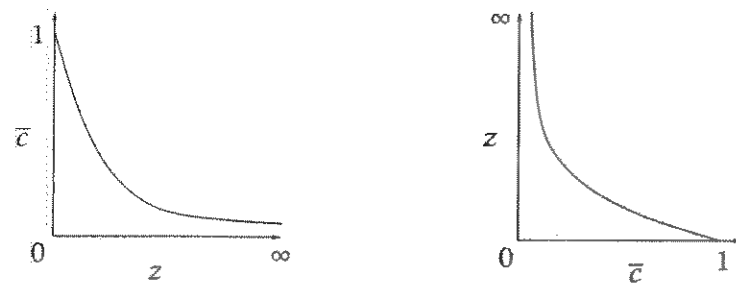


Figure 7.12: Dimensionless concentration  $\bar{c}$  versus  $z$  for large  $\Phi$ ; consider the function  $z(\bar{c})$  in place of  $\bar{c}(z)$ .

Define  $p(z) = d\bar{c}/dz$  to be the desired first derivative. The only nonobvious step in this section is the next one. We make the following variable transformation; instead of considering  $p$  as a function of  $z$ , define a transformation that makes it a function of  $\bar{c}$ . Figure 7.12 illustrates what we have in mind. The variable transformation is

$$p(z) = p(z(\bar{c})) = \tilde{p}(\bar{c})$$

We now use the familiar chain rule of differentiation

$$\frac{dp}{dz} = \frac{d\tilde{p}}{d\bar{c}} \frac{d\bar{c}}{dz} = \frac{d\tilde{p}}{d\bar{c}} p(z) = \frac{d\tilde{p}}{d\bar{c}} \tilde{p}$$

and substitute the mass balance, Equation 7.54, to obtain

$$\frac{d\tilde{p}}{d\bar{c}} \tilde{p} = \frac{\bar{c}}{1 + \Phi \bar{c}}$$

This equation can be separated to yield

$$\int_0^{\tilde{p}} \tilde{p} d\tilde{p} = \int_0^{\bar{c}} \frac{\bar{c}}{1 + \Phi \bar{c}} d\bar{c}$$

Performing the integral on the left-hand side and solving for  $\tilde{p}$  gives

$$\tilde{p} = - \left[ 2 \int_0^{\bar{c}} \frac{\bar{c}}{1 + \Phi \bar{c}} d\bar{c} \right]^{1/2}$$

in which we choose the correct sign to correspond to Figure 7.12. Performing the integral on the right-hand side yields

$$\tilde{p} = - \left[ \frac{2}{\Phi} \left( \bar{c} - \frac{1}{\Phi} \ln(1 + \Phi \bar{c}) \right) \right]^{1/2} \quad (7.56)$$

Evaluating Equation 7.56 at  $\bar{c} = 1$ , which corresponds to  $z = 0$  (see Figure 7.12), and substituting the result into Equation 7.55 produces

$$\eta = \frac{1}{\Phi} \left( \frac{1 + \Phi}{\Phi} \right) \sqrt{2(\Phi - \ln(1 + \Phi))} \quad (7.57)$$

Equation 7.57 tells us what we need to know. We wish to redefine the Thiele modulus so that this equation reads simply

$$\eta = \frac{1}{\Phi}$$

We can accomplish this with a simple rescaling of the Thiele modulus via

$$\Phi = \left( \frac{\phi}{1 + \phi} \right) \frac{1}{\sqrt{2(\phi - \ln(1 + \phi))}} \Phi$$

So, in summary, we have the following two dimensionless groups for this problem

$$\Phi = \left( \frac{\phi}{1 + \phi} \right) \sqrt{\frac{k \bar{c}_m K_A a^2}{2 D_A (\phi - \ln(1 + \phi))}}, \quad \phi = K_A C_{A_s} \quad (7.58)$$

Obviously it does not pay to try to guess an appropriate Thiele modulus for these more complex rate expressions. An asymptotic analysis as presented here is required to find the appropriate scaling. This idea appears to have been discovered independently by three chemical engineers in 1965. To quote from Aris [2, p. 113]

This is the essential idea in three papers published independently in March, May and June of 1965; see Bischoff [6], Aris [1] and Petersen [19]. A more limited form was given as early as 1958 by Stewart in Bird, Stewart and Lightfoot [4, p. 338].

The payoff for this analysis is shown in Figures 7.13 and 7.14. If we use our first guess for the Thiele modulus, Equation 7.51, we obtain Figure 7.13 in which the various values of  $\phi$  have different asymptotes. Using the Thiele modulus defined in Equation 7.58, we obtain the results in Figure 7.14. Figure 7.14 displays things more clearly. First, notice from the mass balance, Equation 7.54, that the dimensionless reaction rate is

$$\frac{\bar{c}}{1 + \Phi \bar{c}} \quad \text{dimensionless reaction rate}$$

Hougen-Watson kinetics



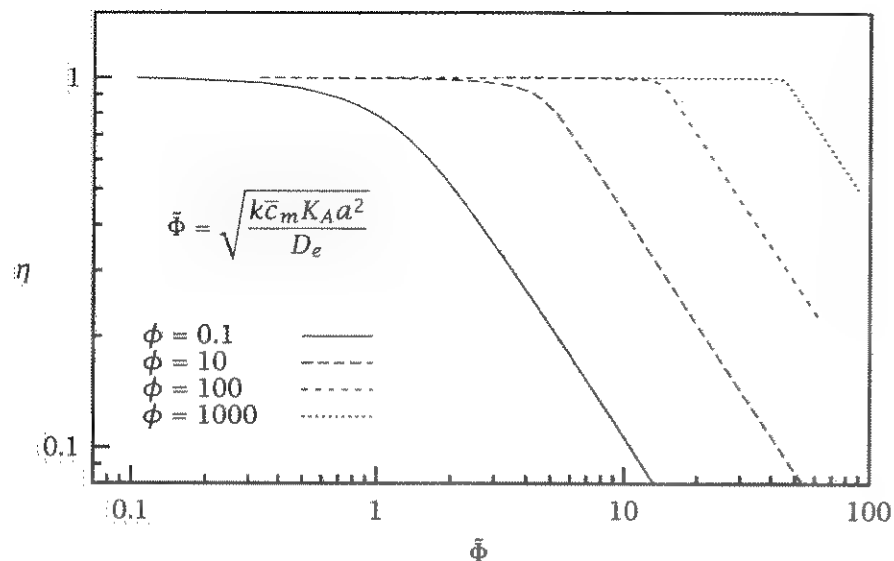


Figure 7.13: Effectiveness factor versus an inappropriate Thiele modulus in a slab; Hougen-Watson kinetics.

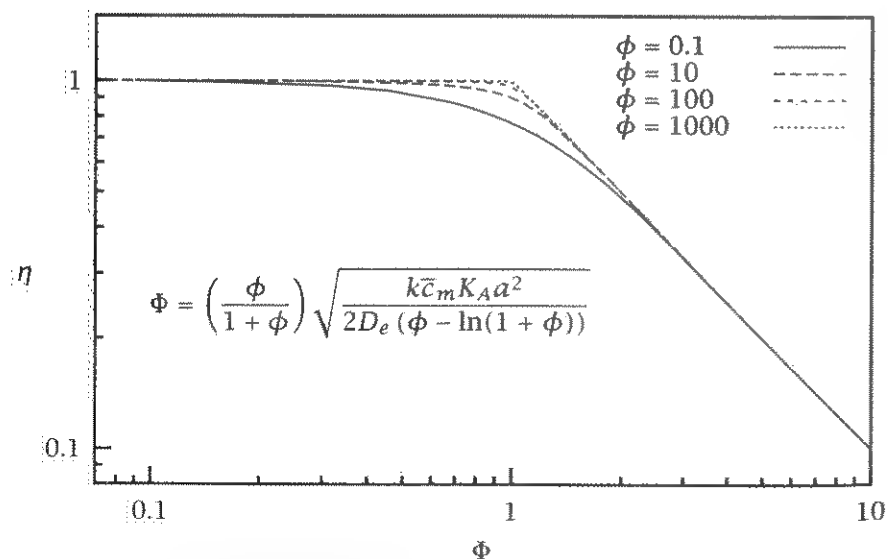


Figure 7.14: Effectiveness factor versus appropriate Thiele modulus in a slab; Hougen-Watson kinetics.

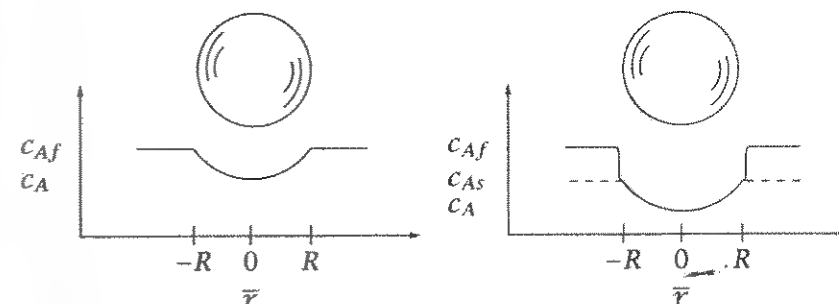


Figure 7.15: Effect of external mass transfer on pellet surface concentration.

and, as we have seen in Chapter 5, the apparent order of the reaction changes depending on the size of the dimensionless adsorption constant  $\phi$ . For small  $\phi$ , we expect first-order reaction behavior, and for large  $\phi$ , we expect zero-order reaction behavior. Comparing Figure 7.14 to Figure 7.10 we see this trend. Recall that Figure 7.10 corresponds to spherical geometry, and we are using the slab geometry in this section, which causes the small differences in the effectiveness factor plots. Notice that we approximate the zero-order reaction discontinuity in the slope of  $\eta$  versus  $\Phi$  for large  $\phi$ . Again we see that as long as we choose an appropriate Thiele modulus, we can approximate the effectiveness factor for all values of  $\phi$  with the first-order reaction. The largest approximation error occurs near  $\Phi = 1$ , and if  $\Phi > 2$  or  $\Phi < 0.2$ , the approximation error is negligible.

#### 7.4.5 External Mass Transfer

If the mass-transfer rate from the bulk fluid to the exterior of the pellet is not high, then the boundary condition

$$c_A(r = R) = c_{Af}$$

is not satisfied. Figure 7.15 provides a depiction of the concentration profile under low and high external mass-transfer rates. If the external mass-transfer rate is low, the concentrations in the bulk fluid and external catalyst surface are significantly different. To obtain a simple model of the external mass transfer, we replace the boundary condition

above with a flux boundary condition

$$D_A \frac{dc_A}{dr} = k_m (c_{Af} - c_A), \quad r = R \quad (7.59)$$

in which  $k_m$  is the external mass-transfer coefficient. If we multiply Equation 7.59 by  $a/c_{Af}D_A$ , we obtain the dimensionless boundary condition

$$\frac{d\bar{c}}{d\bar{r}} = B(1 - \bar{c}), \quad \bar{r} = 3 \quad (7.60)$$

in which

$$B = \frac{k_m a}{D_A} \quad (7.61)$$

is the Biot number or dimensionless mass-transfer coefficient.

Summarizing, for finite external mass transfer, the dimensionless model and boundary conditions are

$$\frac{1}{\bar{r}^2} \frac{d}{d\bar{r}} \left( \bar{r}^2 \frac{d\bar{c}}{d\bar{r}} \right) - \Phi^2 \bar{c} = 0 \quad (7.62)$$

$$\begin{aligned} \frac{d\bar{c}}{d\bar{r}} &= B(1 - \bar{c}) & \bar{r} &= 3 \\ \frac{d\bar{c}}{d\bar{r}} &= 0 & \bar{r} &= 0 \end{aligned}$$

The solution to the differential equation satisfying the center boundary condition can be derived as in Section 7.4 to produce

$$\bar{c}(\bar{r}) = \frac{c_2}{\bar{r}} \sinh \Phi \bar{r} \quad (7.63)$$

in which  $c_2$  is the remaining unknown constant. Evaluating this constant using the external boundary condition gives

$$\bar{c}(\bar{r}) = \frac{3}{\bar{r}} \frac{\sinh \Phi \bar{r}}{\sinh 3\Phi + (\Phi \cosh 3\Phi - (\sinh 3\Phi)/3)/B} \quad (7.64)$$

The effectiveness factor can again be derived by integrating the local reaction rate or computing the surface flux, and the result is

$$\eta = \frac{1}{\Phi} \left[ \frac{1/\tanh 3\Phi - 1/(3\Phi)}{1 + \Phi(1/\tanh 3\Phi - 1/(3\Phi))/B} \right] \quad (7.65)$$

in which

$$\eta = \frac{R_{Ap}}{R_{Ab}}$$

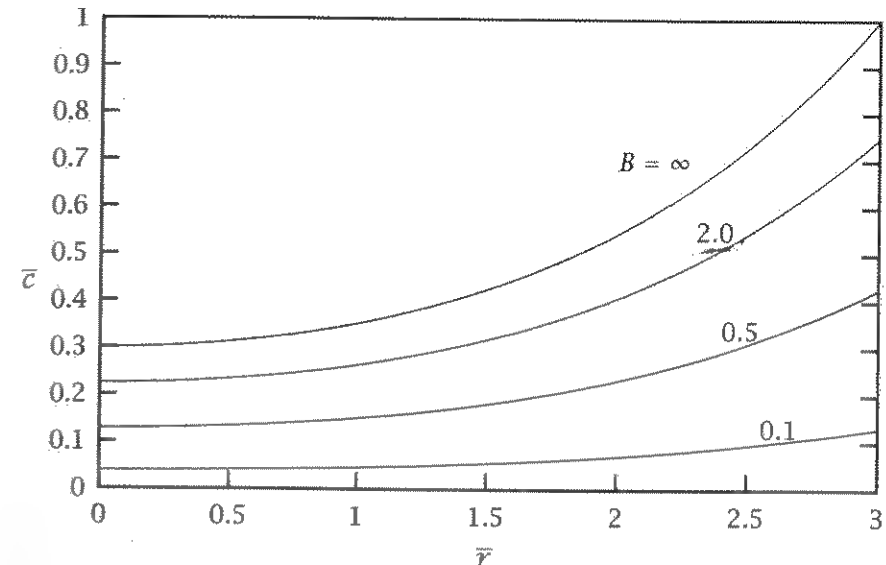
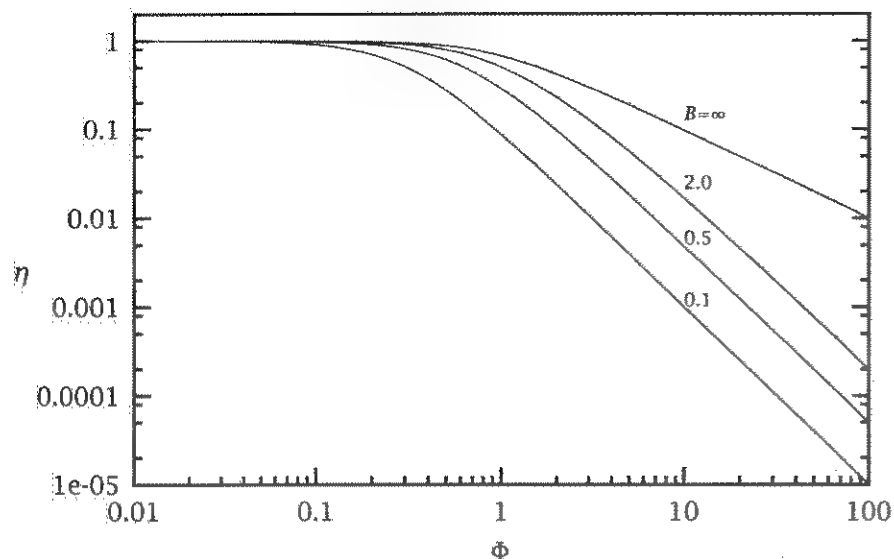


Figure 7.16: Dimensionless concentration versus radius for different values of the Biot number; first-order reaction in a spherical pellet with  $\Phi = 1$ .

Notice we are comparing the pellet's reaction rate to the rate that would be achieved if the pellet reacted at the *bulk fluid concentration* rather than the pellet exterior concentration as before.

Figure 7.16 plots Equation 7.64 for different values of  $B$ . When  $B = \infty$ , Equation 7.64 reduces to Equation 7.25, and the curve for  $\Phi = 1.0$  in Figure 7.4 is the same as the curve for  $B = \infty$  in Figure 7.16. With decreasing  $B$ , corresponding to slower external mass transfer, the concentration profile in the pellet becomes more uniform *and* the dimensionless surface concentration decreases. The lower concentration leads to lower reaction rates. Therefore one normally designs the reactor and chooses operating conditions, such as large gas velocities, to enhance external mass transfer and make the Biot number large.

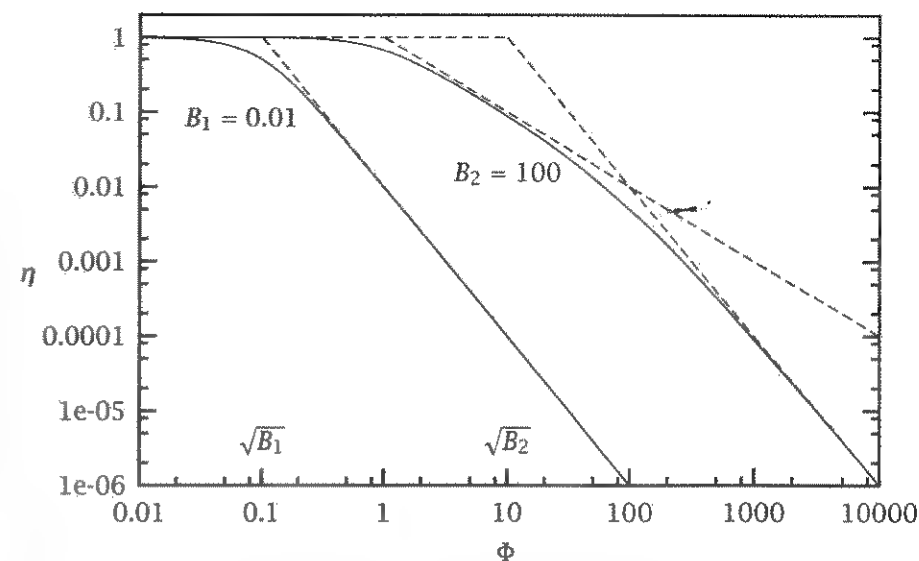
Figure 7.17 shows the effect of the Biot number on the effectiveness factor or total pellet reaction rate. Notice that the slope of the log-log plot of  $\eta$  versus  $\Phi$  has a slope of negative *two* rather than negative *one* as in the case without external mass-transfer limitations ( $B = \infty$ ). Figure 7.18 shows this effect in more detail. If  $B$  is small, the log-log



**Figure 7.17:** Effectiveness factor versus Thiele modulus for different values of the Biot number; first-order reaction in a spherical pellet.

plot corners with a slope of negative two at  $\Phi = \sqrt{B}$ . If  $B$  is large, the log-log plot first corners with a slope of negative one at  $\Phi = 1$ , then it corners again and decreases the slope to negative two at  $\Phi = \sqrt{B}$ . Both mechanisms of diffusional resistance, the diffusion within the pellet and the mass transfer from the fluid to the pellet, show their effect on pellet reaction rate by changing the slope of the effectiveness factor by negative one.<sup>3</sup> Given the value of the Biot number, one can easily sketch the straight line asymptotes shown in Figure 7.18. Then, given the value of the Thiele modulus, one can determine the approximate concentration profile, and whether internal diffusion or external mass transfer or both limit the pellet reaction rate. The possible cases are summarized in Table 7.4.

<sup>3</sup>Students who are taking or already have taken the process control course might want to think about what happens to the amplitude of a signal as it passes through a series of first-order processes.



**Figure 7.18:** Asymptotic behavior of the effectiveness factor versus Thiele modulus; first-order reaction in a spherical pellet.

Biot number	Thiele modulus	Mechanism controlling pellet reaction rate
$B < 1$	$\Phi < B$ $B < \Phi < 1$ $1 < \Phi$	reaction external mass transfer both external mass transfer and internal diffusion
$1 < B$	$\Phi < 1$ $1 < \Phi < B$ $B < \Phi$	reaction internal diffusion both internal diffusion and external mass transfer

**Table 7.4:** The controlling mechanisms for pellet reaction rate given finite rates of internal diffusion and external mass transfer.

#### 7.4.6 Observed versus Intrinsic Kinetic Parameters

We often need to determine a reaction order and rate constant for some catalytic reaction of interest. Assume the following  $n$ th-order reaction takes place in a catalyst particle



We call the values of  $k$  and  $n$  the intrinsic rate constant and reaction order to distinguish them from what we may estimate from data. The typical experiment is to change the value of  $c_A$  in the bulk fluid, measure the rate  $r_1$  as a function of  $c_A$ , and then find the values of the parameters  $k$  and  $n$  that best fit the measurements. We explain this procedure in much more detail in Chapter 9. Here we show only that one should exercise caution with this estimation if we are measuring the rates with a solid catalyst. The effects of reaction, diffusion and external mass transfer may all manifest themselves in the measured rate. We express the reaction rate as

$$r_1 = \eta kc_{Ab}^n \quad (7.66)$$

We also know that at steady state, the rate is equal to the flux of A into the catalyst particle

$$r_1 = k_{mA}(c_{Ab} - c_{As}) = \frac{D_A}{a} \frac{dc_A}{dr} \Big|_{r=R} \quad (7.67)$$

We now study what happens to our experiment under different rate-limiting steps.

**Reaction limited.** First assume that both the external mass transfer and internal pellet diffusion are fast compared to the reaction. Then  $\eta = 1$ , and we would estimate the intrinsic parameters correctly in Equation 7.66

$$\begin{aligned} k_{ob} &= k \\ n_{ob} &= n \end{aligned}$$

Everything goes according to plan when we are reaction limited.

**Diffusion limited.** Next assume that the external mass transfer and reaction are fast, but the internal diffusion is slow. In this case we have

$\eta = 1/\Phi$ , and using the definition of Thiele modulus and Equation 7.66

$$r_1 = k_{ob} c_{As}^{(n+1)/2} \quad (7.68)$$

$$k_{ob} = \frac{1}{a} \sqrt{\frac{2}{n+1}} D_A \sqrt{k} \quad (7.69)$$

$$n_{ob} = (n+1)/2 \quad (7.70)$$

So we see two problems. The rate constant we estimate,  $k_{ob}$ , varies as the square root of the intrinsic rate constant,  $k$ . The diffusion has affected the measured rate of the reaction and disguised the rate constant. We even get an incorrect reaction order: a first-order reaction appears half-order, a second-order reaction appears first-order, and so on.

Also consider what happens if we vary the temperature and try to determine the reaction's activation energy. Let the temperature dependence of the diffusivity,  $D_A$ , be represented also in Arrhenius form, with  $E_{diff}$  the activation energy of the diffusion coefficient. Let  $E_{rxn}$  be the intrinsic activation energy of the reaction. The observed activation energy from Equation 7.69 is

$$E_{ob} = \frac{E_{diff} + E_{rxn}}{2}$$

so both activation energies show up in our estimated activation energy. Normally the temperature dependence of the diffusivity is much smaller than the temperature dependence of the reaction,  $E_{diff} \ll E_{rxn}$ , so we would estimate an activation energy that is one-half the intrinsic value.

**Mass transfer limited.** Finally, assume the reaction and diffusion are fast compared to the external mass transfer. Then we have  $c_{Ab} \gg c_{As}$  and Equation 7.67 gives

$$r_1 = k_{mA} c_{Ab} \quad (7.71)$$

If we vary  $c_{Ab}$  and measure  $r_1$ , we would find the mass transfer coefficient instead of the rate constant, and a first-order reaction instead of the true reaction order

$$\begin{aligned} k_{ob} &= k_{mA} \\ n_{ob} &= 1 \end{aligned}$$

Normally, mass-transfer coefficients also have fairly small temperature dependence compared to reaction rates, so the observed activation energy would be almost zero, independent of the true reaction's activation energy.

The moral to this story is that mass transfer and diffusion resistances disguise the reaction kinetics. We can solve this problem in two ways. First, we can arrange the experiment so that mass transfer and diffusion are fast and do not affect the estimates of the kinetic parameters. We can accomplish this by making the catalyst particles small ( $\Phi$  small) and using a flow reactor with high fluid flowrates to ensure a high mass-transfer coefficient. If this experimental design is impractical or too expensive, we can alternatively model the effects of the mass transfer and diffusion, and estimate the parameters  $D_A$  and  $k_{mA}$  simultaneously with  $k$  and  $n$ . We develop techniques in Chapter 9 to handle this more complex estimation problem.

## 7.5 Nonisothermal Particle Considerations

We now consider situations in which the catalyst particle is not isothermal. Given an exothermic reaction, for example, if the particle's thermal conductivity is not large compared to the rate of heat release due to chemical reaction, the temperature rises inside the particle. We wish to explore the effects of this temperature rise on the catalyst performance. We have already written the general mass and energy balances for the catalyst particle in Section 7.3. Consider the single-reaction case, in which we have  $R_A = -r$  and Equations 7.14 and 7.15 reduce to

$$\begin{aligned} D_A \nabla^2 c_A &= r \\ \hat{k} \nabla^2 T &= \Delta H_R r \end{aligned}$$

We can eliminate the reaction term between the mass and energy balances to produce

$$\nabla^2 T = \frac{\Delta H_R D_A}{\hat{k}} \nabla^2 c_A$$

which relates the conversion of the reactant to the rise (or fall) in temperature. Because we have assumed constant properties, we can integrate this equation twice to give the relationship between temperature and A concentration

$$T - T_s = \frac{-\Delta H_R D_A}{\hat{k}} (c_{As} - c_A) \quad (7.72)$$

We now consider a first-order reaction and assume the rate constant has an Arrhenius form,

$$k(T) = k_s \exp \left[ -E \left( \frac{1}{T} - \frac{1}{T_s} \right) \right]$$

in which  $T_s$  is the pellet exterior temperature, and we assume fast external mass transfer. Substituting Equation 7.72 into the rate constant expression gives

$$k(T) = k_s \exp \left[ \frac{E}{T_s} \left( 1 - \frac{T_s}{T_s + \Delta H_R D_A (c_A - c_{As}) / \hat{k}} \right) \right]$$

It now simplifies matters to define three dimensionless variables

$$\gamma = \frac{E}{T_s}, \quad \beta = \frac{-\Delta H_R D_A c_{As}}{\hat{k} T_s}, \quad \tilde{\Phi}^2 = \frac{k(T_s)}{D_A} a^2$$

in which  $\gamma$  is a dimensionless activation energy,  $\beta$  is a dimensionless heat of reaction, and  $\tilde{\Phi}$  is the usual Thiele modulus. Again we use the tilde to indicate we will find a better Thiele modulus subsequently. With these variables, we can express the rate constant as

$$k(T) = k_s \exp \left[ \frac{\gamma \beta (1 - \bar{c})}{1 + \beta (1 - \bar{c})} \right] \quad (7.73)$$

We then substitute the rate constant into the mass balance, and assume a spherical particle to obtain the final dimensionless model

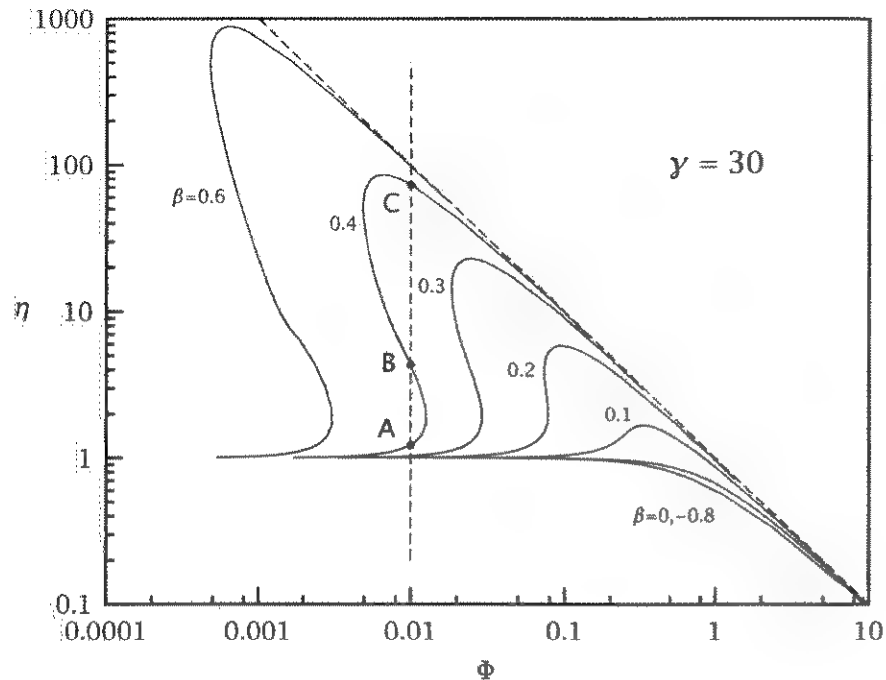
$$\begin{aligned} \frac{1}{\bar{r}^2} \frac{d}{d\bar{r}} \left( \bar{r}^2 \frac{d\bar{c}}{d\bar{r}} \right) &= \tilde{\Phi}^2 \bar{c} \exp \left( \frac{\gamma \beta (1 - \bar{c})}{1 + \beta (1 - \bar{c})} \right) \\ \frac{d\bar{c}}{d\bar{r}} &= 0 \quad \bar{r} = 3 \\ \bar{c} &= 1 \quad \bar{r} = 0 \end{aligned} \quad (7.74)$$

Equation 7.74 is sometimes called the Weisz-Hicks problem in honor of Weisz and Hicks's outstanding paper in which they computed accurate numerical solutions to this problem [22]. Given the solution to Equation 7.74, we can compute the effectiveness factor for the nonisothermal pellet using the usual relationship

$$\eta = \frac{1}{\tilde{\Phi}^2} \left. \frac{d\bar{c}}{d\bar{r}} \right|_{\bar{r}=3} \quad (7.75)$$

If we perform the same asymptotic analysis of Section 7.4.4 on the Weisz-Hicks problem, we find, however, that the appropriate Thiele modulus for this problem is

$$\Phi = \tilde{\Phi} / I(\gamma, \beta), \quad I(\gamma, \beta) = \left[ 2 \int_0^1 \bar{c} \exp \left( \frac{\gamma \beta (1 - \bar{c})}{1 + \beta (1 - \bar{c})} \right) d\bar{c} \right]^{1/2} \quad (7.76)$$



**Figure 7.19:** Effectiveness factor versus normalized Thiele modulus for a first-order reaction in a nonisothermal spherical pellet.

The normalizing integral  $I(\gamma, \beta)$  can be expressed as a sum of exponential integrals [2] or evaluated by quadrature.

Figure 7.19 shows the effectiveness factor versus Thiele modulus for activation energy  $\gamma = 30$  and a variety of heats of reaction,  $\beta$ . Note that  $\Phi$  is well chosen in Equation 7.76 because the large  $\Phi$  asymptotes are the same for all values of  $\gamma$  and  $\beta$ . The first interesting feature of Figure 7.19 is that the effectiveness factor is *greater than unity* for some values of the parameters. Notice that feature is more pronounced as we increase the exothermic heat of reaction. If we consider what is happening within the pellet, this effect may not be too surprising. For the highly exothermic case, the pellet's interior temperature is significantly *higher* than the exterior temperature  $T_s$ . The rate constant inside the pellet is therefore much larger than the value at the exterior,  $k_s$ . Even though the concentration is lower inside the pellet because A is con-

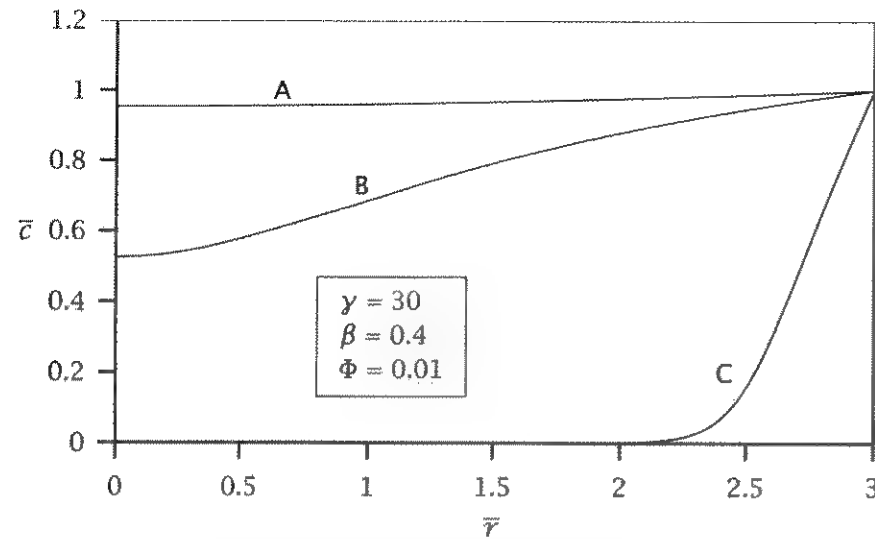
sumed, the increase in the rate constant is more significant and the product  $r = kc_A$  is *larger* inside the pellet. Because the effectiveness factor compares the actual rate in the pellet to the rate at the surface conditions, it is possible for the effectiveness factor to exceed unity in a nonisothermal pellet, which we see in Figure 7.19.

A second striking feature of the nonisothermal pellet is that multiple steady states are possible. Consider the case  $\Phi = 0.01$ ,  $\beta = 0.4$  and  $\gamma = 30$  shown in Figure 7.19. The effectiveness factor has three possible values for this case. We show in Figures 7.20 and 7.21 the solution to Equation 7.74 for this case. The three temperature and concentration profiles correspond to an ignited steady state (C), an extinguished steady state (A), and an unstable intermediate steady state (B). As we showed in Chapter 6, whether we achieve the ignited or extinguished steady state in the pellet depends on how the reactor is started. Aris provides further discussion of these cases and shows that many steady-state solutions are possible in some cases [3, p. 51]. For realistic values of the catalyst thermal conductivity, however, the pellet can often be considered isothermal and the energy balance can be neglected [17]. Multiple steady-state solutions in the particle may still occur in practice, however, if there is a large external heat transfer resistance.

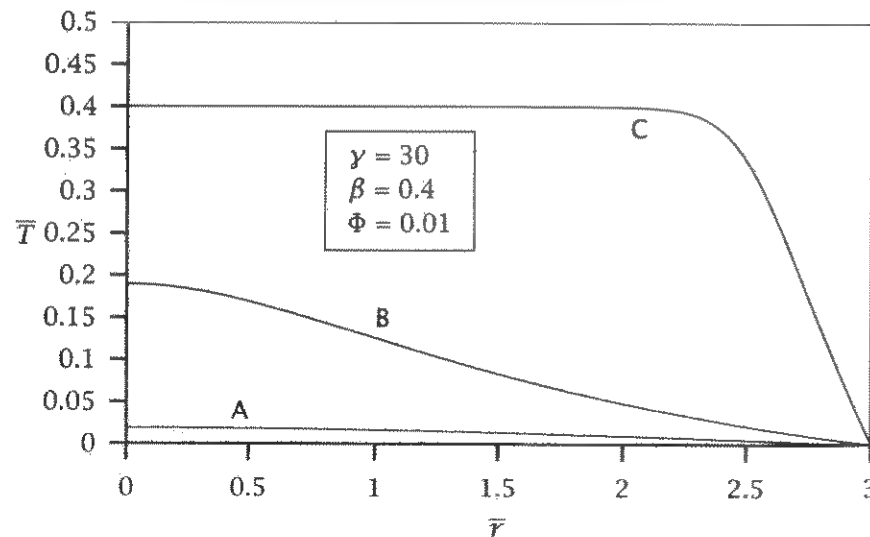
## 7.6 Multiple Reactions

As the next step up in complexity, we consider the case of multiple reactions. Some analytical solutions are available for simple cases with multiple reactions, and Aris provides a comprehensive list [2], but the scope of these is limited. We focus on numerical computation as a general method for these problems. Indeed, we find that even numerical solution of some of these problems is challenging for two reasons. First, steep concentration profiles often occur for realistic parameter values, and we wish to compute these profiles accurately. It is not unusual for species concentrations to change by 10 orders of magnitude within the pellet for realistic reaction and diffusion rates. Second, we are solving boundary-value problems because the boundary conditions are provided at the center and exterior surface of the pellet. Boundary-value problems (BVPs) are generally much more difficult to solve than initial-value problems (IVPs).

A detailed description of numerical methods for this problem is out of place here. We use the collocation method, which is described in more detail in Appendix A. The next example involves five species,



**Figure 7.20:** Dimensionless concentration versus radius for the non-isothermal spherical pellet: lower (A), unstable middle (B), and upper (C) steady states.

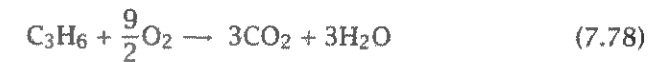


**Figure 7.21:** Dimensionless temperature versus radius for the non-isothermal spherical pellet: lower (A), unstable middle (B), and upper (C) steady states.

two reactions with Hougen-Watson kinetics, and both diffusion and external mass-transfer limitations.

### Example 7.2: Catalytic converter

Consider the oxidation of CO and a representative volatile organic such as propylene in a automobile catalytic converter containing spherical catalyst pellets with particle radius 0.175 cm. The particle is surrounded by a fluid at 1.0 atm pressure and 550 K containing 2% CO, 3% O<sub>2</sub> and 0.05% (500 ppm) C<sub>3</sub>H<sub>6</sub>. The reactions of interest are



with rate expressions given by Oh et al. [16]

$$r_1 = \frac{k_1 c_{\text{CO}} c_{\text{O}_2}}{(1 + K_{\text{CO}} c_{\text{CO}} + K_{\text{C}_3\text{H}_6} c_{\text{C}_3\text{H}_6})^2} \quad (7.79)$$

$$r_2 = \frac{k_2 c_{\text{C}_3\text{H}_6} c_{\text{O}_2}}{(1 + K_{\text{CO}} c_{\text{CO}} + K_{\text{C}_3\text{H}_6} c_{\text{C}_3\text{H}_6})^2} \quad (7.80)$$

The rate constants and the adsorption constants are assumed to have Arrhenius form. The parameter values are given in Table 7.5 [16]. The mass-transfer coefficients are taken from DeAcetis and Thodos [9]. The pellet may be assumed to be isothermal. Calculate the steady-state pellet concentration profiles of all reactants and products.

### Solution

We solve the steady-state mass balances for the three reactant species,

$$D_j \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dc_j}{dr} \right) = -R_j \quad (7.81)$$

with the boundary conditions

$$\frac{dc_j}{dr} = 0 \quad r = 0 \quad (7.82)$$

$$D_j \frac{dc_j}{dr} = k_{mj} (c_{jf} - c_j) \quad r = R \quad (7.83)$$

$j = \{\text{CO}, \text{O}_2, \text{C}_3\text{H}_6\}$ . The model is solved using the collocation method. The reactant concentration profiles are shown in Figures 7.22 and 7.23.



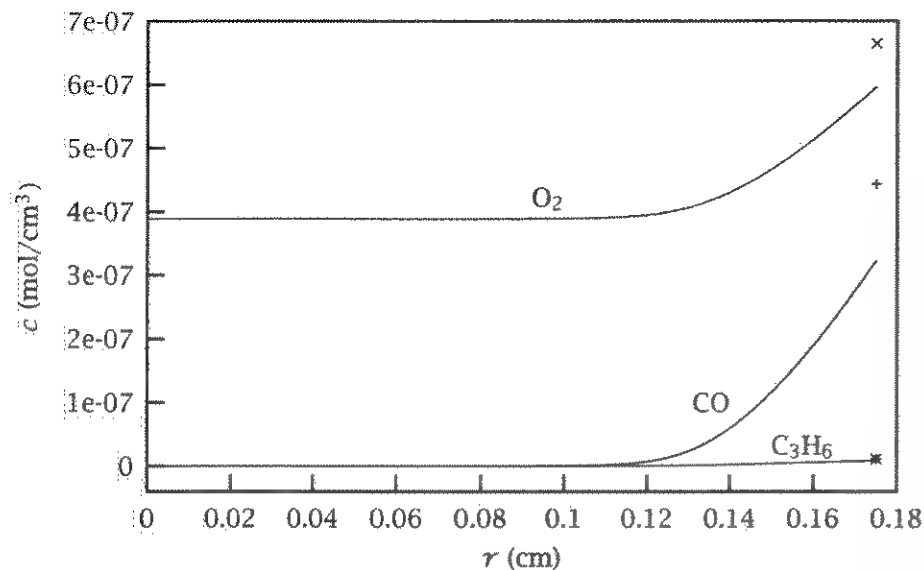


Figure 7.22: Concentration profiles of reactants; fluid concentration of  $O_2$  (x),  $CO$  (+),  $C_3H_6$  (\*).

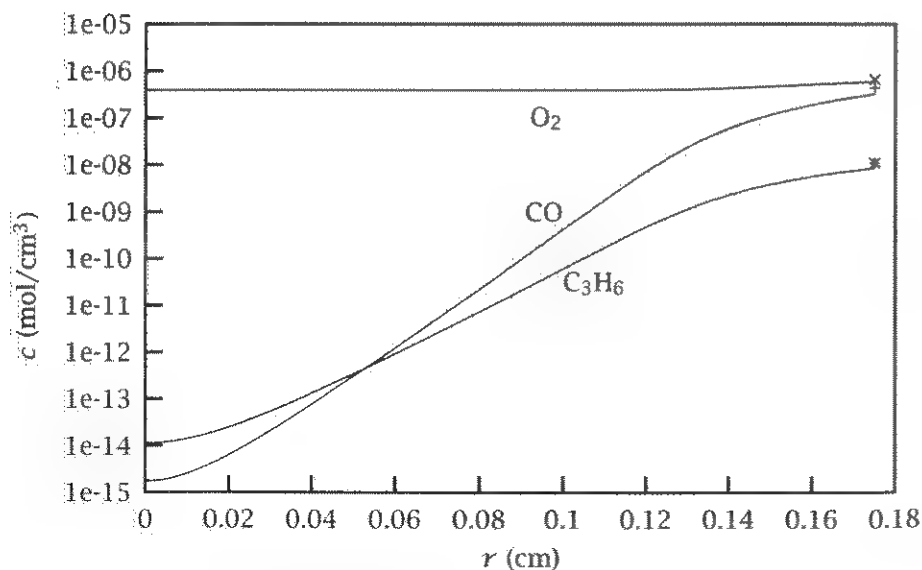


Figure 7.23: Concentration profiles of reactants (log scale); fluid concentration of  $O_2$  (x),  $CO$  (+),  $C_3H_6$  (\*).

Parameter	Value	Units	Parameter	Value	Units
$P$	$1.013 \times 10^5$	N/m <sup>2</sup>	$k_{10}$	$7.07 \times 10^{19}$	mol/cm <sup>3</sup> · s
$T$	550	K	$k_{20}$	$1.47 \times 10^{21}$	mol/cm <sup>3</sup> · s
$R$	0.175	cm	$K_{COO}$	$8.099 \times 10^6$	cm <sup>3</sup> /mol
$E_1$	13,108	K	$K_{C_3H_6O}$	$2.579 \times 10^8$	cm <sup>3</sup> /mol
$E_2$	15,109	K	$D_{CO}$	0.0487	cm <sup>2</sup> /s
$E_{CO}$	-409	K	$D_{O_2}$	0.0469	cm <sup>2</sup> /s
$E_{C_3H_6}$	191	K	$D_{C_3H_6}$	0.0487	cm <sup>2</sup> /s
$c_{COf}$	2.0 %		$k_{mCO}$	3.90	cm/s
$c_{O_2f}$	3.0 %		$k_{mO_2}$	4.07	cm/s
$c_{C_3H_6f}$	0.05 %		$k_{mC_3H_6}$	3.90	cm/s

Table 7.5: Kinetic and mass-transfer parameters for the catalytic converter example.

Notice that  $O_2$  is in excess and both  $CO$  and  $C_3H_6$  reach very low values within the pellet. The log scale in Figure 7.23 shows that the concentrations of these reactants change by seven orders of magnitude. Obviously the consumption rate is large compared to the diffusion rate for these species. The external mass-transfer effect is noticeable, but not dramatic.

The product concentrations could simply be calculated by solving their mass balances along with those of the reactants. Because we have only two reactions, however, the concentrations of the products are computable from the stoichiometry and the mass balances. If we take the following mass balances

$$\begin{aligned}
 D_{CO} \nabla^2 c_{CO} &= -R_{CO} = r_1 \\
 D_{C_3H_6} \nabla^2 c_{C_3H_6} &= -R_{C_3H_6} = r_2 \\
 D_{CO_2} \nabla^2 c_{CO_2} &= -R_{CO_2} = -r_1 - 3r_2 \\
 D_{H_2O} \nabla^2 c_{H_2O} &= -R_{H_2O} = -3r_2
 \end{aligned}$$

and form linear combinations to eliminate the reaction-rate terms for the two products, we obtain

$$\begin{aligned}
 D_{CO_2} \nabla^2 c_{CO_2} &= -D_{CO} \nabla^2 c_{CO} - 3D_{C_3H_6} \nabla^2 c_{C_3H_6} \\
 D_{H_2O} \nabla^2 c_{H_2O} &= -3D_{C_3H_6} \nabla^2 c_{C_3H_6}
 \end{aligned}$$

Because the diffusivities are assumed constant, we can integrate these

once on  $(0, r)$  to obtain for the products

$$D_{\text{CO}_2} \frac{dc_{\text{CO}_2}}{dr} = -D_{\text{CO}} \frac{dc_{\text{CO}}}{dr} - 3D_{\text{C}_3\text{H}_6} \frac{dc_{\text{C}_3\text{H}_6}}{dr}$$

$$D_{\text{H}_2\text{O}} \frac{dc_{\text{H}_2\text{O}}}{dr} = -3D_{\text{C}_3\text{H}_6} \frac{dc_{\text{C}_3\text{H}_6}}{dr}$$

The exterior boundary condition can be rearranged to give

$$c_j - c_{jf} = \frac{D_j}{k_{mj}} \frac{dc_j}{dr}$$

Substituting in the relationships for the products gives

$$c_{\text{CO}_2} = c_{\text{CO}_2f} - \frac{1}{k_{m\text{CO}_2}} \left[ D_{\text{CO}} \frac{dc_{\text{CO}}}{dr} + 3D_{\text{C}_3\text{H}_6} \frac{dc_{\text{C}_3\text{H}_6}}{dr} \right]$$

$$c_{\text{H}_2\text{O}} = c_{\text{H}_2\text{O}f} - \frac{1}{k_{m\text{H}_2\text{O}}} \left[ 3D_{\text{C}_3\text{H}_6} \frac{dc_{\text{C}_3\text{H}_6}}{dr} \right]$$

The right-hand sides are available from the solution of the material balances of the reactants. Plotting these results for the products gives Figure 7.24. We see that  $\text{CO}_2$  is the main product. Note the products flow out of the pellet, unlike the reactants shown in Figures 7.22 and 7.23, which are flowing into the pellet. □

## 7.7 Fixed-Bed Reactor Design

Given our detailed understanding of the behavior of a single catalyst particle, we now are prepared to pack a tube with a bed of these particles and solve the fixed-bed reactor design problem. In the fixed-bed reactor, we keep track of two phases. The fluid-phase streams through the bed and transports the reactants and products through the reactor. The reaction-diffusion processes take place in the solid-phase catalyst particles. The two phases communicate to each other by exchanging mass and energy at the catalyst particle exterior surfaces. We have constructed a detailed understanding of all these events, and now we assemble them together.

### 7.7.1 Coupling the Catalyst and Fluid

We make the following assumptions:

1. Uniform catalyst pellet exterior. Particles are small compared to the length of the reactor.

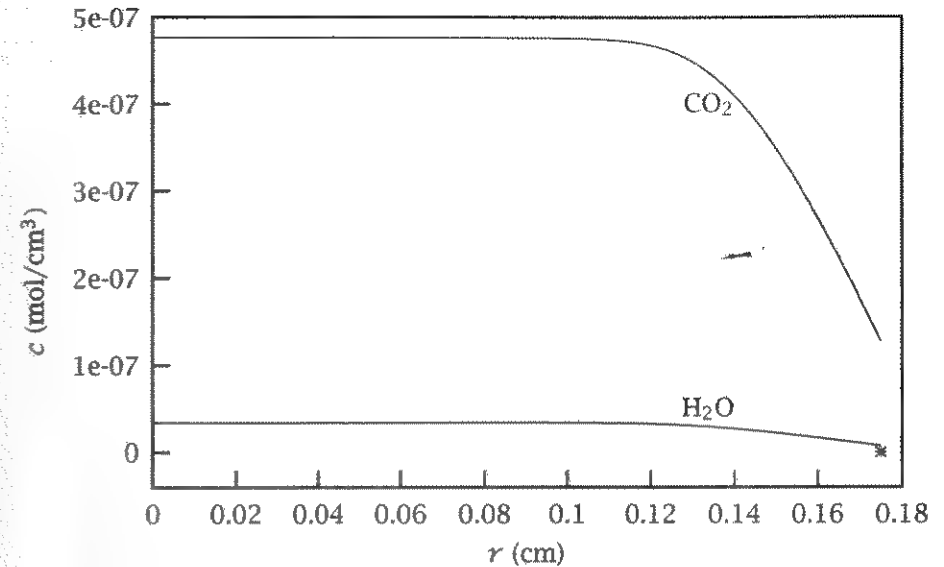


Figure 7.24: Concentration profiles of the products; fluid concentration of  $\text{CO}_2$  (x),  $\text{H}_2\text{O}$  (+).

2. Plug flow in the bed, no radial profiles.
3. Neglect axial diffusion in the bed.
4. Steady state.

**Fluid.** In the fluid phase, we track the molar flows of all species, the temperature and the pressure. We can no longer neglect the pressure drop in the tube because of the catalyst bed. We use an empirical correlation to describe the pressure drop in a packed tube, the well-known Ergun equation [10]. Therefore, we have the following differential equations for the fluid phase

$$\frac{dN_j}{dV} = R_j \quad (7.84)$$

$$Q\rho\hat{C}_p \frac{dT}{dV} = -\sum_i \Delta H_{Ri} \gamma_i + \frac{2}{R} U^o (T_a - T) \quad (7.85)$$

$$\frac{dP}{dV} = -\frac{(1-\epsilon_B)}{D_p \epsilon_B^3} \frac{Q}{A_c^2} \left[ 150 \frac{(1-\epsilon_B)\mu_f}{D_p} + \frac{7}{4} \frac{\rho Q}{A_c} \right] \quad (7.86)$$

The fluid-phase boundary conditions are provided by the known feed conditions at the tube entrance

$$N_j = N_{jf}, \quad z = 0 \quad (7.87)$$

$$T = T_f, \quad z = 0 \quad (7.88)$$

$$P = P_f, \quad z = 0 \quad (7.89)$$

**Catalyst particle.** Inside the catalyst particle, we track the concentrations of all species and the temperature. We neglect any pressure effect inside the catalyst particle. We have the following differential equations for the catalyst particle

$$D_j \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\tilde{c}_j}{dr} \right) = -\tilde{R}_j \quad (7.90)$$

$$\hat{k} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\tilde{T}}{dr} \right) = \sum_i \Delta H_{Ri} \tilde{r}_i \quad (7.91)$$

The boundary conditions are provided by the mass-transfer and heat-transfer rates at the pellet exterior surface, and the zero slope conditions at the pellet center

$$\frac{d\tilde{c}_j}{dr} = 0 \quad r = 0 \quad (7.92)$$

$$D_j \frac{d\tilde{c}_j}{dr} = k_{jm}(c_j - \tilde{c}_j) \quad r = R \quad (7.93)$$

$$\frac{d\tilde{T}}{dr} = 0 \quad r = 0 \quad (7.94)$$

$$\hat{k} \frac{d\tilde{T}}{dr} = k_T(T - \tilde{T}) \quad r = R \quad (7.95)$$

**Coupling equations.** Finally, we equate the production rate  $R_j$  experienced by the fluid phase to the production rate inside the particles, which is where the reaction takes place. Analogously, we equate the enthalpy change on reaction experienced by the fluid phase to the enthalpy change on reaction taking place inside the particles. These ex-

pressions are given below

$$\underbrace{R_j}_{\text{rate } j / \text{vol}} = - \underbrace{(1 - \epsilon_B)}_{\text{vol cat / vol}} \underbrace{\frac{S_p}{V_p} D_j \frac{d\tilde{c}_j}{dr}}_{\text{rate } j / \text{vol cat}} \bigg|_{r=R} \quad (7.96)$$

$$\underbrace{\sum_i \Delta H_{Ri} r_i}_{\text{rate heat / vol}} = \underbrace{(1 - \epsilon_B)}_{\text{vol cat / vol}} \underbrace{\frac{S_p}{V_p} \hat{k} \frac{d\tilde{T}}{dr}}_{\text{rate heat / vol cat}} \bigg|_{r=R} \quad (7.97)$$

Notice we require the bed porosity to convert from the rate per volume of particle to the rate per volume of reactor. The bed porosity or void fraction,  $\epsilon_B$ , is defined as the volume of voids per volume of reactor. The volume of catalyst per volume of reactor is therefore  $1 - \epsilon_B$ . This information can be presented in a number of equivalent ways. We can easily measure the density of the pellet,  $\rho_p$ , and the density of the bed,  $\rho_B$ . From the definition of bed porosity, we have the relation

$$\rho_B = (1 - \epsilon_B) \rho_p$$

or if we solve for the volume fraction of catalyst

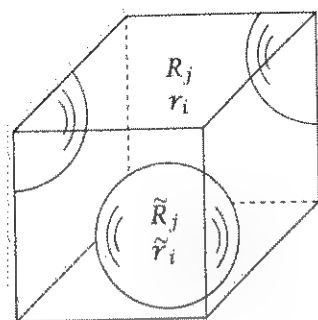
$$1 - \epsilon_B = \rho_B / \rho_p$$

Figure 7.25 shows the particles and fluid, and summarizes the coupling relations between the two phases.

Equations 7.84–7.97 provide the full packed-bed reactor model given our assumptions. We next examine several packed-bed reactor problems that can be solved without solving this full set of equations. Finally, we present Example 7.7, which requires numerical solution of the full set of equations.

### Example 7.3: First-order, isothermal fixed-bed reactor

Use the rate data presented in Example 7.1 to find the fixed-bed reactor volume and the catalyst mass needed to convert 97% of A. The feed to the reactor is pure A at 1.5 atm at a rate of 12 mol/s. The 0.3 cm pellets are to be used, which leads to a bed density  $\rho_B = 0.6 \text{ g/cm}^3$ . Assume the reactor operates isothermally at 450 K and that external mass-transfer limitations are negligible.



Mass

$$R_j = (1 - \epsilon_B) \tilde{R}_{jp}$$

$$\tilde{R}_{jp} = -\frac{S_p}{V_p} D_j \left. \frac{d\tilde{c}_j}{dr} \right|_{r=R}$$

Energy

$$\sum_i \Delta H_{Ri} \tilde{r}_i = (1 - \epsilon_B) \sum_i \Delta H_{Ri} \tilde{r}_{ip}$$

$$\sum_i \Delta H_{Ri} \tilde{r}_{ip} = \frac{S_p}{V_p} \hat{k} \left. \frac{dT}{dr} \right|_{r=R}$$

**Figure 7.25:** Fixed-bed reactor volume element containing fluid and catalyst particles; the equations show the coupling between the catalyst particle balances and the overall reactor balances.

### Solution

We solve the fixed-bed design equation

$$\frac{dN_A}{dV} = R_A = -(1 - \epsilon_B) \eta k c_A$$

between the limits  $N_{Af}$  and  $0.03N_{Af}$ , in which  $c_A$  is the A concentration in the fluid. For the first-order, isothermal reaction, the Thiele modulus is independent of A concentration, and is therefore independent of axial position in the bed

$$\Phi = \frac{R}{3} \sqrt{\frac{k}{D_A}} = \frac{0.3 \text{ cm}}{3} \sqrt{\frac{2.6 \text{ s}^{-1}}{0.007 \text{ cm}^2/\text{s}}} = 1.93$$

The effectiveness factor is also therefore a constant

$$\eta = \frac{1}{\Phi} \left[ \frac{1}{\tanh 3\Phi} - \frac{1}{3\Phi} \right] = \frac{1}{1.93} \left[ 1 - \frac{1}{5.78} \right] = 0.429$$

In Chapter 4, Equation 4.75, we express the concentration of A in terms of molar flows for an ideal-gas mixture

$$c_A = \frac{P}{RT} \left( \frac{N_A}{N_A + N_B} \right)$$

The total molar flow is constant due to the reaction stoichiometry so  $N_A + N_B = N_{Af}$  and we have

$$c_A = \frac{P}{RT} \frac{N_A}{N_{Af}}$$

Substituting these values into the material balance, rearranging and integrating over the volume gives

$$V_R = -(1 - \epsilon_B) \left( \frac{RT N_{Af}}{\eta k P} \right) \int_{N_{Af}}^{0.03 N_{Af}} \frac{dN_A}{N_A}$$

$$V_R = - \left( \frac{0.6}{0.85} \right) \frac{(82.06)(450)(12)}{(0.429)(2.6)(1.5)} \ln(0.03) = 1.32 \times 10^6 \text{ cm}^3$$

and

$$W_c = \rho_B V_R = \frac{0.6}{1000} (1.32 \times 10^6) = 789 \text{ kg}$$

We see from this example that if the Thiele modulus and effectiveness factors are constant, finding the size of a fixed-bed reactor is no more difficult than finding the size of a plug-flow reactor.  $\square$

### Example 7.4: Mass-transfer limitations in a fixed-bed reactor

Reconsider Example 7.3 given the following two values of the mass-transfer coefficient

$$k_{m1} = 0.07 \text{ cm/s}$$

$$k_{m2} = 1.4 \text{ cm/s}$$

### Solution

First we calculate the Biot numbers from Equation 7.61 and obtain

$$B_1 = \frac{(0.07)(0.1)}{(0.007)} = 1$$

$$B_2 = \frac{(1.4)(0.1)}{(0.007)} = 20$$

Inspection of Figure 7.17 indicates that we expect a significant reduction in the effectiveness factor due to mass-transfer resistance in the first case, and little effect in the second case. Evaluating the effectiveness factors with Equation 7.65 indeed shows

$$\eta_1 = 0.165$$

$$\eta_2 = 0.397$$

which we can compare to  $\eta = 0.429$  from the previous example with no mass-transfer resistance. We can then easily calculate the required catalyst mass from the solution of the previous example without mass-transfer limitations, and the new values of the effectiveness factors

$$V_{R1} = \left( \frac{0.429}{0.165} \right) (789) = 2051 \text{ kg}$$

$$V_{R2} = \left( \frac{0.429}{0.397} \right) (789) = 852 \text{ kg}$$

As we can see, the first mass-transfer coefficient is so small that more than twice as much catalyst is required to achieve the desired conversion compared to the case without mass-transfer limitations. The second mass-transfer coefficient is large enough that only 8% more catalyst is required.  $\square$

### Example 7.5: Second-order, isothermal fixed-bed reactor

Estimate the mass of catalyst required in an isothermal fixed-bed reactor for the second-order, heterogeneous reaction.



$$r = kc_A^2 \quad k = 2.25 \times 10^5 \text{ cm}^3/\text{mol s}$$

The gas feed consists of A and an inert, each with molar flowrate of 10 mol/s, the total pressure is 4.0 atm and the temperature is 550 K. The desired conversion of A is 75%. The catalyst is a spherical pellet with a radius of 0.45 cm. The pellet density is  $\rho_p = 0.68 \text{ g/cm}^3$  and the bed density is  $\rho_B = 0.60 \text{ g/cm}^3$ . The effective diffusivity of A is  $0.008 \text{ cm}^2/\text{s}$  and may be assumed constant. You may assume the fluid and pellet surface concentrations are equal.

### Solution

We solve the fixed-bed design equation

$$\frac{dN_A}{dV} = R_A = -(1 - \epsilon_B)\eta kc_A^2$$

$$N_A(0) = N_{Af} \quad (7.98)$$

between the limits  $N_{Af}$  and  $0.25N_{Af}$ . We again express the concentration of A in terms of the molar flows

$$c_A = \frac{P}{RT} \left( \frac{N_A}{N_A + N_B + N_I} \right)$$

As in the previous example, the total molar flow is constant and we know its value at the entrance to the reactor

$$N_T = N_{Af} + N_{Bf} + N_{If} = 2N_{Af}$$

Therefore,

$$c_A = \frac{P}{RT} \frac{N_A}{2N_{Af}} \quad (7.99)$$

Next we use the definition of  $\Phi$  for  $n$ th-order reactions given in Equation 7.47

$$\Phi = \frac{R}{3} \left[ \frac{(n+1)kc_A^{n-1}}{2D_e} \right]^{1/2} = \frac{R}{3} \left[ \frac{(n+1)k}{2D_e} \left( \frac{P}{RT} \frac{N_A}{2N_{Af}} \right)^{n-1} \right]^{1/2} \quad (7.100)$$

Substituting in the parameter values gives

$$\Phi = 9.17 \left( \frac{N_A}{2N_{Af}} \right)^{1/2} \quad (7.101)$$

For the second-order reaction, Equation 7.101 shows that  $\Phi$  varies with the molar flow, which means  $\Phi$  and  $\eta$  vary along the length of the reactor as  $N_A$  decreases. We are asked to estimate the catalyst mass needed to achieve a conversion of A equal to 75%. So for this particular example,  $\Phi$  decreases from 6.49 to 3.24. As shown in Figure 7.9, we can *approximate* the effectiveness factor for the second-order reaction using the analytical result for the first-order reaction, Equation 7.42,

$$\eta = \frac{1}{\Phi} \left[ \frac{1}{\tanh 3\Phi} - \frac{1}{3\Phi} \right] \quad (7.102)$$

Summarizing so far, to compute  $N_A$  versus  $V_R$ , we solve one differential equation, Equation 7.98, in which we use Equation 7.99 for  $c_A$ , and Equations 7.101 and 7.102 for  $\Phi$  and  $\eta$ . We march in  $V_R$  until  $N_A = 0.25N_{Af}$ . The solution to the differential equation is shown in Figure 7.26. The required reactor volume and mass of catalyst are:

$$V_R = 361 \text{ L}, \quad W_c = \rho_B V_R = 216 \text{ kg}$$

As a final exercise, given that  $\Phi$  ranges from 6.49 to 3.24, we can make the large  $\Phi$  approximation

$$\eta = \frac{1}{\Phi} \quad (7.103)$$

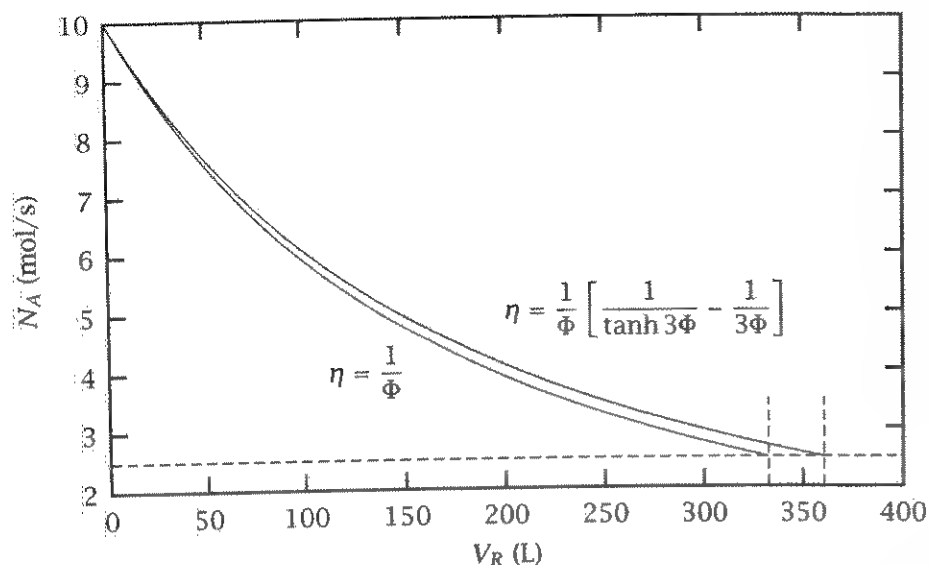


Figure 7.26: Molar flow of A versus reactor volume for second-order, isothermal reaction in a fixed-bed reactor.

to obtain a closed-form solution. If we substitute this approximation for  $\eta$ , and Equation 7.100 into Equation 7.98 and rearrange we obtain

$$\frac{dN_A}{dV} = \frac{-(1 - \epsilon_B) \sqrt{k} (P/RT)^{3/2}}{(R/3) \sqrt{3/D_A} (2N_{Af})^{3/2}} N_A^{3/2}$$

Separating and integrating this differential equation gives

$$V_R = \frac{4[(1 - x_A)^{-1/2} - 1] N_{Af} (R/3) \sqrt{3/D_A}}{(1 - \epsilon_B) \sqrt{k} (P/RT)^{3/2}} \quad (7.104)$$

Large  $\Phi$  approximation

The results for the large  $\Phi$  approximation also are shown in Figure 7.26. Notice from Figure 7.9 that we are slightly overestimating the value of  $\eta$  using Equation 7.103, so we underestimate the required reactor volume. The reactor size and the percent change in reactor size are

$$V_R = 333 \text{ L}, \quad \Delta = -7.7\%$$

Given that we have a result valid for all  $\Phi$  that requires solving only a single differential equation, one might question the value of this closed-

form solution. One advantage is purely practical. We may not have a computer available. Instructors are usually thinking about in-class examination problems at this juncture. The other important advantage is insight. It is not readily apparent from the differential equation what would happen to the reactor size if we double the pellet size, or halve the rate constant, for example. Equation 7.104, on the other hand, provides the solution's dependence on all parameters. As shown in Figure 7.26 the approximation error is small. Remember to check that the Thiele modulus is large for the entire tube length, however, before using Equation 7.104.  $\square$

### Example 7.6: Hougen-Watson kinetics in a fixed-bed reactor

The following reaction converting CO to CO<sub>2</sub> takes place in a catalytic, fixed-bed reactor operating isothermally at 838 K and 1.0 atm



The following rate expression and parameters are adapted from a different model given by Oh et al. [16]. The rate expression is assumed to be of the Hougen-Watson form

$$r = \frac{k c_{\text{CO}} c_{\text{O}_2}}{1 + K c_{\text{CO}}} \quad \text{mol/s cm}^3 \text{ pellet}$$

The constants are provided below

$$k = 8.73 \times 10^{12} \exp(-13,500/T) \text{ cm}^3/\text{mol s}$$

$$K = 8.099 \times 10^6 \exp(409/T) \text{ cm}^3/\text{mol}$$

$$D_{\text{CO}} = 0.0487 \text{ cm}^2/\text{s}$$

in which  $T$  is in Kelvin. The catalyst pellet radius is 0.1 cm. The feed to the reactor consists of 2 mol% CO, 10 mol% O<sub>2</sub>, zero CO<sub>2</sub> and the remainder inerts. Find the reactor volume required to achieve 95% conversion of the CO.

### Solution

Given the reaction stoichiometry and the excess of O<sub>2</sub>, we can neglect the change in c<sub>O<sub>2</sub></sub> and approximate the reaction as pseudo-first order

in CO

$$r = \frac{k' c_{\text{CO}}}{1 + K c_{\text{CO}}} \quad \text{mol/s cm}^3 \text{ pellet}$$

$$k' = k c_{\text{O}_2 f}$$

which is of the form analyzed in Section 7.4.4. We can write the mass balance for the molar flow of CO,

$$\frac{dN_{\text{CO}}}{dV} = -(1 - \epsilon_B) \eta r(c_{\text{CO}})$$

in which  $c_{\text{CO}}$  is the fluid CO concentration. From the reaction stoichiometry, we can express the remaining molar flows in terms of  $N_{\text{CO}}$

$$N_{\text{O}_2} = N_{\text{O}_2 f} + 1/2(N_{\text{CO}} - N_{\text{CO}f})$$

$$N_{\text{CO}_2} = N_{\text{CO}f} - N_{\text{CO}}$$

$$N = N_{\text{O}_2 f} + 1/2(N_{\text{CO}} + N_{\text{CO}f})$$

The concentrations follow from the molar flows assuming an ideal-gas mixture

$$c_j = \frac{P}{RT} \frac{N_j}{N}$$

To decide how to approximate the effectiveness factor shown in Figure 7.14, we evaluate  $\phi = K c_{\text{CO}}$  at the entrance and exit of the fixed-bed reactor. With  $\phi$  evaluated, we compute the Thiele modulus given in Equation 7.58 and obtain

$$\begin{array}{lll} \phi = 32.0 & \Phi = 79.8, & \text{entrance} \\ \phi = 1.74 & \Phi = 326, & \text{exit} \end{array}$$

It is clear from these values and Figure 7.14 that  $\eta = 1/\Phi$  is an excellent approximation for this reactor. Substituting this equation for  $\eta$  into the mass balance and solving the differential equation produces the results shown in Figure 7.27. The concentration of  $\text{O}_2$  is nearly constant, which justifies the pseudo-first-order rate expression. Reactor volume

$$V_R = 233 \text{ L}$$

is required to achieve 95% conversion of the CO. Recall that the volumetric flowrate varies in this reactor so conversion is based on molar flow, not molar concentration. Figure 7.28 shows how  $\Phi$  and  $\phi$  vary with position in the reactor. □

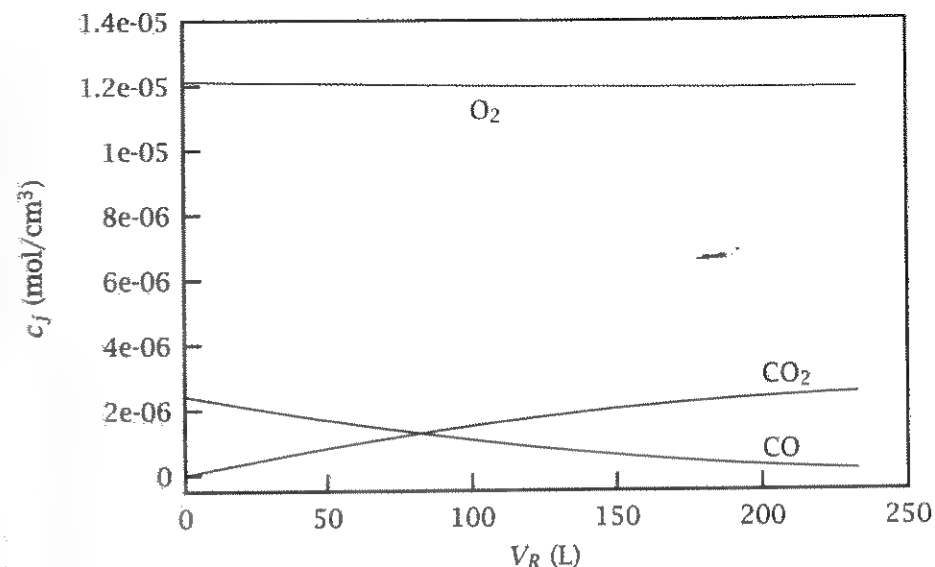


Figure 7.27: Molar concentrations versus reactor volume.

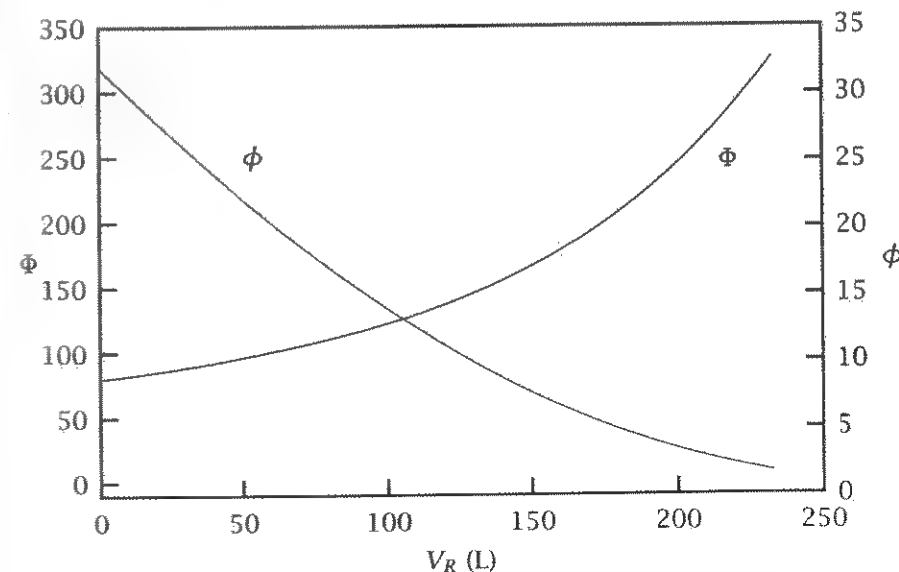


Figure 7.28: Dimensionless equilibrium constant and Thiele modulus versus reactor volume. Values indicate  $\eta = 1/\Phi$  is a good approximation for entire reactor.



Parameter	Value	Units
$P_f$	$2.02 \times 10^5$	N/m <sup>2</sup>
$T_f$	550	K
$R_t$	5.0	cm
$u_f$	75	cm/s
$T_a$	325	K
$U^o$	$5.5 \times 10^{-3}$	cal/(cm <sup>2</sup> Ks)
$\Delta H_{R1}$	$-67.63 \times 10^3$	cal/(mol CO K)
$\Delta H_{R2}$	$-460.4 \times 10^3$	cal/(mol C <sub>3</sub> H <sub>6</sub> K)
$\hat{C}_p$	0.25	cal/(g K)
$\mu_f$	$0.028 \times 10^{-2}$	g/(cm s)
$\rho_b$	0.51	g/cm <sup>3</sup>
$\rho_p$	0.68	g/cm <sup>3</sup>

**Table 7.6:** Feed flowrate and heat-transfer parameters for the fixed-bed catalytic converter.

In the previous examples, we have exploited the idea of an effectiveness factor to reduce fixed-bed reactor models to the same form as plug-flow reactor models. This approach is useful and solves several important cases, but this approach is also limited and can take us only so far. In the general case, we must contend with multiple reactions that are not first order, nonconstant thermochemical properties, and nonisothermal behavior in the pellet and the fluid. For these cases, we have no alternative but to solve numerically for the temperature and species concentrations profiles in both the pellet and the bed. As a final example, we compute the numerical solution to a problem of this type.

We use the collocation method to solve the next example, which involves five species, two reactions with Hougen-Watson kinetics, both diffusion and external mass-transfer limitations, and nonconstant fluid temperature, pressure and volumetric flowrate.

#### Example 7.7: Multiple-reaction, nonisothermal fixed-bed reactor

Evaluate the performance of the catalytic converter in converting CO and propylene. Determine the amount of catalyst required to convert 99.6% of the CO and propylene. The reaction chemistry and pellet mass-transfer parameters are given in Table 7.5. The feed conditions and heat-transfer parameters are given in Table 7.6.

#### Solution

The fluid balances govern the change in the fluid concentrations, temperature and pressure. The pellet concentration profiles are solved with the collocation approach. The pellet and fluid concentrations are coupled through the mass-transfer boundary condition. The fluid concentrations are shown in Figure 7.29. A bed volume of 1098 cm<sup>3</sup> is required to convert the CO and C<sub>3</sub>H<sub>6</sub>. Figure 7.29 also shows that oxygen is in slight excess.

The reactor temperature and pressure are shown in Figure 7.30. The feed enters at 550 K, and the reactor experiences about a 130 K temperature rise while the reaction essentially completes; the heat losses then reduce the temperature to less than 500 K by the exit. The pressure drops from the feed value of 2.0 atm to 1.55 atm at the exit. Notice the catalytic converter exit pressure of 1.55 atm must be large enough to account for the remaining pressure drops in the tail pipe and muffler.

In Figures 7.31 and 7.32, the pellet CO concentration profile at several reactor positions is displayed. The feed profile, marked by ① in Figure 7.32, is similar to the one shown in Figure 7.23 of Example 7.2 (the differences are caused by the different feed pressures). We see that as the reactor heats up, the reaction rates become large and the CO is rapidly converted inside the pellet. By 490 cm<sup>3</sup> in the reactor, the pellet exterior CO concentration has dropped by two orders of magnitude, and the profile inside the pellet has become very steep. As the reactions go to completion and the heat losses cool the reactor, the reaction rates drop. At 890 cm<sup>3</sup>, the CO begins to diffuse back into the pellet. Finally, the profiles become much flatter near the exit of the reactor.

It can be numerically challenging to calculate rapid changes and steep profiles inside the pellet. The good news, however, is that accurate pellet profiles are generally *not* required for an accurate calculation of the overall pellet reaction rate. The reason is that when steep profiles are present, essentially all of the reaction occurs in a thin shell near the pellet exterior. We can calculate accurately down to concentrations on the order of 10<sup>-15</sup> as shown in Figure 7.32, and by that point, essentially zero reaction is occurring, and we can calculate an accurate overall pellet reaction rate. It is always a good idea to vary the numerical approximation in the pellet profile, by changing the number of collocation points, to ensure convergence in the fluid profiles. □

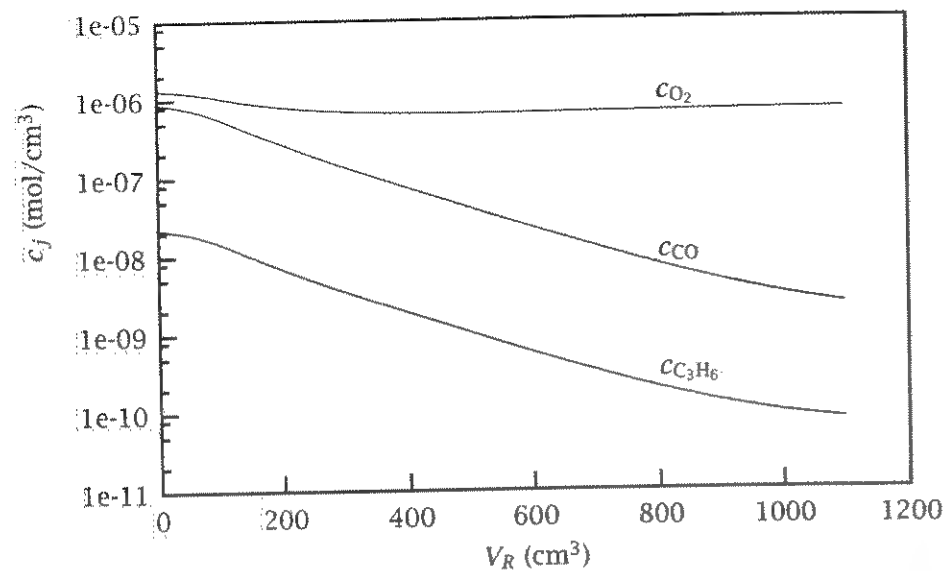


Figure 7.29: Fluid molar concentrations versus reactor volume.

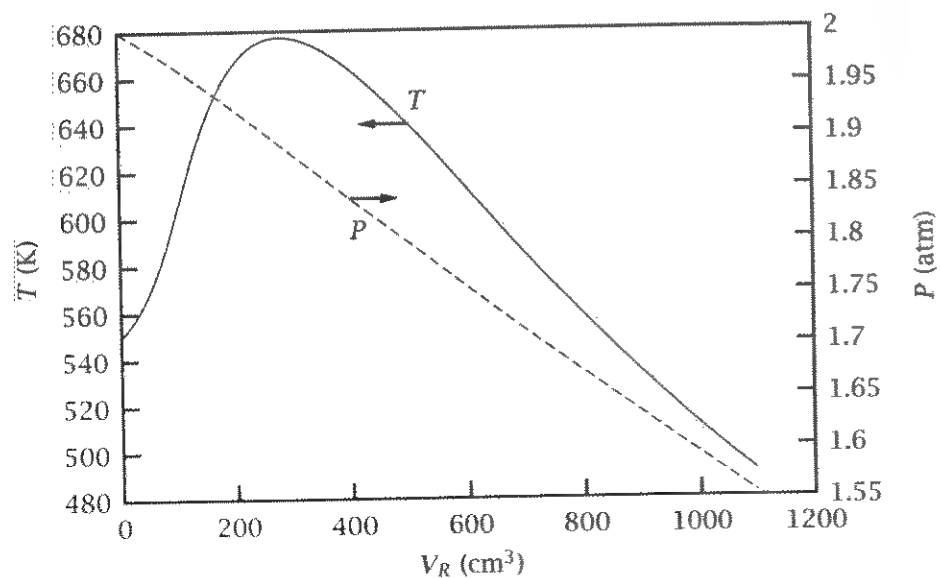


Figure 7.30: Fluid temperature and pressure versus reactor volume.

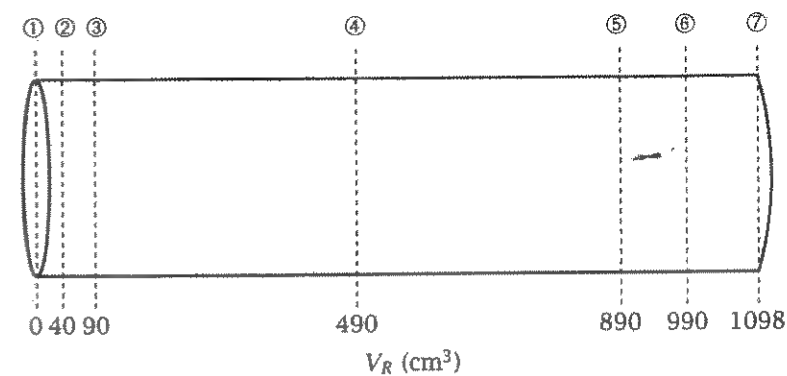


Figure 7.31: Reactor positions for pellet profiles.

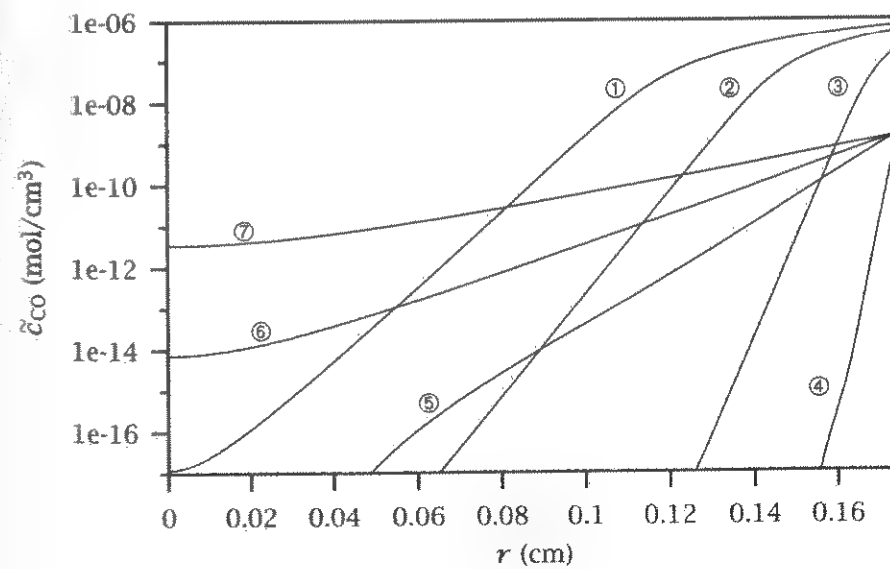


Figure 7.32: Pellet CO profiles at several reactor positions.

## 7.8 Summary

This chapter treated the fixed-bed reactor, a tubular reactor packed with catalyst pellets. We started with a general overview of the transport and reaction events that take place in the fixed-bed reactor: transport by convection in the fluid; diffusion inside the catalyst pores; and adsorption, reaction and desorption on the catalyst surface. We summarized the transport properties of the catalyst particles, and described bulk and Knudsen diffusion phenomena.

In order to simplify the model, we assumed an effective diffusivity could be used to describe diffusion in the catalyst particles. We next presented the general mass and energy balances for the catalyst particle. Next we solved a series of reaction-diffusion problems in a single catalyst particle. These included:

- Single reaction in an isothermal pellet. This case was further divided into a number of special cases.
  - \* First-order, irreversible reaction in a spherical particle.
  - \* Reaction in a semi-infinite slab and cylindrical particle.
  - \*  $n$ th order, irreversible reaction.
  - \* Hougen-Watson rate expressions.
  - \* Particle with significant external mass-transfer resistance.
- Single reaction in a nonisothermal pellet.
- Multiple reactions.

For the single-reaction cases, we performed dimensional analysis and found a dimensionless number, the Thiele modulus, which measures the rate of production divided by the rate of diffusion of some component. A complete analysis of the first-order reaction in a sphere suggested a general approach to calculate the production rate in a pellet in terms of the rate evaluated at the pellet exterior surface conditions. This motivated the definition of the pellet effectiveness factor, which is a function of the Thiele modulus.

For the single-reaction, nonisothermal problem, we solved the so-called Weisz-Hicks problem, and determined the temperature and concentration profiles within the pellet. We showed the effectiveness factor can be greater than unity for this case. Multiple steady-state solutions also are possible for this problem, but for realistic values of the

catalyst thermal conductivity, the pellet often can be considered isothermal and the energy balance can be neglected. Multiple steady-state solutions in the particle may occur in practice, however, if there is a large external heat-transfer resistance.

For complex reactions involving many species, we cannot use the simple Thiele modulus and effectiveness factor approach, and must solve numerically the complete reaction-diffusion problem. These problems are challenging because of the steep pellet profiles that are possible.

Finally, we showed several ways to couple the mass and energy balances over the fluid flowing through a fixed-bed reactor to the balances within the pellet. For simple reaction mechanisms, we were still able to use the effectiveness factor approach to solve the fixed-bed reactor problem. For complex mechanisms, we solved numerically the full problem given in Equations 7.84–7.97. We solved the reaction-diffusion problem in the pellet coupled to the mass and energy balances for the fluid, and we used the Ergun equation to calculate the pressure in the fluid.

## Notation

$a$	characteristic pellet length, $V_p/S_p$
$A_c$	reactor cross-sectional area
$B$	Biot number for external mass transfer
$c$	constant for the BET isotherm
$c_j$	concentration of species $j$
$c_{js}$	concentration of species $j$ at the catalyst surface
$\bar{c}$	dimensionless pellet concentration
$\bar{c}_m$	total number of active surface sites
$D_{AB}$	binary diffusion coefficient
$D_j$	effective diffusion coefficient for species $j$
$D_{jK}$	Knudsen diffusion coefficient for species $j$
$D_{jm}$	diffusion coefficient for species $j$ in the mixture
$D_p$	pellet diameter
$E_{diff}$	activation energy for diffusion
$E_{obs}$	experimental activation energy
$E_{rxn}$	intrinsic activation energy for the reaction
$\Delta H_{Ri}$	heat of reaction $i$
$I_j$	rate of transport of species $j$ into a pellet
$I_0$	modified Bessel function of the first kind, zero order

$I_1$	modified Bessel function of the first kind, first order
$k_e$	effective thermal conductivity of the pellet
$k_{jm}$	mass-transfer coefficient for species $j$
$k_n$	$n$ th-order reaction rate constant
$L$	pore length
$M_j$	molecular weight of species $j$
$n_r$	number of reactions in the reaction network
$N$	total molar flow, $\sum_j N_j$
$N_j$	molar flow of species $j$
$P$	pressure
$Q$	volumetric flowrate
$r$	radial coordinate in catalyst particle
$r_a$	average pore radius
$r_i$	rate of reaction $i$ per unit reactor volume
$r_{\text{obs}}$	observed (or experimental) rate of reaction in the pellet
$r_{ip}$	total rate of reaction $i$ per unit catalyst volume
$\bar{r}$	dimensionless radial coordinate
$R$	spherical pellet radius
$R$	gas constant
$R_j$	production rate of species $j$
$R_{jf}$	production rate of species $j$ at bulk fluid conditions
$R_{jp}$	total production rate of species $j$ per unit catalyst volume
$R_{js}$	production rate of species $j$ at the pellet surface conditions
$S_g$	BET area per gram of catalyst
$S_p$	external surface area of the catalyst pellet
$T$	temperature
$T_f$	bulk fluid temperature
$T_s$	pellet surface temperature
$U^o$	overall heat-transfer coefficient
$v$	volume of gas adsorbed in the BET isotherm
$v_m$	volume of gas corresponding to an adsorbed monolayer
$V$	reactor volume coordinate
$V_g$	pellet void volume per gram of catalyst
$V_p$	volume of the catalyst pellet
$V_R$	reactor volume
$W_c$	total mass of catalyst in the reactor
$y_j$	mole fraction of species $j$
$z$	position coordinate in a slab
$\epsilon$	porosity of the catalyst pellet

$\epsilon_B$	fixed-bed porosity or void fraction
$\eta$	effectiveness factor
$\lambda$	mean free path
$\mu_f$	bulk fluid density
$\nu_{ij}$	stoichiometric number for the $j$ th species in the $i$ th reaction
$\xi$	integral of a diffusing species over a bounding surface
$\rho$	bulk fluid density
$\rho_B$	reactor bed density
$\rho_p$	overall catalyst pellet density
$\rho_s$	catalyst solid-phase density
$\sigma$	hard sphere collision radius
$\tau$	tortuosity factor
$\Phi$	Thiele modulus
$\Omega_{D,AB}$	dimensionless function of temperature and the intermolecular potential field for one molecule of A and one molecule of B

## 7.9 Exercises

### Exercise 7.1: Isothermal slab with first-order kinetics

A first-order, irreversible reaction



takes place in a catalyst pellet with the slab geometry depicted in Figure 7.33. Assume that the pellet height,  $h$ , and length,  $l$ , are much greater than the width,  $w$ . The catalyst exterior is assumed to be at constant species A concentration,  $c_{As}$ . Species A has an effective diffusivity,  $D_A$ , in the pellet and the catalyst is isothermal.

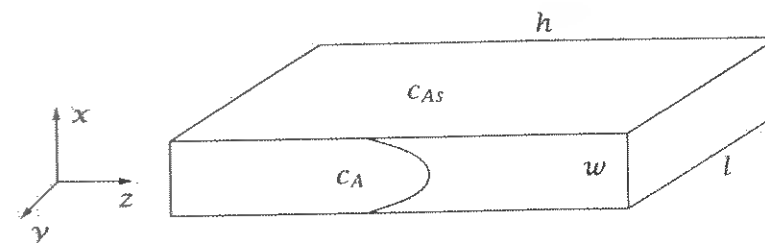


Figure 7.33: Catalyst pellet with slab geometry.

- (a) The steady-state material balance for component A is

$$D_A \nabla^2 c_A + R_A = 0 \quad (7.106)$$

In rectangular coordinates, this becomes

$$D_A \left( \frac{\partial^2 c_A}{\partial x^2} + \frac{\partial^2 c_A}{\partial y^2} + \frac{\partial^2 c_A}{\partial z^2} \right) + R_A = 0$$

What happens to the terms  $\partial^2 c_A / \partial y^2$  and  $\partial^2 c_A / \partial z^2$  if  $h$  and  $l$  are much greater than  $w$ ? Why? Make this simplification and substitute in the production rate for A for the final dimensional model. What are the boundary conditions for the model?

- (b) Now consider writing the model in dimensionless form. What is the appropriate characteristic length,  $a$ ? What is the characteristic concentration? Nondimensionalize the model and boundary conditions, and show the final dimensionless model is

$$\frac{d^2 \bar{c}_A}{d\bar{x}^2} - \Phi^2 \bar{c}_A = 0$$

and the boundary conditions are

$$\begin{aligned} \bar{c}_A = 1 & \quad \bar{x} = 1 \\ \bar{c}_A = 1 & \quad \bar{x} = -1 \end{aligned}$$

What is the definition of the Thiele modulus for this problem?

- (c) What is the solution of this model?
- (d) Perform the integration to compute the effectiveness factor and show

$$\eta = \frac{\tanh \Phi}{\Phi}$$

- (e) Plot  $\bar{c}_A(\bar{x})$ ,  $0 \leq \bar{x} \leq 1$ , for  $\Phi = 1/2, 1, 2, 5, 10$ . Plot  $\bar{c}_A(\bar{r})$  for the sphere for the same  $\Phi$  values. Comment on the differences and similarities.
- (f) Plot  $\eta(\Phi)$ ,  $0.1 \leq \Phi \leq 10$  on a log-log scale for the sphere and slab in the same figure. Comment on the differences and similarities. What is the largest error that you would commit by using the sphere in place of the slab geometry and for what  $\Phi$  value does it occur?

### Exercise 7.2: Pellet profile of product and reactant

Consider the same reaction and slab geometry of Exercise 7.1,



Now let's compute the product concentration profile in the pellet,  $c_B$ . The catalyst exterior is assumed to be at constant species B concentration,  $c_{Bs}$ , and species B has an effective diffusivity,  $D_B$ , which may be different from  $D_A$ .

- (a) The steady-state material balance for component B is

$$D_B \nabla^2 c_B + R_B = 0$$

For the given stoichiometry,  $R_B = -2R_A$ . Using Equation 7.106, show that

$$2(D_A \nabla^2 c_A) + (D_B \nabla^2 c_B) = 0$$

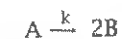
- (b) Solve this equation by performing two integrals and show

$$c_B = c_{Bs} + 2 \frac{D_A}{D_B} c_{As} (1 - \bar{c}_A)$$

- (c) Let's compare the A and B concentrations. Assume the diffusivities are equal and that  $c_{As} = 1$  and  $c_{Bs} = 0$ . Plot  $c_A(x)$  and  $c_B(x)$  for  $\Phi = 10$ . Comment on the differences between  $c_A$  and  $c_B$ .
- (d) Now assume  $D_B = 10D_A$  and replot  $c_A$  and  $c_B$ . Explain what happens to the product B concentration with this larger value of  $D_B$ .

### Exercise 7.3: Slab with finite external mass transfer

Consider the same reaction and slab geometry of Exercise 7.1

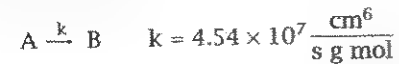


but include the mass-transfer resistance.

- (a) Write the dimensionless material balance and define the Thiele modulus and Biot number for this situation.
- (b) Solve the model and find  $\bar{c}_A(\bar{x})$ . Plot this result for the same values of Biot number as in Figure 7.16 and compare the results.
- (c) Compute the effectiveness factor versus Thiele modulus, plot this result and compare to Figure 7.17. Comment on the effect of pellet geometry on the overall reaction rate.

**Exercise 7.4: Thiele modulus and a second-order reaction**

The following second-order catalytic reaction is currently carried out in a fixed-bed reactor operating isothermally at 650 K.



Pure A is fed to the reactor and a conversion of 93% is achieved at a pressure of 2.0 atm. The catalyst vendor has changed fabrication procedures and has verified that the intrinsic rate constant listed above is unchanged. Using the following data, determine if the mass of catalyst needs to be changed for the new catalyst to maintain the same production rate of B.

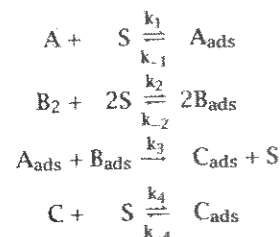
Property	Old catalyst	New catalyst
effective diffusivity	0.0095 cm <sup>2</sup> /s	0.0072 cm <sup>2</sup> /s
particle density	1.75 g/cm <sup>3</sup>	1.79 g/cm <sup>3</sup>
bed density	0.84 g/cm <sup>3</sup>	0.96 g/cm <sup>3</sup>
shape	sphere (0.635 cm diam)	cylinder (0.48 cm diam, 0.79 cm length)

**Exercise 7.5: Thiele modulus and Hougen-Watson kinetics**

The following catalytic reaction is conducted in a 0.25-cm radius spherical pellet.



The mechanism for this reaction is shown below. A, B<sub>2</sub> and C are in adsorption-desorption equilibrium with the surface. The bimolecular surface reaction is irreversible and rate limiting.



Additional data needed for this problem are provided in the following table.

Parameter	Value	Units
$K_1$	90,100	cm <sup>3</sup> /mol
$K_2$	6,500	cm <sup>3</sup> /mol
$K_4$	64,400	cm <sup>3</sup> /mol
$T$	523	K
$D_{eA}$	0.045	cm <sup>2</sup> /s
$c_A$	$5.83 \times 10^{-5}$	mol/cm <sup>3</sup>
$c_{B_2}$	$1.40 \times 10^{-4}$	mol/cm <sup>3</sup>
$c_C$	$1.17 \times 10^{-5}$	mol/cm <sup>3</sup>
$k_3$	$7.41 \times 10^8$	g cat <sup>2</sup> /mol·cm <sup>3</sup> ·s
$c_m$	$1.8 \times 10^{-5}$	mol/g cat

(a) Determine if the concentration of A changes appreciably within the pellet from the value it has at the pellet surface.

(b) Estimate the concentration of A at the center of the pellet.

**Exercise 7.6: Thiele modulus and first-order kinetics**

A catalytic reaction that is first order in the concentration of A



is carried out in a spherical pellet with a 0.20-cm radius. The effective diffusivity of A in the pellet is 0.015 cm<sup>2</sup>/s. The rate of reaction at 398 K is  $2.63 \times 10^{-5}$  mol/cm<sup>3</sup>·s when the concentration of A is  $3.25 \times 10^{-5}$  mol/cm<sup>3</sup>. The intrinsic activation energy is 20 kcal/mol.

(a) Assuming  $D_A$  is independent of temperature, what is the rate at 448 K for the same concentration of A?

**Exercise 7.7: Thiele modulus and rate data**

The second-order catalytic reaction



$$r = kc_A^2 \quad \text{mol/s} \cdot \text{g}$$

is carried out over a spherical catalyst pellet. The reactant and product are gases at the reaction conditions. A series of experiments was performed using 0.70-cm diameter catalyst beads. At 150°C and 1.0–3.0 atm of pure A, a plot of log rate versus log  $c_A$  was a straight line of slope 3/2. The following table lists additional information.

$T$ (°C)	$R_A$ (mol/s·g)	$P_A$ (atm)
150	$-6.17 \times 10^{-4}$	1.0
200	$-2.37 \times 10^{-3}$	1.0

Estimate the value of  $W_c/N_{Af}$  required to achieve 50% conversion of A at 200°C in a fixed-bed reactor using the same catalyst but with a bead diameter of 1.0 cm. Pure A is fed to the reactor at a total pressure of 1.0 atm. Assume the diffusion process is dominated by Knudsen diffusion, and the diffusion coefficient is constant.

### Exercise 7.8: First-order reaction in a fixed-bed reactor

The following first-order catalytic reaction is conducted in a fixed-bed reactor. The reactant and product are gases.



The intrinsic rate constant is  $5.74 \times 10^{13} \exp(-38,000/RT) \text{ s}^{-1}$ , in which the units of  $R$  are cal/mol K, and the units of  $T$  are K. The feed consists of pure A at 1.0 atm, 630 K, at molar flowrate 0.5 mol/s. The catalyst is a cylindrical-shaped pellet with a radius of 0.35 cm and a length of 0.5 cm. The catalyst pellet density  $\rho_p = 0.84 \text{ g/cm}^3$ . The reactor bed density  $\rho_B = 0.52 \text{ g/cm}^3$ . At 630 K the effective diffusivity in the pellet is  $1.40 \times 10^{-3} \text{ cm}^2/\text{s}$ . Determine the mass of catalyst required to achieve 90% conversion of A if the reactor operates isothermally.

### Exercise 7.9: A hollow-cylinder catalyst pellet in a fixed-bed reactor

Estimate the mass of catalyst required in an isothermal fixed-bed reactor for the second-order reaction.

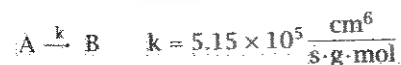


$$r = k_1 c_A^2 \quad k_1 = 1.21 \times 10^5 \text{ cm}^3/\text{mol} \cdot \text{s}$$

The feed is pure A at a molar flowrate of 15 mol/s, the total pressure is 3.0 atm and the temperature is 550 K. The reactor achieves 90% conversion of A. The catalyst is a hollow cylinder with an outer diameter of 1.3 cm, an inner diameter of 0.5 cm, and a length of 0.7 cm. The pellet density  $\rho_p = 0.73 \text{ g/cm}^3$  and the bed density  $\rho_B = 0.58 \text{ g/cm}^3$ . The effective diffusivity of A is  $0.008 \text{ cm}^2/\text{s}$  and may be assumed constant. You may assume the bulk and pellet surface concentrations are equal.

### Exercise 7.10: Catalyst size and production rate

The following second-order catalytic reaction is carried out in a fixed-bed reactor operating isothermally at 650 K.



Pure A is fed to the reactor with a nominal inlet pressure of 2.0 atm. The desired production rate per tube is 18.84 kg/hr of B. If you choose to operate above the nominal inlet pressure, an additional compressor has to be installed upstream of the reactor. The existing tubes are 100-cm long and 20-cm<sup>2</sup> cross-sectional area. The remaining A in the product stream must be separated in a second unit for recycle. For the separation unit to run efficiently, the conversion in the reactor must be at least 93% and the outlet pressure must be at least 1.5 atm. If the reactor effluent pressure is below 1.5 atm, an additional compressor must be installed downstream of the reactor.

The current catalyst has reached the end of its useful lifetime and needs to be replaced. Your project is to make a design change with the new catalyst to improve the reactor efficiency. In particular you are considering changing the catalyst size. The catalyst vendor has told you that they easily can make spherical pellets with diameters of 0.075 cm, 0.15 cm, and 0.30 cm. You also have been assured that the effective diffusivity, bed density, and reaction rate constant do not vary among the catalysts in this size range. One of your team members wants to use the smallest diameter catalyst to minimize the total mass of catalyst required. Another team member wants to use the largest diameter catalyst to minimize the pressure drop.

You have been assigned to model the reactor and decide which is the best choice. Some of the design constraints that you must satisfy include the production rate, the minimal final conversion, the maximal tube length (it is fine to use reactor length less than 100 cm, one simply leaves the end of the tube empty of catalyst), the maximal inlet pressure, and the minimal outlet pressure.

Can you meet these design constraints with all three of the catalysts? Once you have met the constraints, you can optimize the reactor operation over the remaining design decision variables. What is your final choice of catalyst size, and at what nominal inlet pressure will you run the reactor? Include a plot of the pressure, conversion, and Thiele modulus versus reactor length for your final design.

Please write a brief report discussing how the design variables (catalyst size, inlet pressure, bed length) affect both the capital and operating costs of this process. What additional information would you require to perform a better design?

Additional data for the catalyst and reaction are listed in the following table.

Parameter	Value	Units
Effective diffusivity	0.00375	cm <sup>2</sup> /s
Particle density	1.75	g/cm <sup>3</sup>
Bed density	1.12	g/cm <sup>3</sup>
Molecular weight of A	100	kg/kmol
Viscosity of gas	10 <sup>-4</sup>	g/cm·s



**Exercise 7.11: Mass transfer with reaction**

- (a) Draw a sketch of the effectiveness factor versus the Thiele modulus for a spherical pellet. Be sure to label clearly your axes and put as many numbers on the plot as you can.
- (b) Choose a value of  $\Phi = 1$  and draw a sketch of  $c_A(r)$  and  $c_B(r)$  versus  $r$  for an irreversible, first-order reaction,  $A \rightarrow B$ , in an isothermal pellet. Make this sketch as accurate as possible. If you are using dimensionless variables, be sure to indicate what quantities you have used to make the variables dimensionless.
- (c) Comment on the differences and similarities between the effectiveness factor versus Thiele modulus plot for slabs and cylinders compared to the spherical case. How do you nondimensionalize your length for these other two geometries.
- (d) Provide a concise explanation of the reactor design and operation trade-off involved in choosing between a catalyst pellet that gives a diffusion-controlled reactor versus one that gives a kinetically controlled reactor.

**Exercise 7.12: Mass transfer with reaction for reversible kinetics**

Consider the *reversible* first-order elementary reaction in an isothermal solid catalyst slab of thickness  $w$  surrounded by a fluid



The external mass-transfer rate to the outer surface of the catalyst is very large and the fluid has A and B concentrations  $c_{Af}$  and  $c_{Bf}$ , respectively.

- (a) Write down the steady-state material balance for species A and B considering the reaction and diffusion in the slab. Species A has effective diffusivity  $D_A$  and species B has effective diffusivity  $D_B$ , which may not be the same. What are the boundary conditions for these two material balances. Write down the reaction rate expression.
- (b) Add the two material balances together and find an expression for  $c_B$  in terms of  $c_A$ ,  $c_{Af}$ ,  $c_{Bf}$ , and  $D = D_A/D_B$ .
- (c) At equilibrium, the rate of reaction is zero. Find an expression for the equilibrium A concentration,  $c_{Ae}$ , in terms of  $D$ ,  $c_{Af}$ ,  $c_{Bf}$ , and  $K = k_1/k_{-1}$ .
- (d) Define dimensionless concentration and distance via

$$\bar{c} = \frac{c_A - c_{Ae}}{c_{Af} - c_{Ae}}, \quad \bar{x} = \frac{x}{w/2}$$

and show the material balance for A in dimensionless variables is

$$\frac{d^2 \bar{c}}{d\bar{x}^2} - \Phi^2 \bar{c} = 0$$

What is the Thiele modulus for this problem? How does it differ from the irreversible case? What are the dimensionless boundary conditions?

- (e) Solve your model and show

$$\bar{c} = \left(1 - c_{Ae}/c_{Af}\right) \frac{\cosh \Phi \bar{x}}{\cosh \Phi}$$

**Exercise 7.13: Reaction-limited and diffusion-limited fixed-bed reactors**

A second-order catalytic reaction



is carried out in an isothermal fixed-bed reactor containing spherical catalyst pellets. The exit conversion is 85%. The reactant and product are gases. What happens to the required bed length to achieve 85% conversion if the feed pressure is doubled and the feed volumetric flowrate is held constant? You may assume that the flowrate is small enough that the pressure drop across the bed is unimportant. Consider the following two cases.

- (a) What is the new required bed length if the overall pellet reaction rate is controlled by the intrinsic reaction rate?
- (b) What is the new required bed length if the overall pellet reaction rate is controlled by the pellet diffusion rate?

You may assume the entire bed length is either in the intrinsic reaction rate or pellet diffusion rate-limited regimes and that the external mass-transfer resistance between the fluid and the solid catalyst is negligible.

**Exercise 7.14: Second-order reaction in a fixed-bed reactor**

The following second-order gas-phase catalytic reaction is conducted in an isothermal fixed-bed reactor:



The intrinsic rate constant is 15.0 L/mol·s at 800 K. The feed is 15,000 L/hr of pure A at 10.0 atm and 800 K. Spherical catalyst pellets of radius 1.2 cm are used to pack the reactor. The catalyst has a pellet density of 0.90 g/cm<sup>3</sup>, and the reactor bed density is 0.52 g/cm<sup>3</sup>. The effective diffusivity of A inside the catalyst pellet is  $2.06 \times 10^{-4}$  cm<sup>2</sup>/s.

Calculate the mass of catalyst necessary to achieve 85% conversion of A.

### Exercise 7.15: Using experimental data from a fixed-bed reactor

The irreversible catalytic reaction



is first order in the concentration of A. Use the data in the following table to determine the mass of catalyst required for 95% conversion of A in a fixed-bed reactor that can be modeled as a PFR. The experimental rate was measured in a 0.25-cm diameter spherical catalyst pellet at 1.0 atm of A and 510°C. The fixed-bed reactor you are asked to size is isothermal (510°C), constant pressure (1.0 atm), and the feed enters at  $N_{Af} = 1.5$  mol/s.

Name	Symbol	Value	Units
experimental rate	—	$1.04 \times 10^{-4}$	mol/(s·cm <sup>3</sup> )
catalyst radius	$R$	0.125	cm
effective diffusivity of A	$D_A$	0.007	cm <sup>2</sup> /s
catalyst pellet density	$\rho_p$	0.85	g/cm <sup>3</sup>
catalyst bed density	$\rho_B$	0.55	g/cm <sup>3</sup>

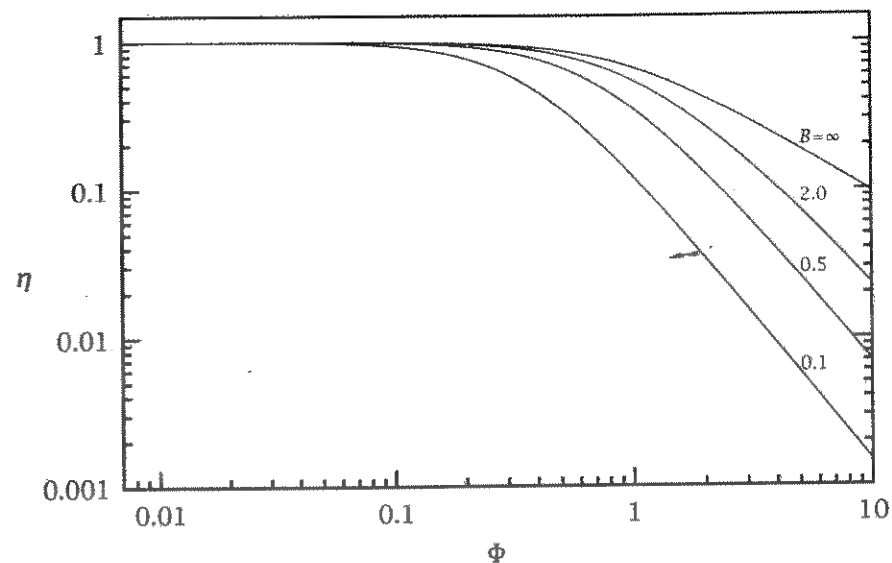


Figure 7.34: Effectiveness factor versus Thiele modulus for different values of the Biot number; second-order reaction in a cylindrical pellet.

### Exercise 7.16: External mass-transfer effects on a second-order reaction

Consider the irreversible, second-order reaction



taking place in a semi-infinite cylindrical pellet. Compute the effectiveness factor versus Thiele modulus for the following values of the Biot number,  $B = \infty, 2, 0.5, 0.1$ . Compare your calculation to Figure 7.34.

### Exercise 7.17: Hougen-Watson kinetics

Consider the Hougen-Watson kinetics of Section 7.4.4. Using Equation 7.58, verify the following limits

$$\Phi = \phi, \quad \phi \text{ small} \quad (7.109)$$

$$\Phi = \frac{\phi}{\sqrt{2}}, \quad \phi \text{ large} \quad (7.110)$$

Verify the result in Equation 7.110 using Figures 7.13 and 7.14, and  $\phi = 1000$ .

### Exercise 7.18: General kinetics

Consider the large  $\Phi$  reaction-diffusion equation, Equation 7.54, with general kinetic expression

$$\frac{d^2 \bar{c}}{dz^2} - f(\bar{c}) = 0 \quad (7.111)$$

$$\begin{aligned} \bar{c} &= 1 & z &= 0 \\ \frac{d\bar{c}}{dz} &= 0 & z &= \infty \end{aligned}$$

in which  $f(\bar{c})$  is the dimensionless reaction rate. Notice the Hougen-Watson and  $n$ th-order reaction kinetics are special cases of this more general form.

- (a) Apply the change of variables of Section 7.4.4 and show the solution is [4, p. 338][18]

$$z(\bar{c}) = \int_{\bar{c}}^1 \frac{d\bar{c}'}{\sqrt{2 \int_0^{\bar{c}'} f(\bar{c}'') d\bar{c}''}} \quad (7.112)$$

- (b) Verify Equation 7.111 is satisfied by  $z(\bar{c})$  by taking two derivatives with respect to  $\bar{c}$  of Equation 7.112.

Hint: recall the Leibniz rule for differentiating an integral in which the limits of integration are variable.

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx = -\frac{da}{dt} f(a, t) + \frac{db}{dt} f(b, t) + \int_a^b \frac{\partial f}{\partial t} dx$$

**Exercise 7.19: Multiple-reaction, nonisothermal fixed-bed reactor**

Consider again the catalytic converter problem of Exercises 7.2 and 7.7. Farrauto and Bartholomew provide a table of exhaust gas velocities for various driving conditions [11]. The gas velocities range from 0.54–25.2 m/s as the engine operation varies from idling to full load. We used 0.75 m/s in Exercise 7.7, which corresponds to a low engine load.

Calculate the fluid profiles in the bed for  $u_f = 5$  m/s feed gas velocity. What is the conversion of CO and C<sub>3</sub>H<sub>6</sub> if the bed volume found in Example 7.7 is used? How much more catalyst is required to reach 99.6% conversion of CO and C<sub>3</sub>H<sub>6</sub> under the new engine operation?

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## 8

# Mixing in Chemical Reactors

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## 8.1 Introduction

The three main reactor types developed thus far — batch, continuous-stirred-tank, and plug-flow reactors — are useful for modeling many complex chemical reactors, and to this point we have neglected a careful treatment of the fluid flow pattern within the reactor. In this chapter we explore some of the limits of this approach and develop methods to address and overcome some of the more obvious limitations.

**Scope of problem.** The general topic of mixing, even in the restricted context of chemical reactors, is an impossibly wide one to treat comprehensively. Obviously the phases of matter that are mixed play a large role and a comprehensive treatment would consider: liquid/liquid, liquid/solid, liquid/gas, and gas/solid mixing. Even in the contacting of two liquid phases, the miscibility of the two phases plays a large role in determining whether or not mixing occurs. We could also consider the mixing of different solid particles, in which particle size plays a large role. In this chapter, we will restrict ourselves to fluid-phase systems.

One natural approach to describing mixing is to solve the equations of motion of the fluid. In fluid systems, the type of fluid flow is obviously important, and we should consider both laminar and turbulent flow, and various mechanisms of diffusion (molecular diffusion, eddy diffusion). Using fluid mechanics to describe all cases of interest is a difficult problem, both from the modeling and computational perspectives. Rapid developments in computational fluid dynamics (CFD), however, make this approach increasingly attractive [1].

A second, classical approach to describing mixing is to use simple tests to experimentally probe the system of interest. These empirical testing approaches do not use any of the structure of the equations of motion, but they can provide some rough features of the mixing taking

place in the system under study. In this chapter we first develop this classical approach, and find out what kinds of understanding it can provide. We also identify some of the limitations of this approach. Nauman and Buffham provide a more in-depth treatment of many of the classical topics covered in this chapter, and provide many further citations to the research literature [18].

**Convection and diffusion.** We should also mention one point to bear in mind when thinking about how convection and diffusion affect mixing. One might intuitively expect that to enhance mixing and reduce spatial variation in concentration, one should seek conditions that maximize the rate of diffusion. Although this notion is correct for mixing on the finest length scales, it is generally much more important in macroscopic scale processes to decrease variations on the larger length scales. Mixing in this regime is enhanced primarily by improving the convection, and diffusion plays only a small role. In simple terms, one does not expect to appreciably decrease the time required to mix the cream in one's coffee by increasing the coffee temperature (enhanced diffusion); one instead turns a spoon a few times (enhanced convection). On the finest length scales, mixing is accomplished readily for small molecules by the random process of molecular diffusion; in fact, the random molecular motions are the only effective mixing processes taking place on the finest length scales.

## 8.2 Residence-Time Distribution

### 8.2.1 Definition

Consider an arbitrary reactor with single feed and effluent streams depicted in Figure 8.1. Without solving for the entire flow field, which might be quite complex, we would like to characterize the flow pattern established in the reactor at steady state. The residence-time distribution (RTD) of the reactor is one such characterization or measure of the flow pattern. Imagine we could slip some inert tracer molecules into the feed stream and could query these molecules on their exit from the reactor as to how much time they had spent in the reactor. We assume that we can add a small enough amount of tracer in the feed so that we do not disturb the established flow pattern. Some of the tracer molecules might happen to move in a very direct path to the exit; some molecules might spend a long time in a poorly mixed zone before finally finding their way to the exit. Due to their random motions as well

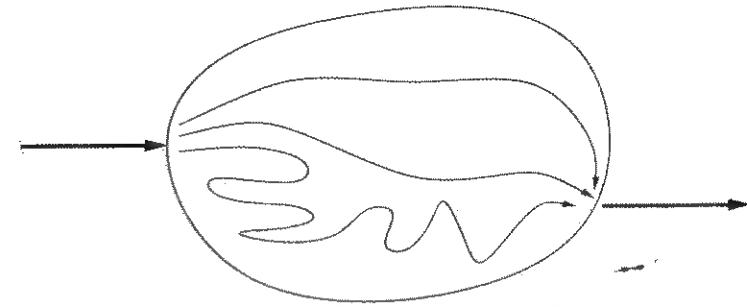


Figure 8.1: Arbitrary reactor with steady flow profile.

as convection with the established flow, which itself might be turbulent, we would start recording a distribution of residence times and we would create the residence-time probability density or residence-time distribution. If the reactor is at steady state, and after we had collected sufficient residence-time statistics, we expect the residence-time distribution to also settle down to a steady function. Let  $p(\theta)$  represent the probability density or residence-time distribution, and  $P(\theta)$  the integrated form so

$$p(\theta)d\theta, \quad \text{probability that a feed molecule spends time } \theta \text{ to } \theta + d\theta \text{ in the reactor} \quad (8.1)$$

$$P(\theta), \quad \text{probability that a feed molecule spends time zero to } \theta \text{ in the reactor} \quad (8.2)$$

The two versions of the probability function obviously contain the same information and are related by

$$P(\theta) = \int_0^\theta p(\theta')d\theta', \quad p(\theta) = \frac{dP(\theta)}{d\theta}$$

Some prefer to use exclusively  $P(\theta)$  because if  $P(\theta)$  is not differentiable, then even defining  $p(\theta)$  is somewhat problematic. We sidestep this issue by introducing the impulse or delta function to describe  $p(\theta)$  in the case where  $P(\theta)$  has jump discontinuities.

### 8.2.2 Measuring the RTD

As a thought experiment to define the RTD, querying tracer molecules on their exit from the reactor is a fine concept. But we plan to actually

measure the RTD, so we require an implementable experiment with actual measurements. We cannot measure the time spent by a particular tracer molecule in the reactor; to us, all tracer molecules are identical. We can measure concentration of tracer molecules in the effluent, however, and that will prove sufficient to measure the RTD. Imagine an experiment in which we measure the concentration of tracer in the feed and effluent streams over some time period, while the reactor maintains a steady flow condition. From the definition of the RTD in the previous section, the effluent tracer concentration at some time  $t$  is established by the combined exit of many tracer molecules with many different residence times. The concentration of molecules that enter the reactor at time  $t'$  and spend time  $t - t'$  in the reactor before exiting is given by  $c_f(t')p(t - t')$ . These molecules are the ones leaving the reactor at time  $t$  that establish effluent concentration  $c_e(t)$ , so we have

$$c_e(t) = \int_{-\infty}^t c_f(t')p(t - t')dt' \quad (8.3)$$

The inlet and outlet concentrations are connected through this convolution integral with the residence-time distribution. If we conduct the experiment so that the feed tracer concentration is zero before an initial time  $t = 0$ , then the integral reduces to

$$c_e(t) = \int_0^t c_f(t')p(t - t')dt', \quad c_f(t) = 0, t \leq 0 \quad (8.4)$$

Notice we can change the variable of integration in Equation 8.4 to establish an equivalent representation

$$c_e(t) = \int_0^t c_f(t - t')p(t')dt' \quad (8.5)$$

which is sometimes a convenient form. This connection between the inlet and outlet concentrations, and the RTD, allows us to determine the RTD by measuring only tracer concentrations. We next describe some of the convenient experiments to determine the RTD.

**Step response.** Although we can in principle use any feed concentration time function to determine the RTD, some choices are convenient for ease of data analysis. One of these is the step response. In the step-response experiment, at time zero we abruptly change the feed tracer concentration from steady value  $c_0$  to steady value  $c_f$ . For convenience we assume  $c_0 = 0$ . Exercise 8.4 shows that we can easily remove this

assumption using deviation variables. Because the feed concentration is constant at  $c_f$  after time zero, we can take it outside the integral in Equation 8.5 and obtain

$$c_e(t) = c_f \int_0^t p(t')dt' = c_f P(t)$$

So for a step-response experiment, the effluent concentration versus time provides immediately the integrated form of the residence-time distribution

$$P(\theta) = c_e(\theta)/c_f, \quad \text{step response} \quad (8.6)$$

**Pulse and impulse responses.** An impulse response is an idealized experiment, but is a useful concept. As we will see it provides the RTD directly rather than in the integrated form. To motivate the impulse-response experiment, imagine we abruptly change the inlet tracer concentration from zero to a large value and return it to zero after a short time as sketched in Figure 8.2. Such a test is called a pulse test. The pulse test is no more difficult to implement than the step test; it is merely two step changes in feed concentration in rapid succession. In some ways it is a superior test to the step response, because by returning the tracer concentration to zero, we use less tracer in the experiment and we cause less disruption of the normal operation of the reactor.

The impulse response is an idealized limit of the pulse response. Consider a family of pulse tests as sketched in Figure 8.2 of shorter and shorter duration  $\Delta t$ . We maintain constant total tracer addition by spiking the feed with higher and higher concentrations so that the product  $c_f \Delta t = a$  is constant. The impulse response is the limit of this experiment as  $\Delta t \rightarrow 0$ . We call this limiting feed concentration versus time function the delta function,  $a\delta(t)$ . It is also called

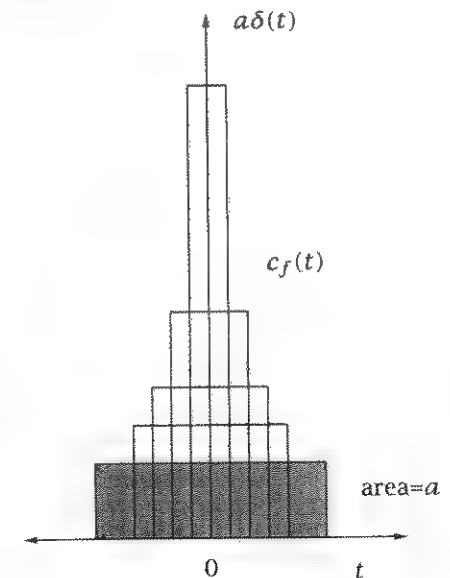


Figure 8.2: Family of pulses and the impulse or delta function.

the Dirac delta function or an impulse, hence the name, impulse response. The constant  $a$  is the amplitude of the delta function. The mathematical problem with this approach is that the limit is not a well-defined function. The values at all points other than  $t = 0$  go to zero as we take narrow enough pulses, and the value at  $t = 0$  is unbounded. But mathematicians have taken a bold step and defined this new kind of function by what it *does* rather than its numerical values. The main property of the delta function is that, because it is so narrowly focused, it extracts the value of an integrand at a point in the interval of integration,

$$\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0), \quad \text{all } g(t) \quad (8.7)$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1, \quad \text{normalized} \quad (8.8)$$

So if we can approximate  $c_f(t) = a\delta(t)$ , then we have from Equation 8.3

$$c_e(t) = a \int_{-\infty}^t \delta(t')p(t-t')dt' = ap(t)$$

So for an experiment approximating an impulse, the effluent concentration versus time provides the residence-time distribution directly

$$p(\theta) = c_e(\theta)/a, \quad \text{impulse response} \quad (8.9)$$

We do not require any sophisticated mathematics to use these generalized functions such as the delta function. We introduce them here because they make our work simpler, not more complicated. You will also find these functions useful in the process control course when you analyze the dynamics and feedback control of processes. Moreover, empirical tests, such as the step- and impulse-response experiments, provide a general method for modeling the dynamic behavior of many processes. Using these methods to determine a reactor RTD is just one small example of their uses in chemical process modeling.

**Arbitrary feed concentration experiment.** It may be inconvenient or impossible to perform a step or impulse test on a reactor. In some situations it is possible to measure a low-concentration impurity in the feed and effluent streams, and construct the RTD from those measurements. In these cases the feed concentration  $c_f(t)$  is an arbitrary, but measured and known, function of time. One must take care that the noise in the measurements does not obscure the RTD, and replication

of experiment and assessment of uncertainty in the RTD is highly recommended. Shinnar [22] provides a detailed discussion for further reading on the science and art of determining and using RTDs to solve industrial reactor problems.

### 8.2.3 Continuous-Stirred-Tank Reactor (CSTR)

We next examine again the well-stirred reactor introduced in Chapter 4. Consider the following step-response experiment: a clear fluid with flowrate  $Q_f$  enters a well-stirred reactor of volume  $V_R$  as depicted in Figure 8.3. At time zero we start adding a small flow of a tracer to the feed stream and measure the tracer concentration in the effluent stream. We assume we keep the flow of the tracer small enough that we do not disturb the existing flow pattern in the reactor. We expect to see a continuous change in the concentration of the effluent stream until, after a long time, it matches the concentration of the feed stream. We are by now experts on solving this type of problem, especially in this simple situation without chemical reaction to complicate matters. Assuming constant density, the differential equation governing the concentration of dye in the reactor follows from Equation 4.38

$$\frac{dc}{dt} = \frac{Q_f}{V_R}(c_f - c), \quad c(0) = 0 \quad (8.10)$$

in which  $c$  is the concentration of the dye in the reactor and effluent stream. In Chapter 4, we named the parameter  $V_R/Q_f$  the "mean residence time" and gave it the symbol  $\theta$ . For our purposes here, we have already reserved  $\theta$  for the distribution of reactor residence times, so we define  $\tau = V_R/Q_f$ . We show subsequently that  $\tau$  is indeed the mean of the RTD for a CSTR. We already derived the solution to Equation 8.10, where

$$c(t) = (1 - e^{-t/\tau})c_f$$

so we have immediately

$$P(\theta) = 1 - e^{-\theta/\tau} \quad (8.11)$$

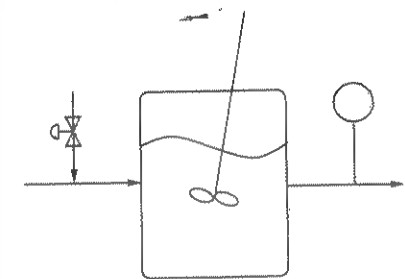
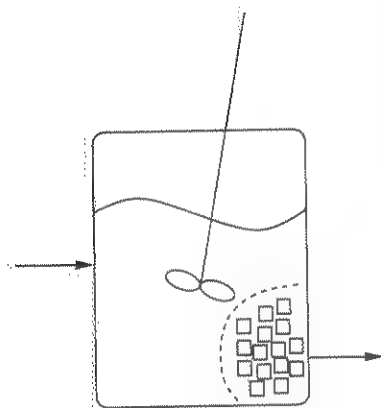


Figure 8.3: CSTR and tracer experiment.



which upon differentiation gives

$$p(\theta) = \frac{1}{\tau} e^{-\theta/\tau}, \quad \text{CSTR residence-time distribution} \quad (8.12)$$



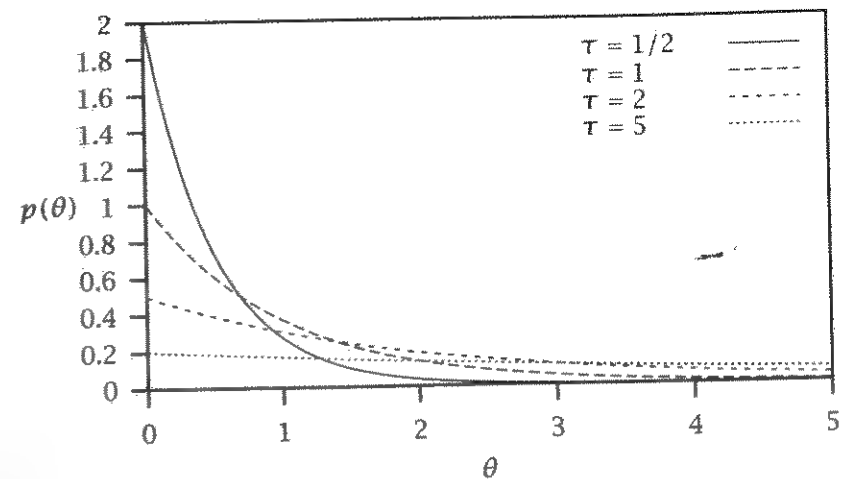
**Figure 8.4:** CSTR and volume elements in a game of chance.

It is often helpful to have two different views of the same result, so we next consider the well-mixed reactor as a game of chance and re-derive the RTD for the CSTR. Imagine we divide the reactor into a number of volume elements, depicted in Figure 8.4. At each time instant, some group of volume elements is withdrawn from the reactor, and new volume elements enter from the feed stream. We assign a probability law to model which volume elements are withdrawn from the reactor. The well-mixed assumption is equivalent to the statement that all volume elements are equally likely to be withdrawn from the reactor. One can view the mixing as quickly (instantaneously in the case of perfect mixing) randomizing the locations of the volume elements so that each one is equally likely to be nearest to the exit when the next element is withdrawn in the effluent stream. We also see why perfect mixing is a physical idealization; it is not possible to instantaneously randomize the locations of the volume elements with finite power to the stirrer.

We have  $n$  volume elements, each with volume  $V_n = V_R/n$ . In time interval  $\Delta t$ , a volume of  $Q_f \Delta t$  leaves with the effluent stream. The number of elements leaving in the time interval is therefore  $Q_f \Delta t / V_n$  out of  $n$ , and the fraction of elements withdrawn is

$$f = \frac{Q_f \Delta t}{V_R} = \frac{\Delta t}{\tau}$$

Now consider the probability that a particular volume element is still in the reactor after  $i$  time intervals. The probability of removal is  $f$ , so the probability of survival is  $1 - f$ . After  $i$  trials, each of which is independent, the probability of survival is the product  $(1 - f)^i$ . Therefore



**Figure 8.5:** CSTR residence-time distribution.

the probability that an element has residence (survival) time  $\theta = i\Delta t$  is

$$\bar{p}(\theta) = \left(1 - \frac{\Delta t}{\tau}\right)^{\theta/\Delta t} \quad (8.13)$$

We now take the limit as  $\Delta t \rightarrow 0$ . Recall from calculus

$$\lim_{x \rightarrow 0} (1 + ax)^{1/x} = e^a$$

Taking the limit and using this result in Equation 8.13 gives

$$\bar{p}(\theta) = e^{-\theta/\tau}$$

which, after normalization, is again the residence-time distribution of the CSTR. The normalization constant is computed by integration

$$\int_0^{\infty} \bar{p}(\theta) d\theta = \int_0^{\infty} e^{-\theta/\tau} d\theta = \tau$$

so the residence-time distribution is

$$p(\theta) = \frac{1}{\tau} e^{-\theta/\tau} \quad (8.14)$$

which is plotted in Figure 8.5 for a variety of mean residence times. We

next compute the mean of this distribution. An integration by parts produces

$$\begin{aligned}\bar{\theta} &= \int_0^{\infty} \theta p(\theta) d\theta = \frac{1}{\tau} \int_0^{\infty} \theta e^{-\theta/\tau} d\theta \\ &= \frac{1}{\tau} \left[ -\tau \theta e^{-\theta/\tau} - (\tau)^2 e^{-\theta/\tau} \right] \Big|_0^{\infty} \\ &= \tau\end{aligned}$$

and we have established that the mean of the RTD for the CSTR is indeed  $\tau = V_R/Q_f$ . We can therefore compactly write Equation 8.14 as

$$p(\theta) = \frac{1}{\bar{\theta}} e^{-\theta/\bar{\theta}} \quad (8.15)$$

Notice the exponential distribution tells us that it is unlikely for a volume element to remain in this reactor for long because at each instant there is a constant probability that the element is withdrawn in the effluent. It is difficult to win for long in this kind of game of chance, which explains the exponential decrease in  $p(\theta)$ . In fact, zero is the most likely residence time, a classic and slightly counter-intuitive case: the most likely value of  $\theta$ , the  $\max_{\theta} p(\theta)$  occurs at  $\theta = 0$ , which is different from the mean of the distribution,  $\bar{\theta} = V_R/Q_f$ . Again, if the mixing is not perfect, this maximum does not occur at  $\theta = 0$  because it requires finite time for a volume element entering with the feed to be convected to the exit. Moreover, our derivation of the RTD of the CSTR shows the following general principle: given an event with constant probability of occurrence, the *time* until the next occurrence of the event is distributed as a decreasing exponential function. This principle is used, for example, to choose the time of the next reaction in the stochastic simulation of chemical kinetics in Chapter 4.

### 8.2.4 Plug-Flow Reactor (PFR) and Batch Reactor

The simple flow pattern in the PFR produces a simple residence-time distribution. Consider a step test in which the reactor is initially free of tracer and we increase the feed tracer concentration from zero to  $c_f$  at time zero. As shown in Figure 8.6, the tracer travels in a front that first reaches the reactor exit at time  $t = l/v$ , in which  $v$  is the velocity of the axial flow and  $l$  is the reactor length. From these physical considerations, we can write the reactor tracer concentration immediately

$$c(t, z) = \begin{cases} 0, & z - vt > 0 \\ c_f, & z - vt \leq 0 \end{cases} \quad (8.16)$$

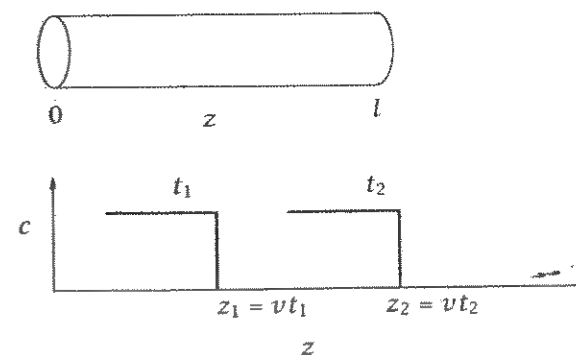


Figure 8.6: Plug-flow reactor with moving front of tracer.

For  $z - vt > 0$ , the tracer front has not reached location  $z$  at time  $t$  so the tracer concentration is zero. For  $z - vt < 0$ , the front has passed location  $z$  at time  $t$  and the tracer concentration is equal to the feed value  $c_f$ .

It is convenient to introduce the unit step or Heaviside function to summarize this result. The Heaviside function is defined as follows

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \quad (8.17)$$

Because we are armed with the delta function, we can even consider differentiating this discontinuous function to obtain the relationship

$$\frac{dH(t)}{dt} = \delta(t) \quad (8.18)$$

Equation 8.16 can then be summarized compactly by

$$c(t, z) = c_f H(t - z/v)$$

so the effluent tracer concentration is given by

$$c_e(t) = c_f H(t - l/v) = c_f H(t - V_R/Q_f) \quad (8.19)$$

and the integrated form of the residence-time distribution is therefore

$$P(\theta) = H(\theta - V_R/Q_f) \quad (8.20)$$

We can differentiate this result using Equation 8.18 to obtain

$$p(\theta) = \delta(\theta - V_R/Q_f)$$

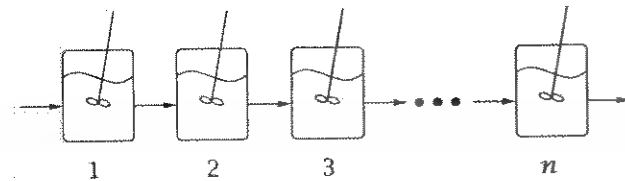


Figure 8.7: Series of  $n$  equal-sized CSTRs with residence time  $\tau/n$ .

In other words, all tracer molecules spend exactly the same time  $V_R/Q_f$  in the reactor. The mean of this distribution is then also  $V_R/Q_f$ , which is verified by using Equation 8.7

$$\bar{\theta} = \int_0^{\infty} \theta \delta(\theta - V_R/Q_f) d\theta = V_R/Q_f$$

which shows that  $V_R/Q_f$  is the mean residence time for the PFR as well as the CSTR, even though the residence-time distributions of these two reactors are quite different. We can compactly summarize the RTD for the plug-flow reactor by

$$\begin{aligned} p(\theta) &= \delta(\theta - \bar{\theta}) && \text{PFR and} \\ P(\theta) &= H(\theta - \bar{\theta}) && \text{batch reactors} \end{aligned} \quad (8.21)$$

Likewise, the RTD for a batch reactor is immediate. All material is charged to the reactor at time  $t = 0$  and remains in the reactor until the final batch time, which we may call  $\bar{\theta}$ . Then Equations 8.21 also apply to the batch reactor.

### 8.2.5 CSTRs in Series

Consider dividing the volume of a single CSTR into  $n$  equal-sized CSTRs in series as shown in Figure 8.7. If the single CSTR has volume  $V_R$  and residence time  $\tau = V_R/Q_f$ , each of the CSTRs in series has volume  $V_R/n$  and residence time  $\tau/n$ . If we solve for the effluent concentration after an impulse at time zero, we can show

$$p(\theta) = \left(\frac{n}{\tau}\right)^n \frac{\theta^{n-1}}{(n-1)!} e^{-n\theta/\tau} \quad (8.22)$$

which is plotted in Figure 8.8 for  $\tau = 2$ . Notice the residence-time distribution becomes more narrowly focused at  $\tau$  as  $n$  increases. In

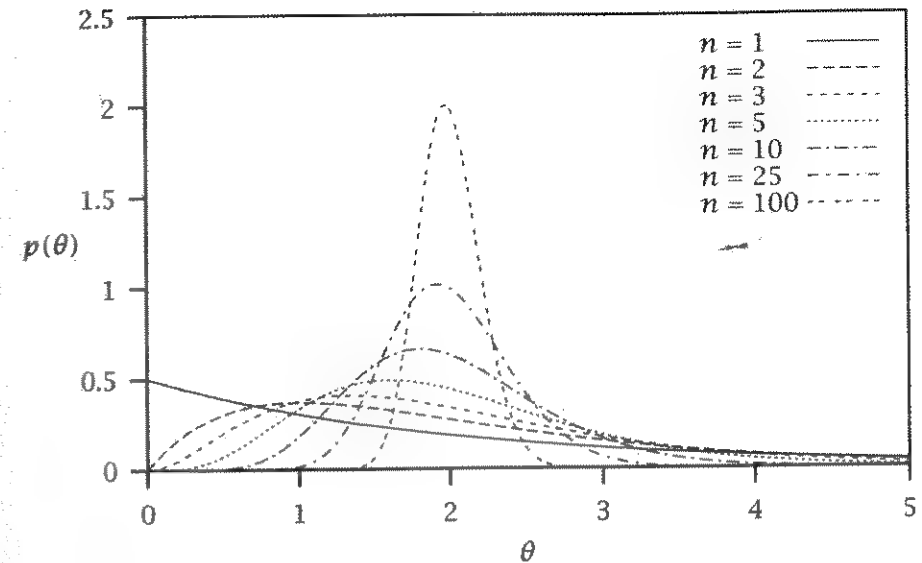


Figure 8.8: Residence-time distribution  $p(\theta)$  versus  $\theta$  for  $n$  CSTRs in series,  $\tau = 2$ .

fact Equation 8.22 for large  $n$  is another approximation for  $\delta(\theta - \tau)$ . If we integrate Equation 8.22 we obtain

$$P(\theta) = \frac{\gamma(n, n\theta/\tau)}{\Gamma(n)} \quad (8.23)$$

in which the gamma function is defined as

$$\Gamma(n) = \int_0^{\infty} t^{(n-1)} e^{-t} dt \quad (8.24)$$

Notice the gamma function is a generalization of the familiar factorial and for  $n$  an integer,

$$\Gamma(n) = (n-1)!$$

The incomplete gamma function is defined by the same integrand on the finite interval

$$\gamma(n, x) = \int_0^x t^{(n-1)} e^{-t} dt \quad (8.25)$$

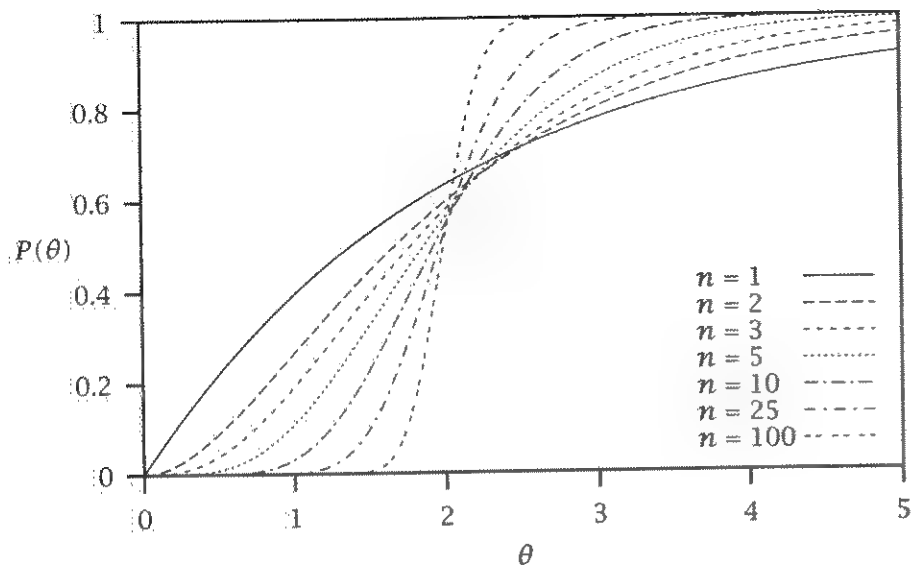


Figure 8.9:  $P(\theta)$  versus  $\theta$  for  $n$  CSTRs in series,  $\tau = 2$ .

You can perform integration by parts on Equation 8.22 to obtain a series representation

$$P(\theta) = 1 - \left( 1 + \frac{n\theta/\tau}{1!} + \frac{(n\theta/\tau)^2}{2!} + \dots + \frac{(n\theta/\tau)^{n-1}}{(n-1)!} \right) e^{-n\theta/\tau} \quad (8.26)$$

Evaluating the incomplete gamma function is numerically better than summing this series when  $n$  is large. Figure 8.9 shows Equation 8.23 for a range of  $n$  values.

### 8.2.6 Dispersed Plug Flow

Ideal plug flow may not be realized in many packed-bed reactors. We develop next a model that allows for deviations from plug flow. As shown in Figure 8.6, in the PFR a change in the feed concentration travels as a perfect front. But measurements in actual packed-bed reactors may show a fair amount of spreading or dispersion as this front travels through the reactor. The dispersed plug-flow model given in Equation 8.27 allows for this spreading phenomenon. A diffusion-like term has been added to the mass balance. The coefficient,  $D_l$ , is called the dispersion coefficient. Note this term is not meant to represent true

molecular diffusion caused by the random motion of the molecules. Molecular diffusion *in the direction of the convected flow* is always negligible in packed bed reactors operating at reasonable flowrates. The spreading of concentration fronts is caused by other phenomena, such as Taylor dispersion [2, p.643].

$$\underbrace{\frac{\partial c}{\partial t}}_{\text{accumulation}} = \underbrace{-v \frac{\partial c}{\partial z}}_{\text{convection}} + \underbrace{D_l \frac{\partial^2 c}{\partial z^2}}_{\text{diffusion}} \quad (8.27)$$

Given diffusion in the tube, the inlet boundary condition is no longer just the feed condition,  $c(0) = c_f$ , that we used in the PFR. To derive the boundary condition at the inlet, we write a material balance over a small region containing the entry point, and consider diffusion and convection terms. The diffusion term introduces a second-order derivative in Equation 8.27, so we now require two boundary conditions. We specify a zero slope condition at the tube exit.

$$\begin{aligned} vc|_{0^-} &= vc|_{0^+} - D_l \left. \frac{\partial c}{\partial z} \right|_{0^+} & z = 0 \\ \frac{\partial c}{\partial z} &= 0 & z = l \end{aligned} \quad (8.28)$$

These two boundary conditions have become known as Danckwerts boundary conditions [6], but they were derived at least 45 years prior to Danckwerts in a classic paper by Langmuir [16]. Further discussion of the applicability and some alternatives to the Danckwerts boundary conditions are given by Bischoff [3], Levenspiel [17, p.272], and Parulekar and Ramkrishna [21].

It is now convenient to introduce a coordinate moving with a scaled velocity

$$x = \frac{z - vt}{\sqrt{4D_l t}} = \frac{\bar{z} - t/\tau}{\sqrt{4Dt/\tau}} \quad (8.29)$$

in which

$$D = \frac{D_l}{vl} = \frac{D_l \tau}{l^2}, \quad \text{dimensionless dispersion number}$$

Transforming variables from  $z$  and  $t$  in Equation 8.27 to  $x$  gives

$$\frac{d^2 c}{dx^2} + 2x \frac{dc}{dx} = 0 \quad (8.30)$$

Rather than use the Danckwerts boundary conditions, we can approximate the behavior with the following simplified boundary conditions

$$\begin{aligned} c &= 1, & x &= -\infty \\ c &= 0, & x &= \infty \end{aligned} \quad (8.31)$$

These boundary conditions correspond to stating that  $c = 0$  for  $z, t$  values corresponding to arbitrarily long times before the step change arrives, and that  $c = 1$ , for  $z, t$  values corresponding to arbitrarily long times after the step change has passed. See Exercise 8.10 for comparing the dispersed PFR RTD with Danckwerts boundary conditions to the one we calculate here with the simplified boundary conditions.

The solution to Equation 8.30 with boundary conditions listed in Equations 8.31 is

$$c(x) = 1/2 \left[ 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \right]$$

The integral can be expressed in terms of the error function, which is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (8.32)$$

Substituting in the original variables and setting  $\bar{z} = 1$  to obtain the response at the reactor outlet as a function of time gives

$$c(\bar{z} = 1, t) = 1/2 \left[ 1 - \operatorname{erf} \left( \frac{1 - t/\tau}{\sqrt{4Dt/\tau}} \right) \right] \quad (8.33)$$

and we have calculated the integrated form of the RTD for the dispersed PFR with simplified boundary conditions

$$P(\theta) = 1/2 \left[ 1 - \operatorname{erf} \left( \frac{1 - \theta/\tau}{\sqrt{4D\theta/\tau}} \right) \right] \quad (8.34)$$

Equation 8.34 is plotted in Figure 8.10 for  $\tau = 2$  and various dispersion numbers  $D$ . We can differentiate Equation 8.34 to obtain the dispersed plug-flow RTD

$$p(\theta) = \frac{1}{4\tau\sqrt{\pi D}} \frac{1 + \tau/\theta}{\sqrt{\theta/\tau}} \exp \left( - \left( \frac{1 - \theta/\tau}{\sqrt{4D\theta/\tau}} \right)^2 \right) \quad (8.35)$$

This RTD is plotted in Figure 8.11.

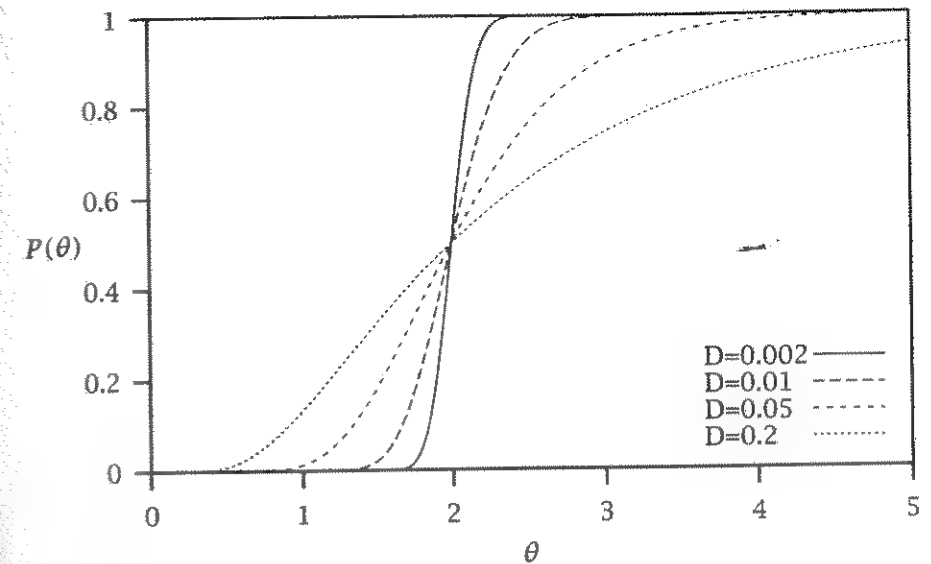


Figure 8.10:  $P(\theta)$  versus  $\theta$  for plug flow with dispersion number  $D$ ,  $\tau = 2$ .

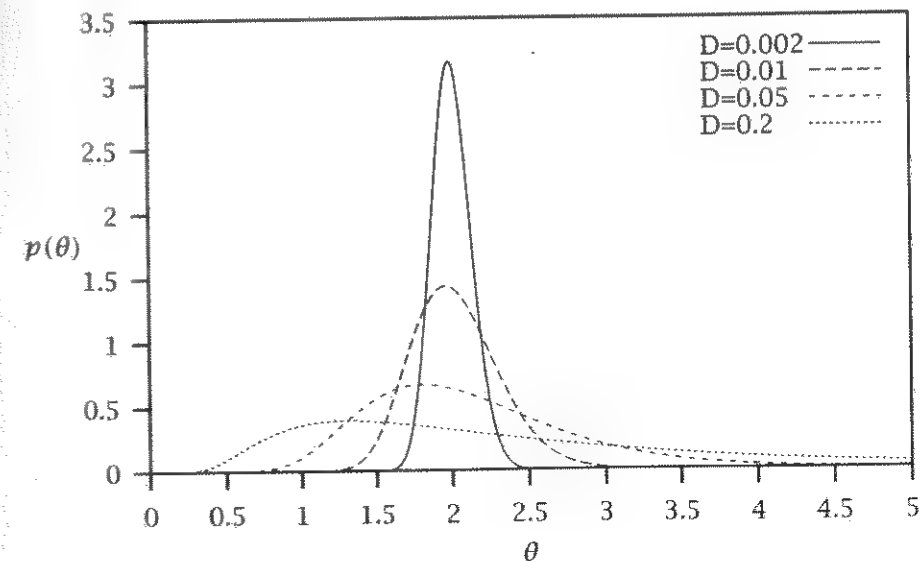


Figure 8.11: Residence-time distribution  $p(\theta)$  versus  $\theta$  for plug flow with dispersion number  $D$ ,  $\tau = 2$ .

The dispersion number,  $D$ , is related to another dimensionless group, the mass-transfer analog of the inverse of the Péclet number,

$$Pe = \frac{vl}{D_A}, \quad \frac{1}{Pe} = \frac{D_A}{vl}$$

which measures the rate of diffusion compared to the rate of convection. The key difference is the Péclet number contains the molecular diffusivity,  $D_A$ , and the dispersion number contains the effective axial dispersion coefficient,  $D_l$ . Levenspiel makes a compelling case that these two quantities have different origins and motivations and deserve different names. To further complicate matters, the inverse of the Péclet number is often called the Bodenstein number in European literature. Weller [26] provides an interesting discussion of the history of the literature on the Bodenstein number, which does not appear to have been defined or used by Bodenstein, but was defined and used by Langmuir [16].

**Dispersed plug flow with reaction.** We modify Equation 8.27 for dispersed plug flow to account for chemical reaction,

$$\frac{\partial c_j}{\partial t} = -v \frac{\partial c_j}{\partial z} + D_{jl} \frac{\partial^2 c_j}{\partial z^2} + R_j \quad (8.36)$$

Danckwerts boundary conditions, as given in Equations 8.28, can be applied without change.

Up to this point in the text, we have solved *exclusively steady-state* profiles in tubular reactors. Obviously tubular reactors experience a start-up transient like every other reactor, and this time-dependent behavior is also important and interesting. Calculating the transient tubular-reactor behavior involves solving the partial differential equation (PDE), Equation 8.36, rather than the usual ODE for the steady-state profile. Appendix A describes the method we use for this purpose, which is called orthogonal collocation.

#### Example 8.1: Transient start-up of a PFR

Compute the transient behavior of the dispersed plug-flow reactor for the isothermal, liquid-phase, second-order reaction



The reactor is initially filled with solvent. The kinetic and reactor parameters are given in Table 8.1.

Parameter	Value	Units
$k$	0.5	L/mol·min
$c_{Af}$	1	mol/L
$D_{Al}$	0.01	m <sup>2</sup> /min
$v$	0.5	m/min
$l$	1	m

Table 8.1: Mass-transfer and kinetic parameters for Example 8.1.

#### Solution

The mass balance for component A is

$$\frac{\partial c_A}{\partial t} = -v \frac{\partial c_A}{\partial z} + D_{Al} \frac{\partial^2 c_A}{\partial z^2} - 2kc_A^2 \quad (8.38)$$

The boundary conditions are

$$vc_{Af} = vc_A|_{0^+} - D_{Al} \frac{\partial c_A}{\partial z} \Big|_{0^+}, \quad z = 0 \quad (8.39)$$

$$\frac{\partial c_A}{\partial z} = 0, \quad z = l \quad (8.40)$$

Finally, an initial condition is required

$$c_A(z, t) = 0, \quad t = 0$$

Figure 8.12 shows the transient profiles. We see the reactor initially has zero A concentration. The feed enters the reactor and the A concentration at the inlet rises rapidly. Component A is transported by convection and diffusion down the reactor, and the reaction consumes the A as it goes. After about  $t = 2.5$ , the concentration profile has reached its steady value. Given the low value of dispersion in this problem, the steady-state profile is close to the steady-state PFR profile for this problem. □

#### Example 8.2: Order matters

Consider two arrangements of a PFR and CSTR of equal volume in series as shown in Figure 8.13. What are the residence-time distributions for the two reactor systems? What are the overall conversions for the two systems?

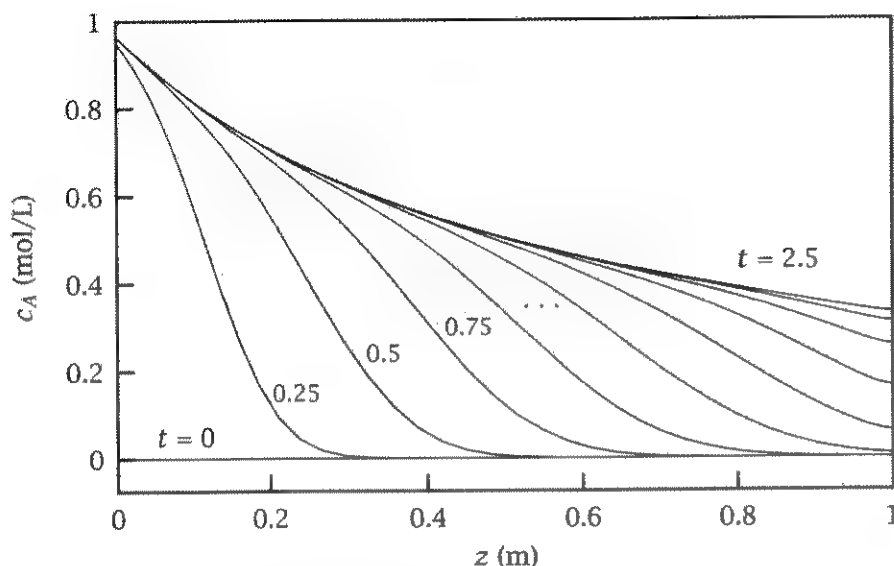


Figure 8.12: Start-up of the tubular reactor;  $c_A(z, t)$  versus  $z$  for various times,  $0 \leq t \leq 2.5$ ,  $\Delta t = 0.25$ .

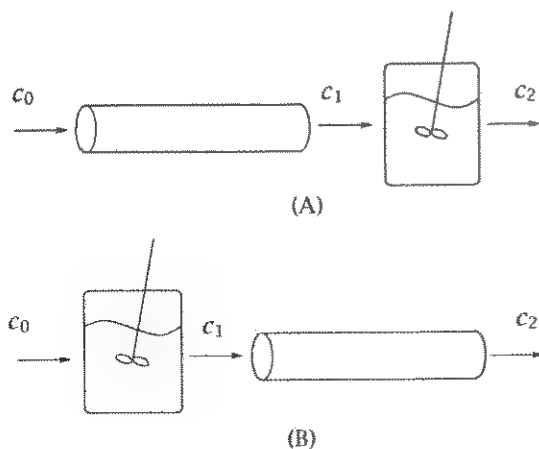


Figure 8.13: PFR followed by CSTR (A), and CSTR followed by PFR (B).

### Solution

Consider a unit step test for the CSTR-PFR arrangement. The outlet concentration for the CSTR is given by

$$c_1(t) = 1 - \exp(-t/\tau)$$

That feed concentration to the PFR is then simply delayed by  $\tau$  time units to give for the CSTR-PFR arrangement.

$$p(\theta) = (1 - \exp(-(t - \tau)/\tau)) H(t - \tau)$$

Next consider a unit step test into the PFR-CSTR arrangement. For this case the intermediate stream is given by a delayed step input

$$c_1(t) = H(t - \tau)$$

With this feed into the CSTR, the effluent is merely the CSTR response to a unit step change after we shift the starting time of the step forward  $\tau$  time units,

$$c_2(t) = (1 - \exp(-(t - \tau)/\tau)) H(t - \tau)$$

so again for this case

$$p(\theta) = (1 - \exp(-(t - \tau)/\tau)) H(t - \tau)$$

and the two residence-time distributions are equal.

The steady-state conversions for both arrangements are also simply calculated. For a single CSTR, the steady-state inlet and outlet concentrations are related by

$$c_0/c_i = \frac{-1 + \sqrt{1 + 4k\tau c_i}}{2k\tau c_i} = C(c_i)$$

For a single PFR, the inlet and outlet concentrations are related by

$$c_0/c_i = \frac{1}{1 + k\tau c_i} = P(c_i)$$

So we wish to compare  $P(C(c_0))$  for the CSTR-PFR case and  $C(P(c_0))$  for PFR-CSTR case. Because we are not even told  $k\tau c_0$ , we check over a range of values. Figure 8.14 displays the result. We see that the conversions are *not* the same and that the PFR-CSTR gives higher conversion (lower outlet concentration) than the CSTR-PFR for all values of  $k\tau c_0$  for a second-order reaction.  $\square$



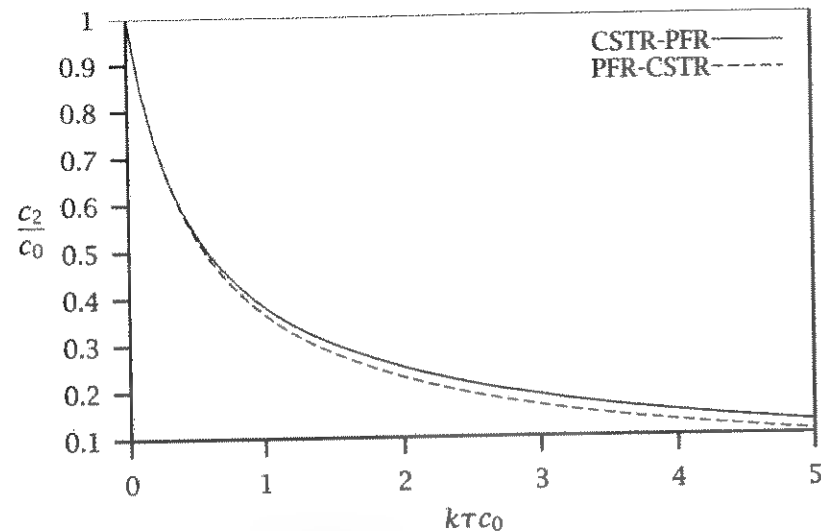


Figure 8.14: Comparison of the effluent concentrations for the two cases shown in Figure 8.13.

### 8.3 Limits of Reactor Mixing

We have seen in the previous section that complete knowledge of the reactor residence-time distribution is insufficient to predict the reactor performance. Although we have characterized completely the time tracer molecules spend in the reactor, we have not characterized their surrounding environment in the reactor during this time.<sup>1</sup> In the literature these two effects are sometimes termed **macromixing**: the distribution of residence times of molecules moving with the flow; and **micromixing**: the exchange of material between different volume elements during their residence times. Although we may find it instructive to separate these two phenomena in the simple reactor mixing models under discussion, in actual reactors this separation may be unrealistic. Accurate prediction of reactor performance may require solution or ap-

<sup>1</sup>If someone were to characterize your learning in this course by measuring your hours spent in the classroom (RTD), they would hopefully obtain a positive correlation between learning and residence time. But we would naturally want to evaluate the environment inside the classroom during these hours if we were going to make more accurate predictions of learning. We would want to know if the instructor was prepared for lecture and saying reasonable things, if the students were attentive or asleep, and so on.

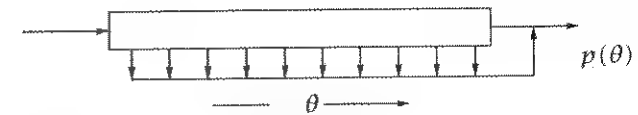


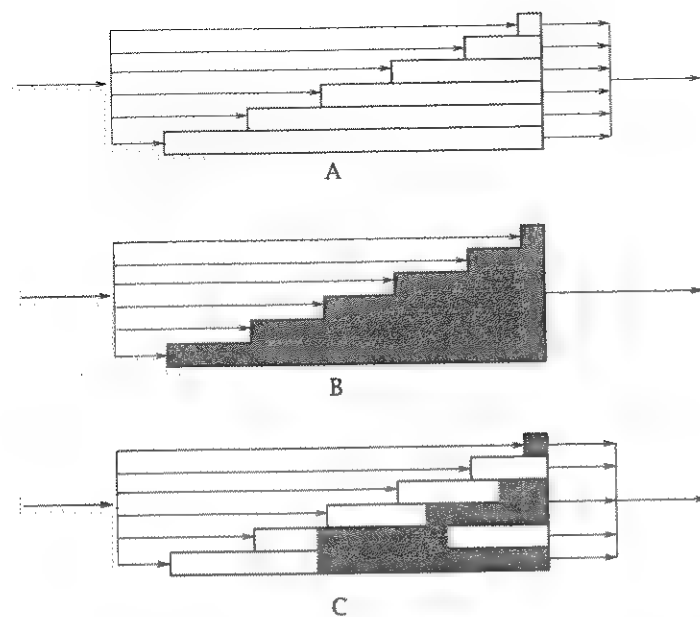
Figure 8.15: Completely segregated flow as a plug-flow reactor with side exits; outlet flows adjusted to achieve given RTD.

proximate solution of the equations of motion for the fluid, including material transport due to diffusion.

In defense of the simple mixing models, however, they do provide another important insight. We can determine the limits of achievable mixing consistent with a measured reactor residence-time distribution. These mixing limits do provide some insight into the limits of achievable reactor performance, although this connection remains an active area of research as discussed in section 8.4.

#### 8.3.1 Complete Segregation

Imagine we know a reactor RTD,  $p(\theta)$ , either through direct measurement in a tracer experiment or solution of the equations of motion or some other means. We know from Example 8.2 that this constraint does not tell us the complete state of mixing in the reactor. We define next the two extreme limits of mixing consistent with the given RTD. These ideas were laid out in influential papers by Danckwerts and Zwietering. The first limit is called complete segregation; it is the limit of no mixing between volume elements. We can realize this limit by considering the ideal reactor depicted in Figure 8.15. As we progress down the plug-flow reactor, the residence time  $\theta$  of the material reaching that location increases. We can imagine withdrawing from the reactor at each location or  $\theta$  a fraction of the flow corresponding to the required RTD value  $p(\theta)$ , although this might be difficult to achieve in practice. A PFR with this removal rate then has the specified RTD. No material in two volume elements with different residence times is ever exchanged because the plug flow has zero backmixing. This last point is perhaps more clear if we redraw the reactor configuration as an equivalent bank of PFRs of different lengths without side exits, as in Figure 8.16.A [25]. Each tube has a single  $\theta$  value according to its length. We feed the fraction  $p(\theta)$  of the total flow into each tube of residence time  $\theta$  so as to achieve the given RTD for the composite reactor system. This reactor



**Figure 8.16:** Alternate representation of completely segregated flow (A), maximum mixed flow (B), and an intermediate mixing pattern (C).

system is called completely segregated because there is no exchange of material between the various tubes. Each tube acts as its own private reactor that processes material for a given amount of time and then discharges it to be mixed with the other reactors at the exit.

It is a simple matter to predict the behavior of this completely segregated reactor. We assume a single reaction and constant density throughout the following discussion. Each tube of specified length or volume  $V$  can be assigned a residence time according to  $\theta = V/Q$ . Let  $c(\theta)$  represent the concentration of a reactant in a volume element that has spent time  $\theta$  in the reactor. Because the reactor is segregated, each tube satisfies the constant density PFR equation, Equation 4.112,

$$\frac{dc}{d\theta} = R(c), \quad c(0) = c_f \quad (8.41)$$

The concentration of the effluent stream is then simply computed by multiplying the concentration of each tube by the fraction of the total

feed passing through that tube

$$c_s = \int_0^\infty p(\theta)c(\theta)d\theta \quad (8.42)$$

in which  $c(\theta)$  comes from the solution of Equations 8.41. It is often convenient to remove the explicit integration required by Equation 8.42. Let  $c_s(\theta)$  represent the effect of combining streams with residence times less than or equal to  $\theta$ , so

$$c_s(\theta) = \int_0^\theta p(\theta)c(\theta)d\theta$$

From this definition it is clear that  $c_s(\theta)$  satisfies the following differential equation and initial condition

$$\frac{dc_s}{d\theta} = p(\theta)c(\theta), \quad c_s(0) = 0$$

and the reactor effluent concentration is the limit of  $c_s(\theta)$  as  $\theta \rightarrow \infty$ . We can combine the two differential equations for convenient numerical solution of the segregated case

$$\boxed{\begin{aligned} \frac{dc}{d\theta} &= R(c) & c(0) &= c_f \\ \frac{dc_s}{d\theta} &= p(\theta)c(\theta) & c_s(0) &= 0 \end{aligned}} \quad (8.43)$$

Notice that this is an initial-value problem, but, in general, we require the solution at  $\theta = \infty$  to determine the effluent concentration of the reactor. Differential equations on semi-infinite domains are termed singular, and require some care in their numerical treatment as we discuss next. On the other hand, if the residence-time distribution is zero beyond some maximum residence time,  $\theta_{\max}$ , then it is straightforward to integrate the initial-value problem on  $0 \leq \theta \leq \theta_{\max}$ .

**Numerical solution.** We can solve Equation 8.43 as an initial-value problem as written with an ODE solver. Because of the semi-infinite domain, we would need to check the solution for a sequence of increasingly large  $\theta$  values and terminate the ODE solver when the value of  $c_s(\theta)$  stops changing. Alternatively, we can map the semi-infinite domain onto a finite domain and let the ODE solver do the work for us. Many transformations are possible, such as  $z = \exp(-\theta)$ , but experience suggests a strongly decreasing function like the exponential

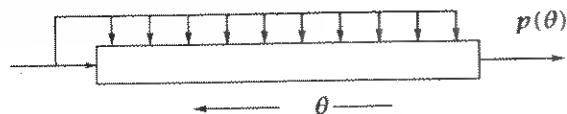


Figure 8.17: Maximum mixed flow as a plug-flow reactor with side entrances; inlet flows adjusted to achieve a given RTD.

causes the right-hand side to go to infinity at  $z = 1$ , and we simply exchange one singularity for another. A more gentle transformation and its inverse are

$$z = \frac{\theta}{1 + \theta}, \quad \theta = \frac{z}{1 - z}$$

Using this change of variable, we rewrite the derivative as

$$\frac{dc}{d\theta} = \frac{dc}{dz} \frac{dz}{d\theta} = (1 - z)^2 \frac{dc}{dz}$$

Using this result, we transform Equation 8.43 to

$$\boxed{\begin{aligned} \frac{dc}{dz} &= \frac{R(c)}{(1 - z)^2} & c(0) &= c_f \\ \frac{dc_s}{dz} &= \frac{p(z/(1 - z)) c}{(1 - z)^2} & c_s(0) &= 0 \end{aligned}} \quad (8.44)$$

Most modern ODE solvers allow the user to specify critical stopping values. These are values of the variable of integration beyond which the ODE solver will not step. We would specify  $z = 1$  as a critical value because the right-hand side is not defined past  $z = 1$ . At the value  $z = 1$ , we would specify the right-hand sides are zero because the reaction will have equilibrated at  $z = 1$ ,  $\theta = \infty$  so  $R(c) = 0$ , and  $p(\theta) = 0$  at  $\theta = \infty$ . Again, some care with must be taken because the denominators are also going to zero. If the ODE solver terminates successfully, that usually indicates the transformation was successful. It is useful to plot  $c(z)$  to make sure the  $z = 1$  end does not exhibit some unusual behavior.

### 8.3.2 Maximum Mixedness

We realize the opposite mixing limit, maximum mixedness, by reversing the flow in the segregated reactor as shown in Figure 8.17 [28]. The feed stream is distributed along the length of the PFR and injected at the appropriate rate at various side entrances corresponding to different  $\theta$  "locations" to achieve the required RTD. Notice that because the flow

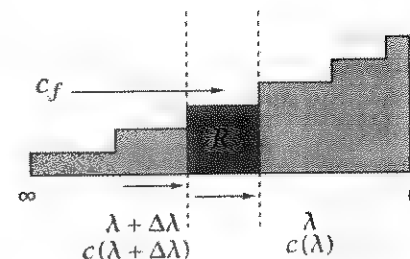


Figure 8.18: Volume element in the state of maximum mixedness.

has been reversed compared to the segregated case, the  $\theta$  locations increase from zero at the exit of the tube to large values at the entrance to the tube. We allow an infinitely long tube if we wish to allow RTDs such as the CSTR defined on a semi-infinite domain. Reactors with these specified sidestream addition policies are conceptually important in understanding recent research on achievable reactor performance as discussed in Section 8.4.

Consider the equivalent representation of maximum mixedness in Figure 8.16.B. The shading means that the material at these locations is completely mixed with the material from the other tubes at these same locations [25]. Notice that we have aligned the exits of the tubes in Figure 8.16. Therefore when we mix material between tubes, we are mixing material not with common time spent in the reactor but rather with a common time-to-go  $\lambda$ . Indeed, the mixing process at a given location is mixing material with different times spent in the reactor. It is not possible to mix material with different times-to-go without changing their exit times for the following reason. If we mix two groups of molecules with different times-to-go,  $\lambda_1, \lambda_2$ , the mixture must later be separated again so that the molecules may exit the reactor at the specified times  $\lambda_1$  and  $\lambda_2$ . Such a separation is not possible because molecules are not distinguishable based on their times-to-go in the reactor. We face the equivalent problem if we mix two identical, pure-component gases initially on opposite sides of a partition. After mixing, we have no process to separate them again based on their initial locations because the molecules are identical. Such a separation process would violate the second law of thermodynamics.

We next derive the differential equation that governs the maximum mixedness reactor [28]. Consider an expanded view of the reactor in a state of maximum mixedness shown in Figure 8.16.B. As depicted in Figure 8.18, feed with flowrate  $Qp(\theta)d\theta$  is added to each tube having

residence time  $\theta$  to achieve the desired RTD. So if we consider a volume element to be a mixed section of the composite reactor between times-to-go  $\lambda$  and  $\lambda + \Delta\lambda$ , the material balance for this element consists of the following terms:

$$\text{entering at } \lambda + \Delta\lambda: \quad Q \left( \int_{\lambda+\Delta\lambda}^{\infty} p(\lambda') d\lambda' \right) c(\lambda + \Delta\lambda) \quad (8.45)$$

$$\text{leaving at } \lambda: \quad Q \left( \int_{\lambda}^{\infty} p(\lambda') d\lambda' \right) c(\lambda) \quad (8.46)$$

$$\text{feed addition:} \quad Q(p(\lambda)\Delta\lambda) c_f \quad (8.47)$$

$$\text{production:} \quad \underbrace{Q \left( \int_{\lambda}^{\infty} p(\lambda') d\lambda' \right) \Delta\lambda}_{\text{volume of reactor element}} \underbrace{R(c(\lambda))}_{\text{rate per volume}} \quad (8.48)$$

Considering the reactor is at steady state, we set the accumulation to zero and obtain

$$c(\lambda + \Delta\lambda) \int_{\lambda+\Delta\lambda}^{\infty} p(\lambda') d\lambda' - c(\lambda) \int_{\lambda}^{\infty} p(\lambda') d\lambda' + c_f p(\lambda) \Delta\lambda + R(c) \Delta\lambda \int_{\lambda}^{\infty} p(\lambda') d\lambda' = 0$$

We can combine the first two integral terms and divide by  $\Delta\lambda$  to obtain

$$\frac{c(\lambda + \Delta\lambda) - c(\lambda)}{\Delta\lambda} \int_{\lambda+\Delta\lambda}^{\infty} p(\lambda') d\lambda' - c(\lambda) p(\lambda) + c_f p(\lambda) + R(c) \int_{\lambda}^{\infty} p(\lambda') d\lambda' = 0$$

Taking the limit as  $\Delta\lambda \rightarrow 0$  and rearranging gives

$$\frac{dc}{d\lambda} = \frac{p(\lambda)}{\int_{\lambda}^{\infty} p(\lambda') d\lambda'} (c(\lambda) - c_f) - R(c)$$

Equivalently we can express the integral in terms of the integrated form of the RTD and write

$$\frac{dc}{d\lambda} = \frac{p(\lambda)}{1 - P(\lambda)} (c(\lambda) - c_f) - R(c) \quad (8.49)$$

We wish to calculate the reactor effluent concentration, which is given by  $c(\lambda)$  at  $\lambda = 0$ . As in the segregated reactor case, this first-order differential equation is singular; we wish to integrate from  $\lambda = \infty$ , the entrance to the longest tube, to the combined tube exits at  $\lambda = 0$ . A

boundary condition is required at  $\lambda = \infty$ . For  $c$  to remain bounded as  $\lambda \rightarrow \infty$ , we stipulate the boundary condition

$$\frac{dc}{d\lambda} = 0, \quad \lambda = \infty$$

Provided we know the limit  $p(\lambda)/(1 - P(\lambda))$  as  $\lambda \rightarrow \infty$ , we can solve Equation 8.49 directly for the boundary condition on  $c$  at  $\lambda = \infty$ ; we call this value  $c_{\infty}$ . Note that  $c_{\infty} \neq c_f$ .

**Numerical solution.** We wish to transform the  $\lambda \in (\infty, 0)$  interval into  $z \in (0, 1)$ . The analogous transformation to the segregated reactor is

$$z = \frac{1}{1 + x}, \quad x = \frac{1 - z}{z}$$

The derivative becomes

$$\frac{dc}{d\lambda} = \frac{dc}{dz} \frac{dz}{d\lambda} = -z^2 \frac{dc}{dz}$$

in which the minus sign arises because we are changing the direction when integrating in the transformed  $z$  variable. Equation 8.49 then becomes

$$\frac{dc}{dz} = -\frac{1}{z^2} \left[ \frac{p((1-z)/z)}{1 - P((1-z)/z)} (c - c_f) - R(c) \right] \quad c(0) = c_{\infty} \quad (8.50)$$

and we integrate from  $z = 0$  to  $z = 1$ . Again, a critical stopping value should be set at  $z = 1$  to avoid an undefined right-hand side. We set the right-hand side to zero at  $z = 0$  because we determined the value of  $c_{\infty}$  such that the bracketed term in Equation 8.50 was zero. Again, care should be exercised at  $z = 0$  because the denominator goes to zero at  $z = 0$  ( $\theta = \infty$ ). Plotting  $c(z)$  and examining the  $z = 0$  end for unusual behavior is recommended.

### Example 8.3: Two CSTRs in series

We illustrate the results of these sections with an example taken from Zwietering [28].

Given the RTD of two equal-sized CSTRs in series for a single, second-order, irreversible reaction, compute the reactor effluent concentration for the following cases: segregated flow, maximum mixedness and two ideal CSTRs.

### Solution

The residence-time distribution for two CSTRs in series is given by Equations 8.22 and 8.26 for  $n = 2$ ,

$$p(\theta) = \frac{4\theta}{\tau^2} e^{-2\theta/\tau}$$

$$1 - P(\theta) = (1 + 2\theta/\tau) e^{-2\theta/\tau}$$

in which  $\tau = V_R/Q_f$  and  $V_R$  is the total volume of the CSTRs. The balance for the maximum mixedness case becomes

$$\frac{dc}{d\theta} = \frac{4\lambda}{\tau(\tau + 2\lambda)} (c - c_f) + kc^2$$

Defining dimensionless variables,  $\bar{c} = c/c_0$  and  $x = \theta/\tau$ , the equation becomes

$$\frac{d\bar{c}}{dx} = \frac{4x}{2x + 1} (\bar{c} - 1) + K\bar{c}^2 \quad (8.51)$$

in which  $K = kc_0\tau$ . Notice that all the physical constants of the reactor combine into the single dimensionless constant  $K$ . If we apply the zero slope condition at  $\lambda = \infty$ , we obtain the quadratic equation

$$2(\bar{c}_\infty - 1) + K\bar{c}_\infty^2 = 0$$

which can be solved for  $\bar{c}_\infty$ . Again we have an equation on a semi-infinite interval, which we can transform via

$$z = \frac{1}{1+x}, \quad x = \frac{1-z}{z}$$

in which  $x \in (\infty, 0)$  is transformed to  $z \in (0, 1)$ . The transformed derivative satisfies

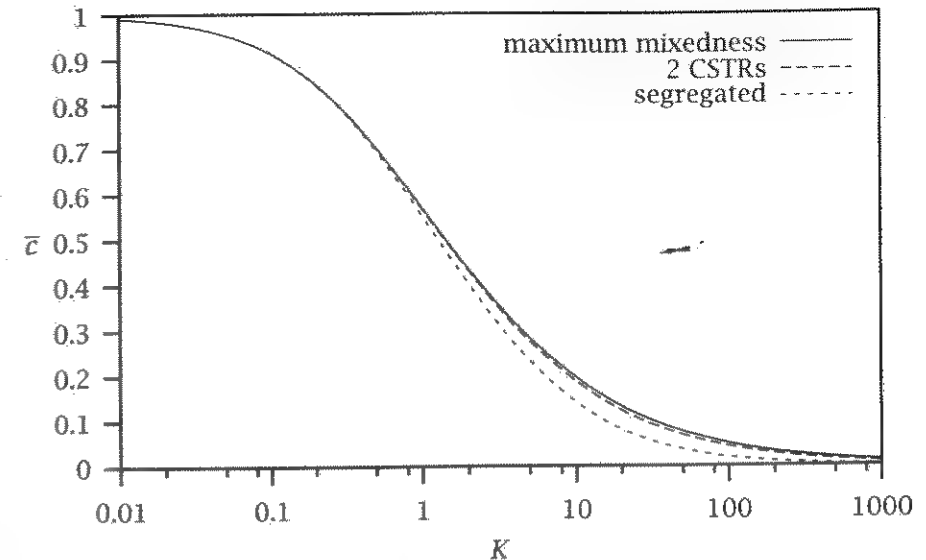
$$\frac{d\bar{c}}{dx} = \frac{d\bar{c}}{dz} \frac{dz}{dx} = -z^2 \frac{d\bar{c}}{dz}$$

so the final differential equation is

$$\frac{d\bar{c}}{dz} = -\frac{1}{z^2} \left[ \frac{4(1-z)/z}{2(1-z)/z + 1} (\bar{c} - 1) + K\bar{c}^2 \right]$$

$$\bar{c}(0) = \bar{c}_\infty$$

The effluent of the maximum mixed reactor is given by the solution  $\bar{c}(z)$  at  $z = 1$ . Figure 8.19 displays the solution to this differential equation for a range of  $K$  values.  $\square$

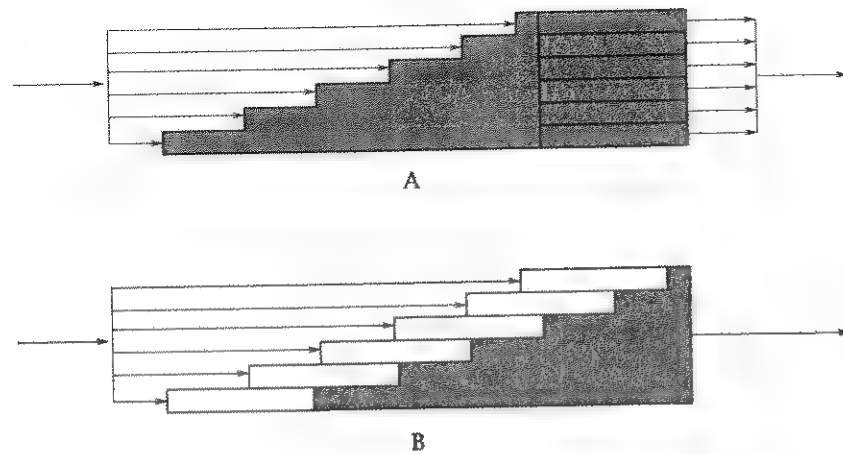


**Figure 8.19:** Dimensionless effluent concentration  $\bar{c} = c/c_0$  versus dimensionless rate constant  $K = k\tau c_0$  for second-order reaction; the RTD for all cases is given by 2 CSTRs in series.

**Intermediate conditions of mixing.** Weinstein and Adler [25] also proposed an interesting general conceptual mixing model by allowing a general mixing pattern between the various tubes as depicted in Figure 8.16.C.

The segregated reactor depicted in Figure 8.15 and Figure 8.16.A is sometimes referred to as late mixing or mixing as late as possible. The material remains segregated until it reaches the common exit where the segregated streams are finally mixed in the reactor effluent. The situation depicted by the maximum mixedness reactor of Figure 8.17 and Figure 8.16.B is sometimes called early mixing. The material is mixed at the earliest possible times as it travels down the tubes; no segregated streams remain to be combined at the reactor exit.

Consider again the two reactors in Example 8.2. The conceptual mixing pattern is sketched in Figure 8.20. The reactors have identical RTDs. Comparing these two reactor configurations, the reactor with the CSTR preceding the PFR is in the condition of maximum mixedness



**Figure 8.20:** CSTR followed by PFR (A) and PFR followed by CSTR (B) as examples of complete and partial mixing; RTDs for the two configurations are equal.

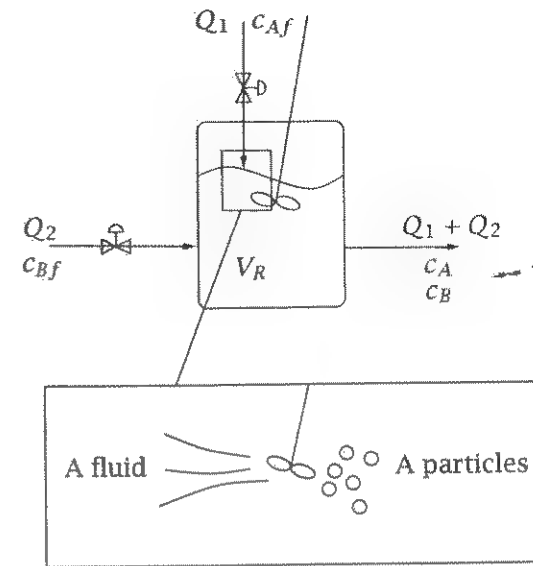
because the CSTR is the condition of maximum mixedness and the feed to the PFR is therefore well mixed, so the different portions of the RTD in the PFR section have identical compositions, and could be considered well mixed or segregated. The PFR preceding the CSTR is not in the condition of maximum mixedness, nor is it segregated. As shown in Figure 8.20, it displays an intermediate state of mixing, similar to case C in Figure 8.16. We show in Section 8.4 that because the reaction rate is second order, complete mixing gives the lowest conversion possible consistent with the given RTD. This conclusion is consistent with the calculation performed in Example 8.2 in which the CSTR-PFR arrangement had lower conversion than the PFR-CSTR.

### 8.3.3 Mass Transfer and Limits of Reactor Mixing

Consider the following physical picture to help in our understanding of segregated flow and maximum mixedness. Figure 8.21 shows the classic situation in which we mix two liquid-phase feed streams in a stirred tank for the second-order reaction



We model the action of the stirrer as shearing the fluid A stream into small, uniformly sized "particles" of component A dispersed in the con-



**Figure 8.21:** Adding two liquid-phase feed streams to a stirred tank; the stirrer is modeled as shearing the liquid A feed stream into small, uniformly sized particles of A and randomly distributing them in the continuous phase containing B.

tinuous phase containing component B dissolved in a solvent. The size of the A "particles" is one measure of how well the stirrer is working. This physical picture, although idealized, is motivated by several types of real reactors, such as suspension and emulsion polymerization reactors. Ottino provides a well-illustrated discussion of the detailed results of fluid shear [20, pp.1-17]. We assume these "particles" of component A move rapidly about the reactor with the fluid flow. We therefore have an ideal CSTR residence-time distribution; if we inject tracer with the A stream or the B stream, we would see the classic step response for the CSTR. In other words, the macromixing is excellent, and determining the residence-time distribution would not indicate anything except an ideally mixed reactor.

Now we model the micromixing. Let the mass transfer between the particles and the continuous phase be described by a mass-transfer coefficient, so the mass balance for components A and B inside the

Parameter	Value	Units
$k$	1	L/mol·min
$k_{mA}$	$1.67 \times 10^{-4}$	cm/min
$k_{mB}$	$1.67 \times 10^{-4}$	cm/min
$\alpha = Q_1/Q_2$	1	
$c_{Af}$	1	mol/L
$c_{Bf}$	1	mol/L
$\bar{\theta} = V_R/(Q_1 + Q_2)$	10	min

**Table 8.2:** Mass-transfer and kinetic parameters for micromixing problem.

particles is given by

$$\begin{aligned} V \frac{dc_A}{d\theta} &= k_{mA}(\bar{c}_A - c_A)S - kc_A c_B V & c_A(0) &= c_{Af} \\ V \frac{dc_B}{d\theta} &= k_{mB}(\bar{c}_B - c_B)S - kc_A c_B V & c_B(0) &= 0 \end{aligned} \quad (8.53)$$

in which  $\theta$  is the time the particle has been in the reactor,  $V$  and  $S$  are the particle volume and area, respectively, and  $k_{mA}$  and  $k_{mB}$  are the A and B mass-transfer coefficients. The variables  $\bar{c}_A$  and  $\bar{c}_B$  are the continuous-phase concentrations of A and B. The initial conditions follow from the fact that the particles are initially formed from the pure A feed stream. Only as  $\theta$  increases do they have time to communicate with the continuous phase. To determine the A and B concentrations in the continuous phase, we write the overall, steady-state mass balances for both components

$$\begin{aligned} 0 &= Q_1 c_{Af} - Q_1 \int_0^\infty c_A(\theta) p(\theta) d\theta - Q_2 \bar{c}_A - \frac{V_R}{1+\alpha} \left[ \alpha \int_0^\infty kc_A c_B p(\theta) d\theta + k \bar{c}_A \bar{c}_B \right] \\ 0 &= Q_2 c_{Bf} - Q_1 \int_0^\infty c_B(\theta) p(\theta) d\theta - Q_2 \bar{c}_B - \frac{V_R}{1+\alpha} \left[ \alpha \int_0^\infty kc_A c_B p(\theta) d\theta + k \bar{c}_A \bar{c}_B \right] \end{aligned} \quad (8.54)$$

We use orthogonal collocation on  $z = \theta/(1+\theta)$  to solve Equations 8.53 simultaneously with Equations 8.54 [24]. Orthogonal collocation is described briefly in Appendix A. The kinetic and mass-transfer parameters are given in Table 8.2. We compute the total A and B concentration

in the effluent by summing over both particle and continuous phases

$$\begin{aligned} c_{At} &= \frac{\alpha}{1+\alpha} \int_0^\infty c_A(\theta) p(\theta) d\theta + \frac{1}{1+\alpha} \bar{c}_A \\ c_{Bt} &= \frac{\alpha}{1+\alpha} \int_0^\infty c_B(\theta) p(\theta) d\theta + \frac{1}{1+\alpha} \bar{c}_B \end{aligned}$$

We next study the effect of particle size. Figure 8.22 shows  $c_A(\theta)$  for particle sizes ranging from 0.1  $\mu\text{m}$  to 1.0 cm. We see that if the stirrer is able to produce A particles of 1.0  $\mu\text{m}$  or less, then the reactor is essentially in the state of maximum mixedness, or, equivalently, operates as an ideally mixed CSTR. At the other extreme, if the A particles are larger than about 1.0 mm, then the reactor operates essentially as a segregated-flow reactor. Segregated flow essentially reduces the reaction rate to zero because the A and B species cannot come into contact.

Figure 8.23 provides a detailed look inside the particles for  $r = 1, 10$  and 100  $\mu\text{m}$ . For  $r = 1 \mu\text{m}$ , the A and B concentrations in the particles rapidly change from the feed values to the continuous phase values as they spend time in the reactor. This equilibration with the continuous phase is rapid because the particles are small, the total surface area and rate of mass transfer are therefore large. This case is close to maximum mixedness. For  $r = 100 \mu\text{m}$ , the particles are 100 times larger, and the total surface area and rate of mass transfer are small. Therefore, these particles remain at the inlet feed conditions for a large time. They are washed out of the reactor before they can produce hardly any reaction rate. This case corresponds to essentially complete segregation.

Summarizing, this example is instructive for two reasons. First the residence-time distribution corresponds to a perfect CSTR regardless of particle size. Residence-time distribution measures the reactor macromixing, which is excellent. The particle size governs the micromixing. Small particles have large mass-transfer rates and equilibrate with the continuous phase and the particles in the reactor with different ages leading to the case of maximum mixedness. Large particles have small mass-transfer rates and do not exchange much material with the continuous phase nor therefore with particles of other ages. This case corresponds to segregated flow, which leads to essentially zero rate of reaction. Particles of intermediate size then describe the reactors in intermediate states of mixing.



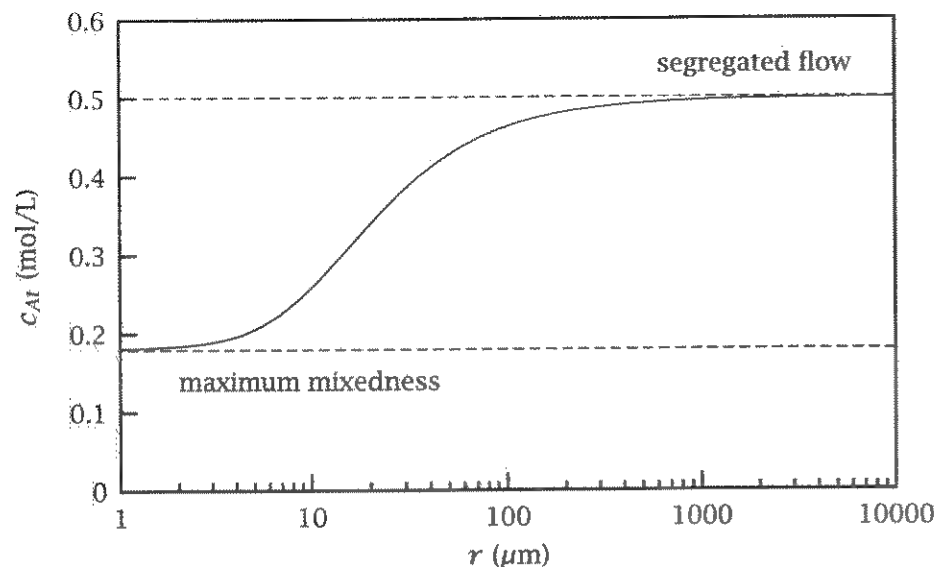


Figure 8.22: Total concentration of A in the reactor effluent versus particle size.

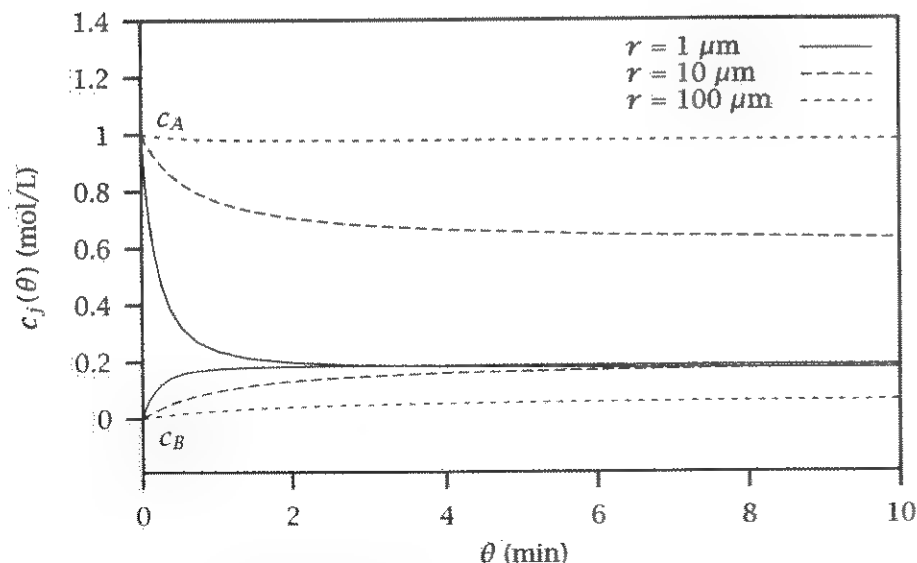


Figure 8.23: Particle concentrations of A and B versus particle age for three different-sized particles.

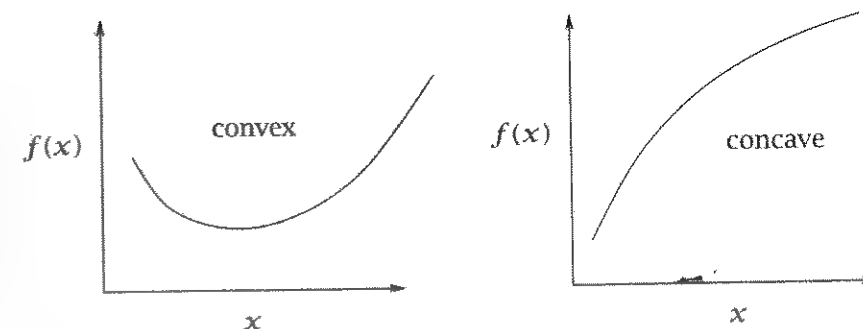


Figure 8.24: Differentiable convex and concave functions.

## 8.4 Limits of Reactor Performance

### 8.4.1 A Single Convex (Concave) Reaction Rate

To generalize the results of Examples 8.2 and 8.3, we define convex and concave functions. As presented in the introductory calculus course, the simplest version pertains to functions having at least two derivatives. In that case, a function is convex (concave upward) if its second derivative is everywhere greater than or equal to zero. A function is concave (concave downward) if its second derivative is everywhere less than or equal to zero, as shown in Figure 8.24

$$\frac{d^2 f(x)}{dx^2} \geq 0, \quad f \text{ convex}$$

$$\frac{d^2 f(x)}{dx^2} \leq 0, \quad f \text{ concave}$$

For example, the  $n$ th-order reaction-rate expression  $r = c^n$ , is *convex* if  $n \geq 1$  and *concave* if  $n \leq 1$ . Note that first-order rate functions are both convex and concave.

The general result for the single reaction is

Given a single reaction with convex (concave) reaction rate expression, the highest (lowest) conversion for a given RTD is achieved by the segregated reactor and the lowest (highest) conversion is achieved by the maximally mixed reactor.

This nonobvious result is a significant generalization of the numerical Examples 8.2 and 8.3, and Exercise 8.6, and requires justification. The argument presented next first appeared in Chauhan et al. [5]; Nauman and Buffham [18] also provide a detailed discussion.

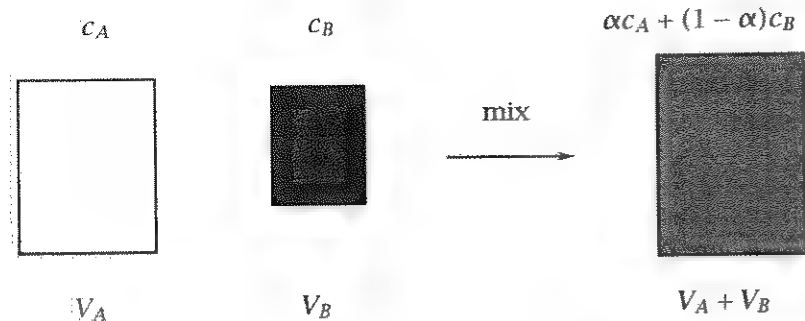


Figure 8.25: Two volume elements before and after mixing.

**Step 1.** To start, consider the two volume elements shown in Figure 8.25. Note that in this discussion  $c_A$  and  $c_B$  represent concentration of the *same reactant species* in volume elements or tubes A and B. When the volume elements are segregated the total reaction rate  $r_s$  is simply

$$r_s = r(c_A)V_A + r(c_B)V_B$$

so that the segregated rate per volume is

$$r_s = \alpha r(c_A) + (1 - \alpha)r(c_B), \quad 0 \leq \alpha \leq 1$$

in which  $\alpha$  is the volume fraction of element A

$$\alpha = \frac{V_A}{V_A + V_B}$$

On the other hand, if we mix the contents, the concentration is

$$c_m = \frac{c_A V_A + c_B V_B}{V_A + V_B} = \alpha c_A + (1 - \alpha)c_B$$

The total reaction rate per volume after mixing is therefore

$$r_m = r(c_m)$$

As shown in Figure 8.26, for all  $c_A$ ,  $c_B$  and  $\alpha$ , if we mix the two volume elements, we lower the overall reaction rate. The opposite conclusion applies if we have a concave rate expression. The rate of the mean  $r_m$  is less than the mean of the rate  $r_s$  for convex reactions, or

$$r(\alpha c_A + (1 - \alpha)c_B) \leq \alpha r(c_A) + (1 - \alpha)r(c_B), \quad \text{all } c_A, c_B, 0 \leq \alpha \leq 1$$

This result is the key to understanding what happens in the general reactor. In fact, this statement can be taken as the definition of convexity (Exercise 8.13).

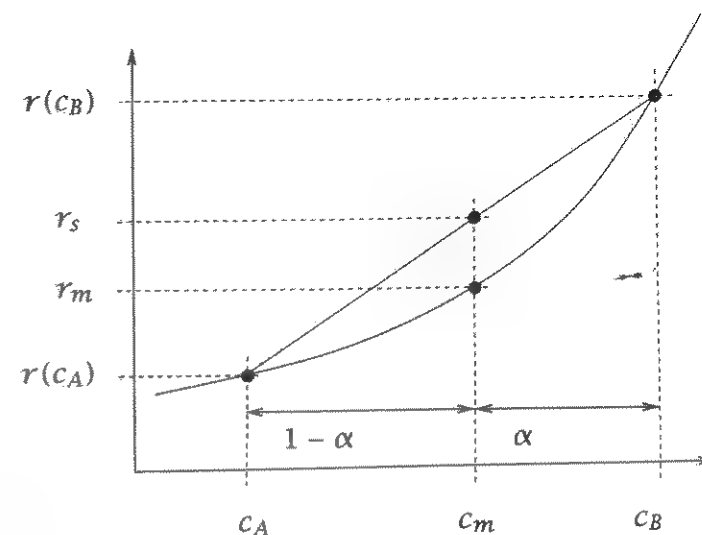


Figure 8.26: Convex rate expression and the effect of mixing; rate of the mean ( $r_m$ ) is less than the mean of the rate ( $r_s$ ).

**Step 2.** Now consider two tubes as shown in Figure 8.27, which we may choose to mix or maintain segregated as material proceeds down their lengths. Again assume a single reaction takes place and the reaction-rate expression is a convex function of a single reactant species. Without loss of generality assume the stoichiometric coefficient for the limiting species is negative one. For constant density, the material balances for the segregated tubes are

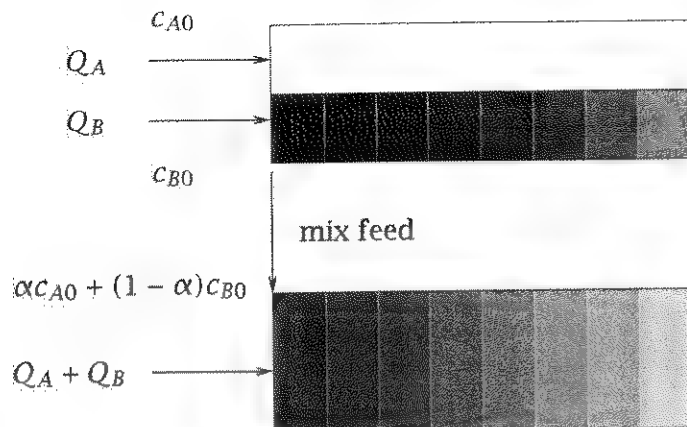
$$\begin{aligned} \frac{dc_A}{d\theta} &= -r(c_A), & c_A(0) &= c_{A0} \\ \frac{dc_B}{d\theta} &= -r(c_B), & c_B(0) &= c_{B0} \end{aligned}$$

in which  $\theta = V/Q_f$ . We can track the mean concentration for the segregated case  $c_s$  by simply summing the molar flows for tubes A and B divided by the total flow

$$c_s = \alpha c_A + (1 - \alpha)c_B \quad (8.55)$$

in which  $\alpha$  is now the flowrate fraction in tube A

$$\alpha = \frac{Q_A}{Q_A + Q_B}$$



**Figure 8.27:** Two tubes before and after mixing the entering feed; averaging the two segregated tubes produces  $c_s$ ; the mixed feed tube produces  $c_m$ .

We also can write a differential equation for  $c_s$  by simply differentiating Equation 8.55

$$\frac{dc_s}{d\theta} = -[\alpha r(c_A) + (1 - \alpha)r(c_B)], \quad c_s(0) = \alpha c_{A0} + (1 - \alpha)c_{B0} \quad (8.56)$$

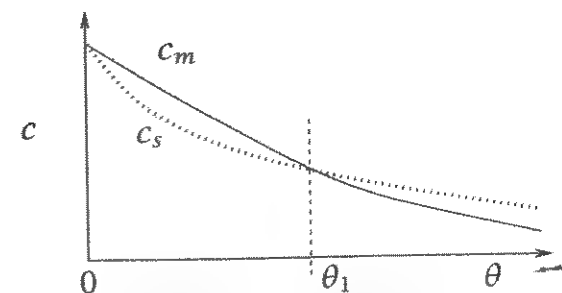
Consider now the mixed case. If the tubes are mixed at some point, which we may call  $\theta = 0$ , then the material balance for the concentration after that point is

$$\frac{dc_m}{d\theta} = -r(c_m), \quad c_m(0) = \alpha c_{A0} + (1 - \alpha)c_{B0} \quad (8.57)$$

Our goal now is to show  $c_m \geq c_s$  for all reactor positions,  $\theta$ , and all feed concentrations and flowrates,  $c_{A0}$ ,  $c_{B0}$  and  $\alpha$ . We know at  $\theta = 0$

$$\frac{dc_m}{d\theta} = -[r(\alpha c_{A0} + (1 - \alpha)c_{B0})] \geq -[\alpha r(c_A) + (1 - \alpha)r(c_B)] = \frac{dc_s}{d\theta}$$

If the initial derivatives have this relationship we know, for at least some small distance down the tube,  $c_m \geq c_s$  as shown in Figure 8.28. How do we know, however, that the curves do not cross each other at some later time? Assume this crossing can happen as shown in Figure 8.28, and we establish a contradiction. Let  $\theta_1$  be the first such crossing time. At  $\theta_1$ ,  $c_A$  and  $c_B$  have some well-defined values and



**Figure 8.28:** Mean segregated and mixed concentrations versus  $\theta$ ; curves crossing at  $\theta_1$  is a contradiction.

$c_s = \alpha c_A + (1 - \alpha)c_B$ . We have assumed that  $c_m = c_s$  at  $\theta_1$  so the differential equation for  $c_m$ , Equation 8.57, gives

$$\frac{dc_m}{d\theta} = -[r(\alpha c_A + (1 - \alpha)c_B)], \quad \theta = \theta_1$$

The differential equation for  $c_s$  still applies and Equation 8.56 gives

$$\frac{dc_s}{d\theta} = -[\alpha r(c_A) + (1 - \alpha)r(c_B)], \quad \theta = \theta_1$$

Comparing the right-hand sides of these two differential equations and using the convexity of  $r(c)$ , we conclude

$$\frac{dc_m}{d\theta} \geq \frac{dc_s}{d\theta}, \quad \theta = \theta_1$$

But this relationship contradicts the assumption that the  $c_s$  and  $c_m$  curves cross each other. Therefore there can be no time  $\theta_1$  at which the curves cross and we conclude

$$c_m(\theta) \geq c_s(\theta), \quad \text{all } \theta$$

This argument and result apply equally well for all  $c_{A0}$ ,  $c_{B0}$  and  $\alpha$ .

**Step 3.** Finally, consider a segregated reactor with arbitrary residence-time distribution as depicted in Figure 8.16.C. We select any pair of tubes, mix them, make the same argument that we made in Step 2, and replace the segregated tubes with mixed tubes that achieve lower conversion than the original system. We continue in this fashion, and after we pairwise mix all the segregated tubes with mixed tubes, we achieve the reactor of maximum mixedness in Figure 8.16.B. and the lowest possible conversion. Note this pairing and mixing procedure does not affect the RTD.

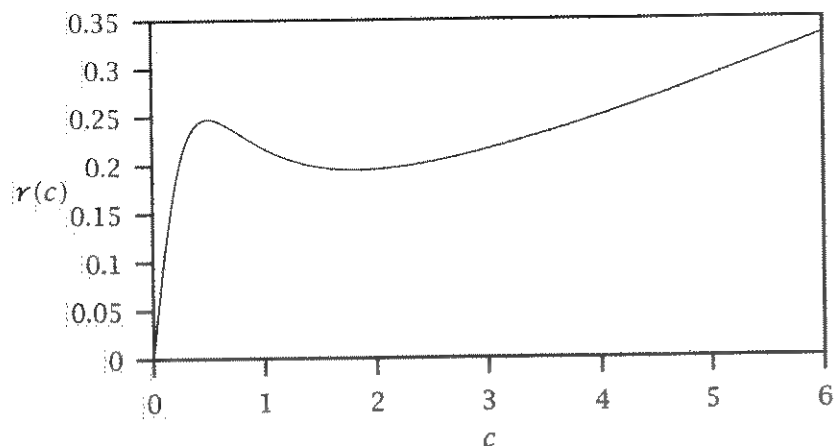


Figure 8.29: Reaction rate versus concentration of limiting reactant; rate expression is neither convex nor concave.

#### 8.4.2 The General Case

One might expect that the limits of reactor mixing determine directly the limits of reactor performance for more general kinetic schemes as well as the single convex or concave rate expression of the last section. Unfortunately nature is more subtle. We present next an example that dispels this notion, and then discuss what is known about the limits of reactor performance. This example is based on one presented by Glasser, Hildebrandt and Godorr [11]. Levenspiel [17] shows how to find the optimal reactor configuration for this type of example.

##### Example 8.4: Optimal is neither segregated nor maximally mixed

Consider the rate expression

$$r(c) = \frac{c}{1 + 5c^2} + 0.05c \quad (8.58)$$

which is plotted in Figure 8.29. For a feed concentration of 5, find the combination of CSTRs and PFRs that achieve 95% conversion with the smallest total reactor volume. Determine the RTD for this reactor configuration. What conversion is achieved in a segregated reactor with this RTD? What conversion is achieved in a maximally mixed reactor with this RTD?

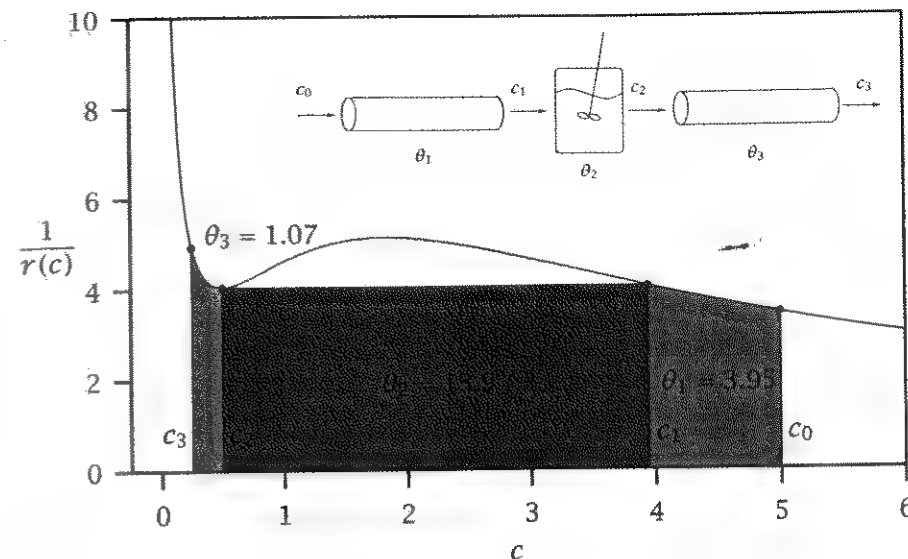


Figure 8.30: Inverse of reaction rate versus concentration; optimal sequence to achieve 95% conversion is PFR-CSTR-PFR.

#### Solution

As we mentioned in Chapter 4, the smallest volume can be achieved with a series combination of CSTRs and PFRs. First we plot the inverse of the rate as shown in Figure 8.30. Then we find any minima in the inverse rate function and construct CSTRs from those values until we intersect the inverse rate curve. In the remaining sections of the curve where the inverse rate is a decreasing function of concentration, we use PFRs. Examining the plotted  $1/r$  function in Figure 8.30, we see the optimal configuration is a PFR-CSTR-PFR; this configuration is sketched in Figure 8.30. We can calculate the sizes of the reactors as follows. We know from the problem statement that  $c_0 = 5$ ,  $c_3 = 0.25$ . We next find the point where  $dr(c)/dc = 0$ . Notice these are also the places where  $d(1/r(c))/dc = 0$ . Setting the derivative of Equation 8.58 to zero gives a quadratic equation with two roots: 0.501 and 1.83. We choose the one corresponding to the minimum in  $1/r$ , which gives

$$c_2 = 0.501, \quad 1/r(c_2) = 4.045$$

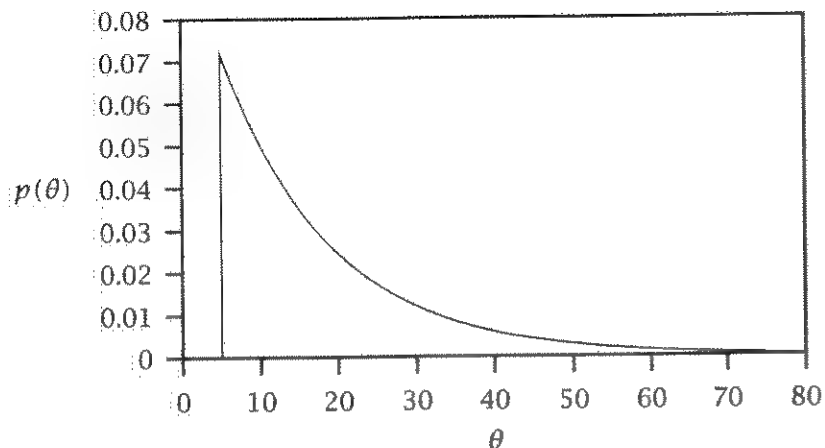


Figure 8.31: Residence-time distribution for the optimal reactor configuration.

Next we find the concentration  $c_1$  such that  $1/r(c_1) = 1/r(c_2)$ . This results in a cubic equation, which we solve numerically. Then the residence time is given by  $\theta_2 = 1/r(c_2)(c_1 - c_2)$  which gives

$$c_1 = 3.94, \quad \theta_2 = 13.9$$

To size the PFRs we simply use the PFR design equation and obtain

$$\theta_1 = -\int_{c_0}^{c_1} \frac{1}{r(c)} dc = 3.95, \quad \theta_3 = -\int_{c_2}^{c_3} \frac{1}{r(c)} dc = 1.07$$

These results are displayed in Figure 8.30. Because we have a series of CSTRs and PFRs, we can write the RTD immediately

$$p(\theta) = \frac{1}{\theta_2} \exp\left[-\frac{\theta - (\theta_1 + \theta_3)}{\theta_2}\right] H(\theta - (\theta_1 + \theta_3))$$

which is plotted in Figure 8.31. With the RTD in hand, we can compute both the maximally mixed, Equation 8.50, and segregated, Equation 8.44, reactor cases. The results of those two calculations are summarized in the following table

Reactor	Conversion
optimal	0.95
segregated	0.68
maximally mixed	0.75

We see that these two mixing limits do *not* bound the performance of the actual reactor sequence with the given RTD. In fact, they are off by more than 20%. Even with a single reaction, if the rate expression is neither convex nor concave, we cannot bound the performance of an actual reactor between the segregated and maximally mixed mixing limits.  $\square$

**The attainable region.** The primary use of the classical mixing models, such as the segregated reactor and the maximally mixed reactor, is to build insight into the effects of mixing on reactor behavior under the constraint of a fixed, and presumably measurable, RTD. As we have seen in Example 8.4, however, if we are mainly interested in determining bounds on achievable reactor states (conversions, yields, etc.), these simple mixing models are insufficient. In this section we would like to provide a brief overview of what is known about finding sharp bounds on reactor performance. The general problem can be defined in this way.

Given a feed stream of known composition and a set of chemical reactions with known rate expressions, determine the set of all possible steady-state species concentrations that can be achieved by *any* combination of chemical reactors.

This set was proposed by Horn almost 40 years ago and named the **attainable region** [13]. Because the set is defined for *all possible reactor combinations*, it seems conceptually difficult to formulate a procedure by which we can calculate this set. We should also note that by considering all reactor combinations, we are also considering all possible residence-time distributions, which is a considerable generalization from the single RTD that was considered in the mixing discussion in previous sections. In spite of the seeming difficulty in finding the attainable region, excellent, recent research progress has been made. Feinberg provides a nice summary overview of the history and many recent developments [7].

Glasser and Hildebrandt revived recent interest in this problem [12, 11]. Feinberg and Hildebrandt [9] characterized the boundary of the attainable region, which is of importance because it bounds the possible steady-state concentrations. They showed, for example, that the extreme points of the attainable region boundary are made up entirely of plug-flow reactor trajectories. They also showed that combinations

of PFRs, CSTRs, and what are called differential side-stream reactors (PFRs with addition of feed along the side of the tube), provide the means to find all of the attainable region extreme points.

In addition to properties and conceptual characterization of the attainable region, researchers have proposed computational procedures to approximate the attainable region and solve reactor synthesis problems. Some of these are based on proposing a superstructure of reactor types and numbers, and optimizing an objective function among the possible reactors [14]. Because the superstructure does not enumerate all possibilities, the solution may not be close to the true attainable region. A person skilled in reactor design may be able to choose reactor numbers and types well and overcome this general difficulty on specific reaction networks of interest.

Some computational methods are based on finding the boundary of the attainable region using the reactor types that characterize the attainable region extreme points. Hybrid methods involving superstructures and geometric considerations have also been proposed [15].<sup>2</sup>

Manousiouthakis has recently proposed an infinite dimensional state-space approach (IDEAS) that requires only PFRs, CSTRs and mixing. The advantage of this approach is that one solves only convex, linear optimization problems. The disadvantage is the problems are infinite dimensional and require a finite dimensional approximation for calculation. A full analysis of the convergence properties of the finite dimensional approximation is not yet available, but the approach shows promise on numerical examples [4].

If we wish to allow separation as well as chemical reaction, and almost all industrial designs would fall into this category, then the problem switches from a pure reactor synthesis problem to a reactor-separator synthesis problem. The CSTR equivalence principle of Chapter 4 is an example of the strikingly simple and general results that recently have been achieved for the reactor-separator synthesis problem [8].

Forecasting is always risky business, but given the rapid pace of recent progress, it seems likely that new and highly useful results on pure reactor and reactor-separator synthesis problems will be forthcoming. These ideas and results may have immediate industrial impact, and certainly fall within the scope of the course in reactor analysis and design.

<sup>2</sup>Biegler, Grossmann and Westerberg provide an overview in Chapter 13 of their text, *Systematic Methods of Chemical Process Design*, Prentice Hall PTR, Upper Saddle River, New Jersey, 1997.

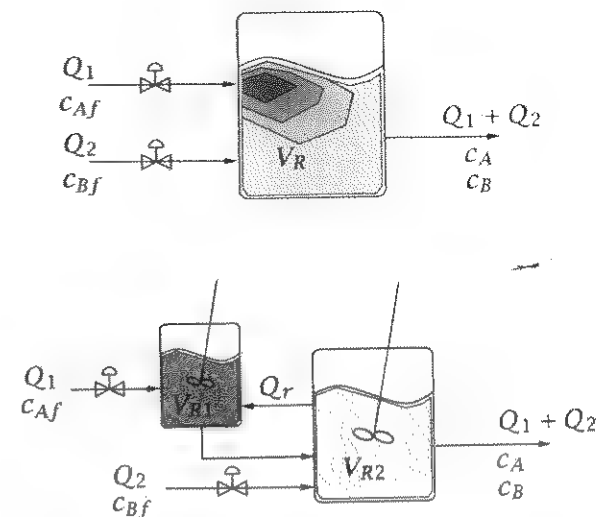


Figure 8.32: Imperfect mixing (top reactor) leads to formation of an A-rich zone, which is modeled as a small CSTR feeding a second CSTR (bottom two reactors).

## 8.5 Examples in Which Mixing is Critical

Returning to the topic of mixing, we would like to close the chapter by presenting a few more chemical mechanisms for which reactor mixing can play a critical role.

### Example 8.5: Mixing two liquid-phase streams in a stirred tank

A classic mixing problem arises when we must bring two liquid-phase feed streams together to perform the second-order reaction



in the presence of the undesirable side reaction



If the rate of the second degradation reaction is fast compared to the rate of mixing of the two feed streams, we can anticipate problems. To understand what happens in this situation, consider the mixing model depicted in Figure 8.32. Component A is assumed to be the limiting

Parameter	Value	Units
$k_1$	1	$\text{min}^{-1}$
$k_2$	2	$\text{L/mol} \cdot \text{min}$
$n$	2	
$\theta_1 = V_{R1}/Q_2$	1	min
$\theta_2 = V_{R2}/Q_2$	2	min
$\theta = V_R/Q_2$ $= \theta_1 + \theta_2$	3	min
$\alpha = Q_1/Q_2$	0.1	
$\rho = Q_r/Q_2$	varies	

**Table 8.3:** Reactor and kinetic parameters for feed-mixing example.

reactant. It is added at a low flowrate to a CSTR that contains an excess of reactant B. In the top figure we depict the ideal-mixing case in which the rate of mixing is arbitrarily fast compared to the rate of either reaction. But this ideal mixing may be impossible to achieve if the reaction rates are reasonably large. So in the bottom figure, we model the formation of an A-rich zone near the feed entry point. This small CSTR exchanges mass with a larger reactor that contains the excess of reactant B. We can vary the recycle flowrate between the two CSTRs,  $Q_r$ , and the sizes of the two reactors,  $V_{R1}$  and  $V_{R2}$ , to vary the degree of mixing. For large  $Q_r$ , we expect the two-reactor mixing model to approach the single, ideally mixed CSTR.

As discussed in Chapter 4, the conversion and yield are the usual quantities of interest in competing parallel reactions of the type given in Reactions 8.59 and 8.60. We assume the density of this liquid-phase system is constant, and define the overall conversion of reactant A and yield of desired product C as follows:

$$x_A = \frac{Q_1 c_{Af} - (Q_1 + Q_2) c_A}{Q_1 c_{Af}} \quad y_C = \frac{(Q_1 + Q_2) c_C}{Q_1 c_{Af} - (Q_1 + Q_2) c_A}$$

Given the parameters and rate constants in Table 8.3, calculate  $x_A$  and  $y_C$  versus  $Q_r$  for the two-reactor mixing model shown in Figure 8.32, and compare the result to the single, well-mixed reactor. Then calculate the residence-time distribution  $P(\theta)$  for tracer injected with the A feed stream for the two models. Discuss whether or not the residence-time distribution is a reliable indicator for problems with yield in the imperfectly mixed reactor.

### Solution

The steady-state mass balance for the single, well-mixed CSTR is

$$0 = Q_1 c_{Af} - (Q_1 + Q_2) c_A - (k_1 c_A c_B + k_2 c_A^n) V_R$$

$$0 = Q_2 c_{Bf} - (Q_1 + Q_2) c_B - k_1 c_A c_B V_R$$

Defining the following parameters

$$\alpha = \frac{Q_1}{Q_2} \quad \theta = \frac{V_R}{Q_2} \quad \rho = \frac{Q_r}{Q_2}$$

allows us to write these as

$$0 = \alpha c_{Af} - (1 + \alpha) c_A - (k_1 c_A c_B + k_2 c_A^n) \theta$$

$$0 = c_{Bf} - (1 + \alpha) c_B - k_1 c_A c_B \theta$$

We can solve numerically the two equations for the two unknowns  $c_A, c_B$ . The concentration of C in the outflow is determined from the change in the concentration of B,

$$(Q_1 + Q_2) c_C = Q_2 c_{Bf} - (Q_1 + Q_2) c_B$$

Using this relationship and the defined parameters gives for conversion and yield,

$$x_A = \frac{\alpha c_{Af} - (1 + \alpha) c_A}{\alpha c_{Af}} \quad y_C = \frac{c_{Bf} - (1 + \alpha) c_B}{\alpha c_{Af} - (1 + \alpha) c_A}$$

For the two-reactor system, we write mass balances for each reactor. Let  $c_{A1}, c_{A2}, c_{B1}, c_{B2}$  be the unknown A and B concentrations in the two-reactors, respectively. The mass balances are

#### Reactor 1:

$$0 = Q_1 c_{Af} - (Q_1 + Q_r) c_{A1} + Q_r c_{A2} - (k_1 c_{A1} c_{B1} + k_2 c_{A1}^n) V_{R1}$$

$$0 = -(Q_1 + Q_r) c_{B1} + Q_r c_{B2} - k_1 c_{A1} c_{B1} V_{R1}$$

#### Reactor 2:

$$0 = (Q_1 + Q_r) c_{A1} - Q_r c_{A2} - (Q_1 + Q_2) c_{A2} - (k_1 c_{A2} c_{B2} + k_2 c_{A2}^n) V_{R2}$$

$$0 = Q_2 c_{Bf} + (Q_1 + Q_r) c_{B1} - Q_r c_{B2} - (Q_1 + Q_2) c_{B2} - (k_1 c_{A2} c_{B2}) V_{R2}$$



We can summarize this case using the previously defined variables as four equations in four unknowns

$$0 = \alpha c_{Af} - (\alpha + \rho) c_{A1} + \rho c_{A2} - (k_1 c_{A1} c_{B1} + k_2 c_{A1}^2) \theta_1$$

$$0 = -(\alpha + \rho) c_{B1} + \rho c_{B2} - k_1 c_{A1} c_{B1} \theta_1$$

$$0 = (\alpha + \rho) c_{A1} - \rho c_{A2} - (1 + \alpha) c_{A2} - (k_1 c_{A2} c_{B2} + k_2 c_{A2}^2) \theta_2$$

$$0 = c_{Bf} + (\alpha + \rho) c_{B1} - \rho c_{B2} - (1 + \alpha) c_{B2} - (k_1 c_{A2} c_{B2} \theta_2$$

Figures 8.33 and 8.34 show the yield and conversion for the two cases as a function of  $Q_r$ . The conversion is not adversely affected by the poor mixing. In fact, the conversion in the two-reactor system is higher than the single, well-mixed reactor. Notice, however, that at low values of  $Q_r$ , which corresponds to poor mixing at the feed location, the yield changes from *more than 90%* to *less than 15%*. Low yield is a qualitatively different problem than low conversion. If the conversion is low, we can design a separation system to remove the unreacted A and recycle it, or use it as feed in a second reactor. With low yield, however, the A has been irreversibly converted to an undesired product D. The raw material is lost and cannot be recovered. It is important to diagnose the low yield as a reactor mixing problem, and fix the problem at the reactor. A yield loss cannot be recovered by downstream processing.

Next we compute the outcome of injecting a unit step change in a tracer in the A feed stream. We solve the transient CSTR balances and calculate the tracer concentration at the outlet. Because the tracer does not take part in any reactions, this can be done analytically or numerically. The result is shown in Figure 8.35. We see the familiar single CSTR step response. For the two-reactor mixing model, when  $\rho = 0$ , which corresponds to the poorest mixing and lowest yield, the step test does reliably indicate the poor mixing. At the end of this chapter and also in Chapter 9 we show how to use this step response to determine the best value of  $\rho$  to model the mixing. When  $\rho$  is reasonably large,  $Q_r = Q_2$ , and the single CSTR and two-reactor cases have similar yields and step responses.

Notice in all three step responses, the tracer concentration reaches only  $c_{Is} = 0.091 = \alpha / (1 + \alpha)$  because we inject tracer in only one of the two feed streams.  $\square$

This example is one of the classic sets of reactions in which mixing has a significant impact on the reactor performance and the product yield. It deserves careful study because it builds intuition and leads us to ask good questions when confronted with more complex cases.

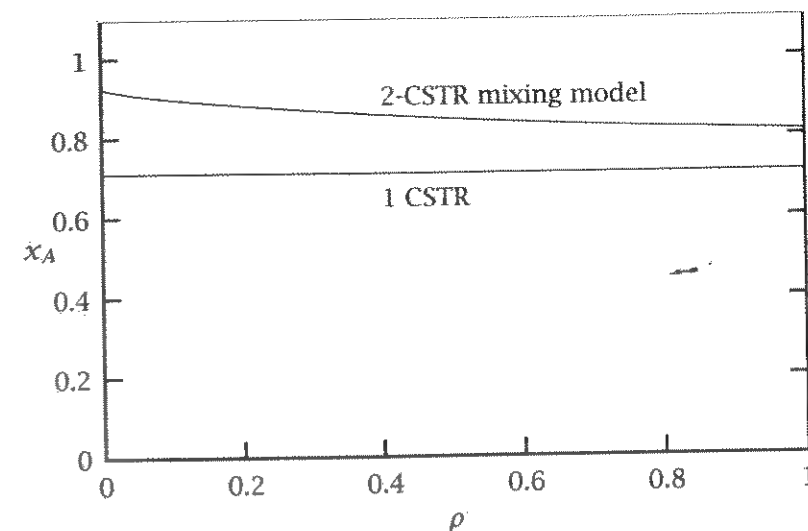


Figure 8.33: Conversion of reactant A for single, ideal CSTR, and as a function of internal flowrate,  $\rho = Q_r/Q_2$ , in a 2-CSTR mixing model.

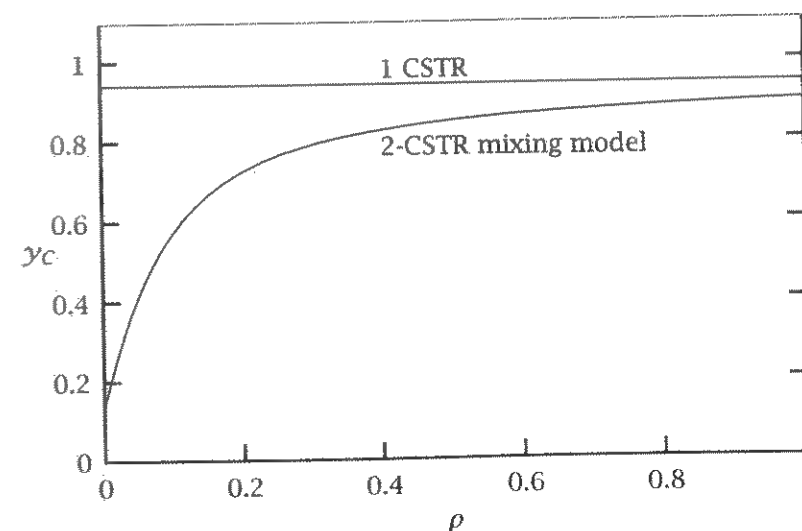


Figure 8.34: Yield of desired product C for single, ideal CSTR, and as a function of internal flowrate,  $\rho = Q_r/Q_2$ , in a 2-CSTR mixing model.

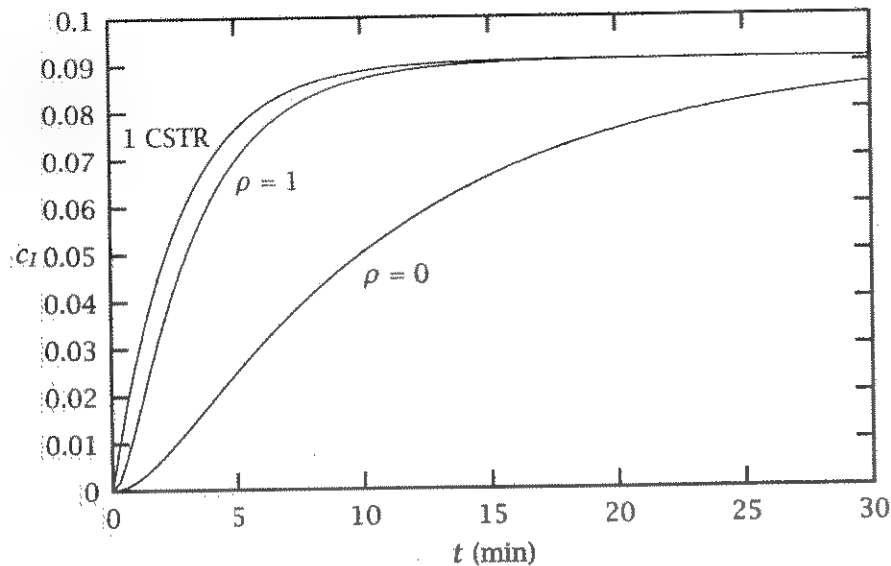
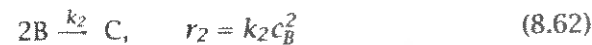


Figure 8.35: Step response for single, ideal CSTR, and 2-CSTR mixing model with  $\rho = 0, 1$ .

For example, Villa et al. [23] discuss similar issues that arise in more complex polymerization reaction engineering problems.

#### Example 8.6: Maximizing yield in dispersed plug flow

Consider the following two liquid-phase reactions in which B is the desired product



The second reaction can represent the first step in a polymerization process of species B, which is undesirable in this case.

Because the second reaction is second order in B, it is desirable to keep the *average* B concentration in the reactor low, to avoid yield losses, but achieve high B concentration near the reactor exit to maximize the production rate. Intuitively the CSTR is a bad choice, because it maintains the same B concentration everywhere in the reactor. A PFR

Parameter	Value	Units
$k_1$	1	$\text{min}^{-1}$
$k_2$	1	$\text{L/mol} \cdot \text{min}$
$c_{Af}$	1	$\text{mol/L}$
$c_{Bf}$	0	
$v$	1	$\text{m/min}$
$l$	0.5	$\text{m}$
$D_l$	varies	$\text{m}^2/\text{min}$

Table 8.4: Parameters for the dispersed PFR example.

should offer higher yield. The B concentration is low near the tube entrance, and increases to its maximum value at the tube exit if we choose the right length or residence time. If we make the tube too long, however, the B is largely converted to C and the yield is again low. In this case, yield is adversely affected by mixing.

Calculate the steady-state conversion of A and yield of B versus PFR length for the kinetic and reactor parameters in Table 8.4. What is an appropriate reactor length to maximize yield of B? Study the effect of dispersion. Approximately how large can the dispersion number be before the advantages of the PFR over the CSTR are lost?

#### Solution

The steady-state mass balances for components A and B are

$$v \frac{dc_A}{dz} - D_l \frac{d^2 c_A}{dz^2} = R_A \quad (8.63)$$

$$v \frac{dc_B}{dz} - D_l \frac{d^2 c_B}{dz^2} = R_B \quad (8.64)$$

in which

$$R_A = -k_1 c_A, \quad R_B = k_1 c_A - 2k_2 c_B^2$$

and we have assumed the dispersion numbers of both species are the same,  $D_{Al} = D_{Bl} = D_l$ . Because the fluid is a liquid, we assume the velocity is constant. We use Danckwerts boundary conditions for both species

$$v c_{jf} = v c_j(0) - D_l \frac{dc_A}{dz}(0), \quad z = 0 \quad (8.65)$$

$$\frac{dc_j}{dz} = 0, \quad z = l \quad (8.66)$$

$j = (A, B)$ . Given the concentrations, and because the flowrate is constant, the conversion and yield are

$$x_A = \frac{c_{Af} - c_A}{c_{Af}} \quad y_B = \frac{c_B}{c_{Af} - c_A}$$

Figures 8.36 and 8.37 show the conversion of A and yield of B versus tube length for a tube designed to maximize the yield of B. A tube length of about 0.5 m is appropriate. As the length increases above this value, the conversion of A increases, but the yield of B drops rapidly, defeating the main purpose of using a PFR. For the kinetic parameters chosen, the CSTR yield can be improved by about 8% with a PFR. As shown in Figure 8.36, the high-dispersion PFR is essentially a CSTR, and achieves  $y_B = 0.79$ . The PFR with  $D = 0.001$  achieves  $y_B = 0.87$ . We see that the dispersion number must be kept less than about 0.1 to maintain this advantage in yield. □

## 8.6 Summary

In this chapter we generalized the two flow assumptions of the idealized reactor models: the perfect mixing assumption of the batch reactor and CSTR, and the plug-flow assumption of the PFR. We defined the residence-time distribution (RTD) of a reactor, and showed how to measure the RTD with simple tracer experiments such as the step test, pulse test and (idealized) impulse test. The RTD gives a rough measure of the flow pattern in the reactor, but it does not determine completely the reactor performance. Indeed, reactors with different flow patterns, and therefore different performances, may have identical RTDs. We showed the CSTR has an exponential RTD. The derivation of the RTD of the CSTR also illustrated the following general principle: given an event with constant probability of occurrence, the time until the next occurrence of the event is distributed as a decreasing exponential function. This principle was used, for example, to choose the time of the next reaction in the stochastic simulations of Chapter 4.

The residence-time distribution of the PFR was shown to be arbitrarily sharp because all molecules spend identical times in the PFR. We introduced the delta function to describe this arbitrarily narrow RTD. We added a dispersion term to the PFR equations to model the spread of the RTD observed in actual tubular reactors. We computed the full, transient behavior of the dispersed plug-flow model, and displayed the evolution of the concentration profile after a step change in the feed concentration.

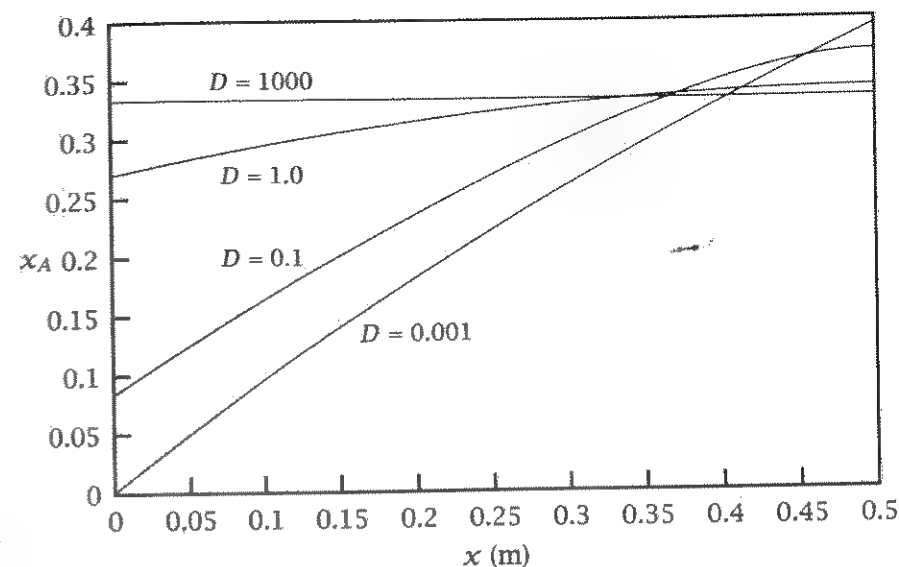


Figure 8.36: Conversion of reactant A versus reactor length for different dispersion numbers.

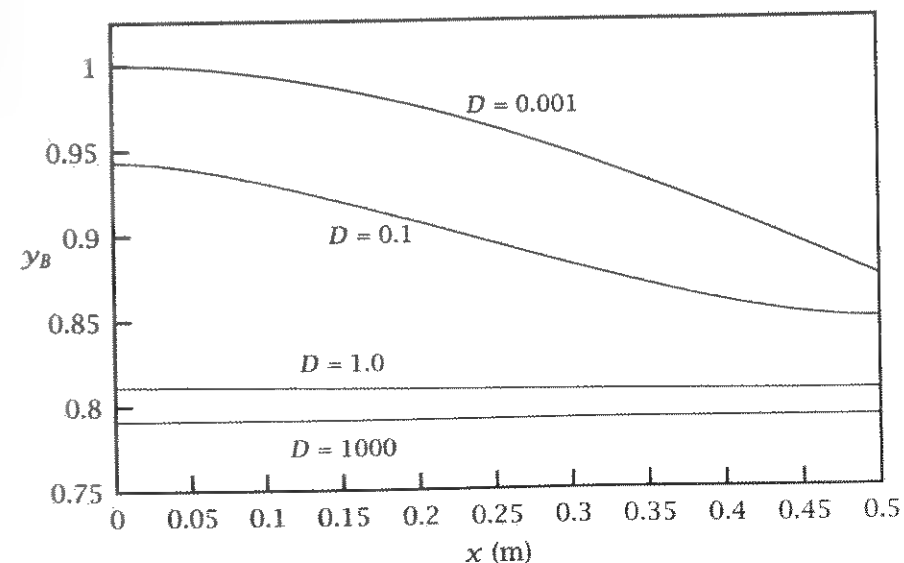


Figure 8.37: Yield of desired product B versus reactor length for different dispersion numbers.

We then examined the limits of reactor mixing consistent with a given RTD. The two limits are segregated flow and maximum mixedness. We showed how a physical process such as mass transfer between a continuous phase and a particle phase can approach segregated flow for large particles (small mass-transfer rates) and can approach maximum mixedness for small particles (high mass-transfer rates).

We also showed that the mixing limits bound the possible reactor behavior for the case of a single, convex reaction-rate expression. For more general reaction networks, however, the mixing limits do not bound the reactor performance. For the general reaction network, recent research on the attainable region has started to shed light on the possible reactor performance. If we consider separation as well as reaction, rather general results such as the CSTR equivalence principle of Chapter 4 have recently been discovered. These topics seem ripe for further research progress.

Next we discussed two contrasting cases in which mixing plays a critical role. In the mixing of two liquid reactants, we showed that formation of a poorly mixed zone can lead to significant yield losses. By contrast, for the kinetics of the second example, good mixing leads to yield losses; in this example the reactor should be designed to approach segregated flow.

Finally, the recent progress in the area of computational fluid dynamics (CFD) gives us reason to believe that direct solution of the equations of motion for the fluid will be a tractable approach for designing reactors and evaluating their performance [1]. It seems reasonable to expect the classical RTD methods and simple flow models to complement the computationally intensive CFD methods. CFD methods may be used to validate simpler mixing models. These validated, simple mixing models may continue to play important roles in reactor analysis, design and optimization.

## Notation

$c_e$	effluent concentration in RTD measurement
$c_f$	feed concentration
$c_j$	concentration of species $j$
$c_m$	concentration in a maximally mixed flow model
$c_s$	concentration in a segregated flow model
$c_\infty$	concentration boundary condition in maximum mixedness model
$D$	dimensionless dispersion number, $D = D_l \tau / l^2$
$D_A$	molecular diffusivity
$D_{jl}$	dispersion coefficient for species $j$
$D_l$	dispersion coefficient
$\text{erf}(x)$	error function, Equation 8.32
$H(x)$	Heaviside or unit step function, Equation 8.17
$k_{mj}$	mass-transfer coefficient
$l$	tubular reactor length
$n$	number of CSTRs in a mixing model
$p(\theta)$	probability that a molecule spends time $\theta$ to $\theta + d\theta$ in reactor, RTD
$P(\theta)$	probability that a molecule spends time zero to $\theta$ in the reactor, integrated form of the RTD
$Pe$	Péclet number, $Pe = \nu l / D_A$
$Q$	volumetric flowrate
$Q_f$	feed volumetric flowrate
$r$	particle radius in mixing model
$\dot{r}$	reaction rate of (single) reaction
$R_j$	production rate of species $j$
$\nu$	fluid axial velocity
$V_R$	reactor volume
$x_j$	molar conversion of component $j$
$y_j$	yield of species $j$
$z$	reactor length variable
$\gamma(n, x)$	incomplete gamma function of order $n$ and argument $x$ , Equation 8.25
$\Gamma(n)$	gamma function of $n$ , Equation 8.24
$\delta(x)$	delta or impulse function, Equations 8.7 and 8.18, and Figure 8.2
$\theta$	residence time of tracer molecule in reactor
$\bar{\theta}$	mean residence time, $\bar{\theta} = \int_0^\infty \theta' p(\theta') d\theta'$
$\lambda$	time-to-go before molecule exits reactor
$\tau$	$V_R / Q_f$

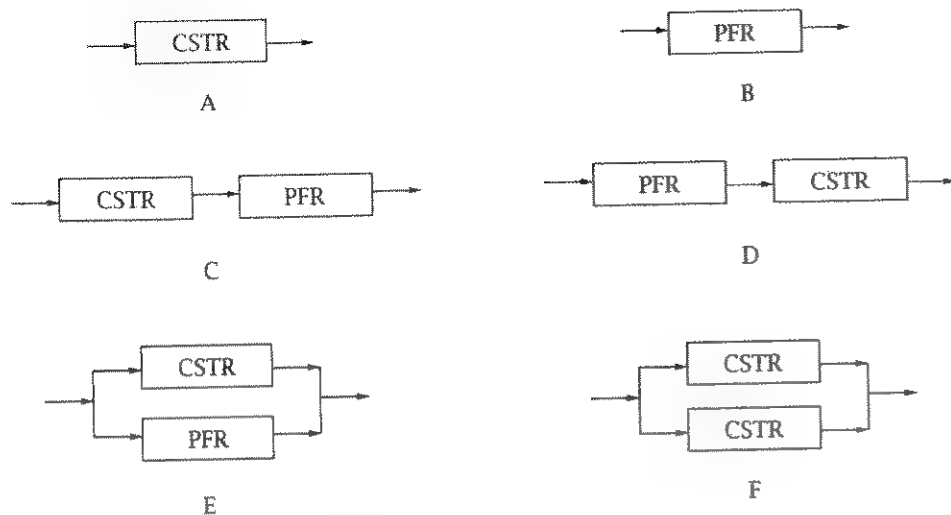


Figure 8.38: Reactor configurations subjected to a step test in tracer concentration.

## 8.7 Exercises

### Exercise 8.1: Configuration to RTD

Make a qualitative sketch of the integrated form of the residence-time distribution  $P(\theta)$  by considering a step change in tracer concentration in each of the reactor configurations shown in Figure 8.38. Be sure to mark the mean of the RTD on your sketches. The total flow into each configuration is the same and each individual reactor has the same volume. Is each of the answers in A-F unique?

### Exercise 8.2: RTD to configuration

Consider the feed and effluent tracer concentrations from a pulse test shown in Figure 8.39. Draw the simplest reactor configuration consisting of PFRs and CSTRs similar to those shown in Figure 8.38 that you would use to model the reactor based on these test data. Determine  $\tau = V_R/Q_f$  for each reactor in your configuration.

Is this configuration unique? Why or why not?

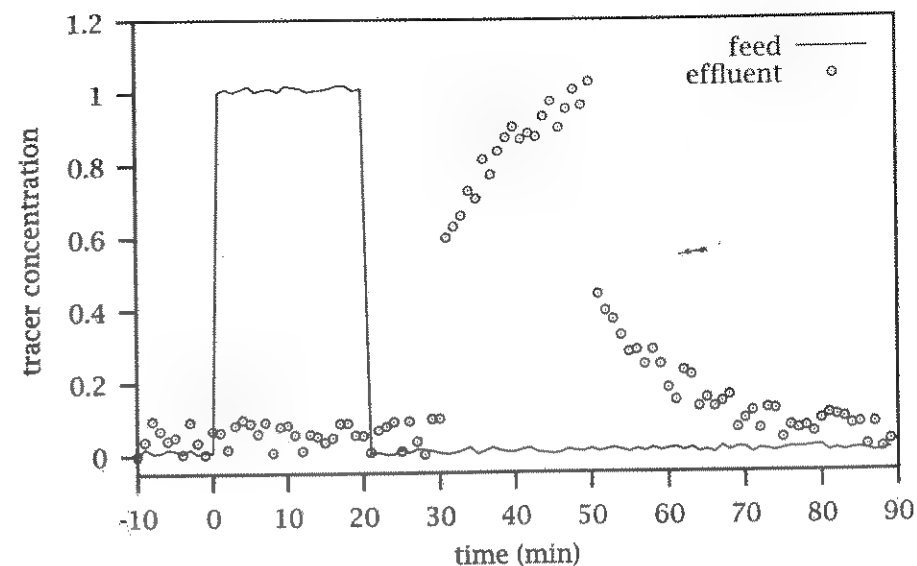


Figure 8.39: Tracer concentrations in the feed and effluent streams versus time.

### Exercise 8.3: More equivalent RTDs with different conversions

Prepare a plot like that in Figure 8.14 for the following reaction rates

$$r = kc, \quad r = kc^{1/2}, \quad r = kc/(1 + kc)$$

Which configuration achieves higher conversion for these cases, CSTR-PFR or PFR-CSTR?

### Exercise 8.4: Deviation variables

Consider again the RTD relationship, Equation 8.3

$$c_e(t) = \int_{-\infty}^t c_f(t') p(t-t') dt'$$

Assume the feed tracer concentration is not initially zero but some steady concentration  $c_0$ . As in the step-response experiment, at time zero the feed is abruptly changed to a new steady value  $c_f \neq c_0$ . Consider a dimensionless deviation variable for effluent tracer concentration.

$$\bar{c}_e(t) = \frac{c_e(t) - c_0}{c_f - c_0}$$

- (a) Show that the results of Section 8.2.2 still apply in this situation. Namely, the RTD is easily determined from the effluent concentration via

$$P(\theta) = \bar{c}_e(\theta)$$

- (b) For a constant-density CSTR, what differential equation and initial condition does the deviation variable  $\bar{c}_e(t)$  satisfy?

### Exercise 8.5: Predicting performance

The residence-time distribution is tabulated below for a flow reactor that has a feed rate of  $16.2 \text{ ft}^3/\text{min}$ . Under the feed conditions ( $c_{Af} = 0.02 \text{ lbmol}/\text{ft}^3$  and  $c_{Bf} = 2.34 \text{ lbmol}/\text{ft}^3$ ) the reaction



is essentially first order in the concentration of A and the rate constant is  $0.23 \text{ min}^{-1}$  at the temperature of the reactor. Determine the effluent concentration of reactant A when the total feed to the reactor is  $16.2 \text{ ft}^3/\text{min}$ .

$\theta$ (min)	$p(\theta)$ ( $\text{min}^{-1}$ )	$\theta$ (min)	$p(\theta)$ ( $\text{min}^{-1}$ )	$\theta$ (min)	$p(\theta)$ ( $\text{min}^{-1}$ )
0	0	18.5	0.0604	37.0	0.0026
3.7	0.0005	22.2	0.0437	40.7	0.0010
7.4	0.0120	25.9	0.0259	44.4	0.0004
11.1	0.0415	29.6	0.0133	48.1	0.0001
14.8	0.0627	33.3	0.0061	51.8	0.0000

### Exercise 8.6: Three CSTRs and maximum mixedness

- (a) Given feed flowrate  $Q$  and total reactor volume  $V$ , let  $\tau = V/Q$  and express the residence-time distribution for three equal-sized CSTRs in series.
- (b) Consider second-order kinetics,  $r = kc^2$ , with feed concentration  $c_0$ . Define dimensionless variables

$$\bar{c} = c/c_0, \quad x = \theta/\tau, \quad K = kc_0\tau$$

and show the maximum mixedness case for the three-CSTR RTD is described by

$$\frac{d\bar{c}}{dx} = K\bar{c}^2 + \frac{27x^2}{9x^2 + 6x + 2}(\bar{c} - 1), \quad d\bar{c}/dx = 0 \text{ for } x = \infty \quad (8.67)$$

What "initial" value  $\bar{c}(\infty)$  replaces the derivative boundary condition in Equation 8.67? Solve Equation 8.67 for a range of  $K$  values and plot  $\bar{c}(0)$  versus  $K$ .

- (c) Compare your results above with Zwietering's results for this problem, Tables 1 and 2 in Zwietering [28]. Why do you suppose Zwietering does not report values for the maximum mixedness case for  $K = 3$  and  $K = 50$ ? Did you experience any difficulties computing these cases? What typographical error do you find in the published tables? See also the results in Example 8.3.

### Exercise 8.7: RTD for laminar-flow reactor

Calculate the residence-time distribution (RTD) for a tubular reactor undergoing steady, laminar flow (Hagen-Poiseuille flow). The velocity profile for Hagen-Poiseuille flow is [2, p. 51]

$$v(r) = \frac{2Q_f}{\pi R^2} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]$$

in which  $v$  is the axial velocity and  $R$  is the tube radius. Plot this RTD and compare to the RTD for the plug-flow reactor.

### Exercise 8.8: Other forms of the rate expression

- (a) Establish that  $r_m \leq r_s$  if the rate expression has the form

$$r(c) = \sum_i \alpha_i c^{\gamma_i} \quad \alpha_i \geq 0 \quad \gamma_i \geq 1$$

- (b) Establish that  $r_m \geq r_s$  if the rate expression has the form

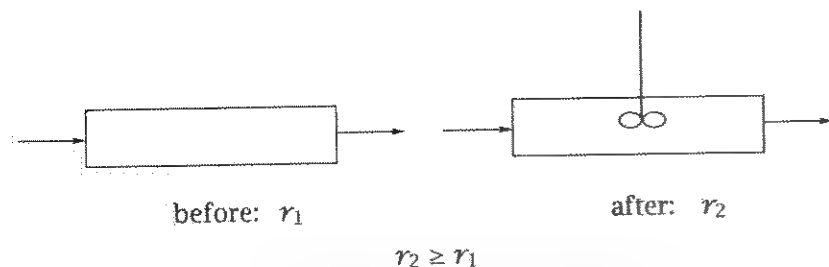
$$r(c) = \sum_i \beta_i c^{\delta_i} \quad \beta_i \geq 0 \quad \delta_i \leq 1$$

### Exercise 8.9: Mixing it up

Another perplexed student shows up at your office hours and complains, "Look, this mixing stuff doesn't make any sense." You and he discuss the simple case of a single, irreversible reaction with a concave rate expression,



with  $n \leq 1$ . Displaying impeccable logic, he continues, "Assume we have a PFR operating at steady state at some overall conversion. If we turn on a stirrer and suddenly mix the contents of the PFR (shown in Figure 8.40), you claimed in class that the reactor's overall reaction rate will suddenly *increase* as well. But you also told us in Chapter 4 that for these same kinetics, the PFR has a *higher* steady-state conversion than a CSTR of the same volume. Both of these statements cannot be true!"

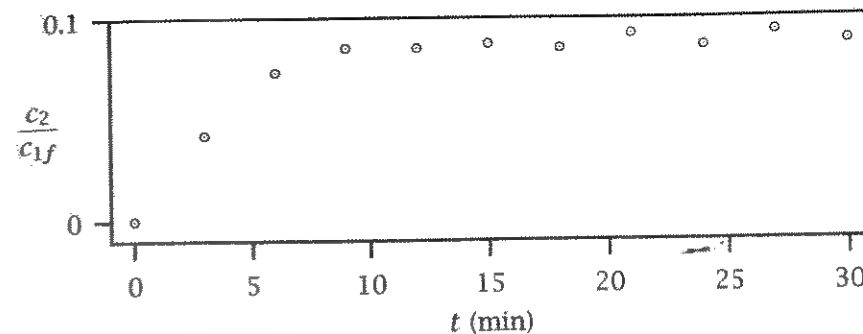


**Figure 8.40:** Reactor before and after stirrer is started; overall reaction rates before and just after mixing are  $r_1$  and  $r_2$ , respectively.

- Resolve this seeming contradiction by drawing a sketch of the overall reaction rate and the effluent concentration of the reactor versus time from the time just before the stirrer is started until the reactor comes to its new steady state with the stirrer running.
- Draw the same sketch if the reaction rate is convex.
- Calculate the exact results for the cases  $k = 1$ ;  $\theta = 1$ ;  $c_0 = 1$ ;  $\pi = 1/2, 1, 2$ ; and compare the numerical results to your sketches. Make the sketches before you perform the computation if you want to check your understanding of the issues.

#### Exercise 8.10: RTD for dispersed PFR with Danckwerts BCs

- Calculate the RTD for the dispersed PFR using the Danckwerts boundary conditions by solving numerically Equation 8.27 subject to Equations 8.28. You may wish to try the same approach used in solving Example 8.1.
- Compare your calculated values to Figures 8.10 and 8.11 for the same values of the dispersion number and mean residence time. These figures were calculated for the dispersed PFR with the simplified boundary conditions given in Equations 8.31. For what values of the dispersion number are the RTDs similar, and for what values of dispersion number are the RTDs different?
- Because the model is linear, the RTD of the dispersed PFR can also be solved analytically. Solve Equation 8.27 subject to Equations 8.28 using either Laplace transforms or an eigenfunction expansion. Compare your result with that given by Otake and Kunigita [19] (see also the discussion



**Figure 8.41:** Effluent concentration versus time after unit step change in the first reactor.

$t$ (min)	0	3	6	9	12	15	18	21	24	27	30
$c_2/c_{1f}$	0.00	0.042	0.073	0.085	0.085	0.087	0.085	0.092	0.086	0.093	0.088

in Westerterp, Van Swaaij and Beenackers [27, p.187]). Compare your analytical solution to your previous calculation. Which approach do you prefer?

#### Exercise 8.11: Fitting a reactor model to a step response

Consider the 2-CSTR reactor model of Example 8.5.

- Write the mass balances for an inert component that is added to the  $Q_1$  feed stream depicted in the bottom of Figure 8.32 and show

$$\begin{aligned}\frac{dc_1}{dt} &= \frac{1}{\theta_1} [\alpha c_{1f} - (\alpha + \rho)c_1 + \rho c_2] \\ \frac{dc_2}{dt} &= \frac{1}{\theta_2} [(\alpha + \rho)c_1 - (1 + \alpha + \rho)c_2]\end{aligned}\quad (8.68)$$

in which  $c_1$  and  $c_2$  are the inert component concentrations in the first and second reactors, respectively.

A unit-step change in inert concentration in the feed is made to the first reactor, and the outlet concentration is shown in Figure 8.41.

- Adjust  $\rho$  by hand, solve Equations 8.68, and compare to the data. The other parameters are given in Table 8.3. Iterate until you have a pretty good fit to the data. What is the best value of  $\rho$ ?



- (c) Predict the conversion of A and yield of C for this reactor.

### Exercise 8.12: Ordering PFRs and CSTRs

Consider a series of CSTRs and PFRs with a single  $n$ th-order reaction taking place in which

$$r(c) = kc^n$$

We showed in Example 8.2 that for a convex reaction-rate expression, the PFR should precede the CSTR to maximize conversion. The reason is shown in Figure 8.20. The CSTR-PFR configuration is a maximally mixed reactor, which has the lowest conversion for a convex rate expression.

- (a) Consider now a series of four reactors: two CSTRs and two PFRs, which has six possible configurations: PPCC, PCPC, PCCP, CPPC, CPCP, and CCPP. For  $n = 2$ , we should be able to safely reason that the lowest conversion is achieved with series: CCPP, because that is also a maximally mixed reactor. Rank order all other configurations from lowest to highest conversion. Be careful with the ordering of PCCP and CPPC. Which of these achieves higher conversion?
- (b) Repeat for  $n = 1/2$ . We know in this case the series CCPP should achieve the highest conversion because the rate expression is concave. Is your order in the two parts completely reversed or not? If not, can you explain why not?
- (c) Can you guess the solution for a series of six reactors: three CSTRs and three PFRs? Note there are 15 possible reactor configurations in this case. Can you suggest a general result for ordering PFRs and CSTRs given a single, convex or concave rate expression? The authors are not aware of a result of this type in the research literature.

### Exercise 8.13: Convex and concave functions

Convexity is a fundamental notion in the theory of optimization. Convexity can be defined to apply to functions that are not differentiable such as cases B and C in Figure 8.42. We say  $f(x)$  is a convex function if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y), \quad \text{all } x, y, 0 \leq \alpha \leq 1$$

In chemical reactor problems, the results stated for convex rate expressions apply to functions like  $r = c^n$  in which  $n \geq 1$ , but also apply to rate expressions like that shown in Figure 8.42, which may be only piecewise differentiable and not monotone.

- (a) Establish the following fact about convex functions, which will prove useful. Given  $f$  is convex and some value of  $x$ , then there exists a constant  $m$  such that

$$f(y) - f(x) \geq m(y - x), \quad \text{for every } y \quad (8.69)$$

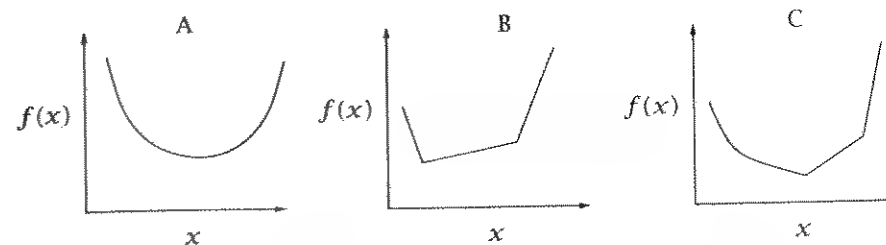


Figure 8.42: Some convex functions; differentiability is not required.

Hint: without loss of generality assume  $x < y$  and consider an arbitrary  $z$  such that  $x < z \leq y$ . Use the definition of convexity of  $f$  to show

$$f(y) - f(x) \geq \left( \frac{f(z) - f(x)}{z - x} \right) (y - x)$$

We now choose  $m$  to maximize the function  $(f(z) - f(x))/(z - x)$  over  $z$  to establish the result.

- (b) Find  $m$  graphically for each of the functions sketched in Figure 8.42. Notice  $m$  depends on the value of  $x$ .

### Exercise 8.14: Jensen's inequality

Jensen's inequality [10] states that given a convex function  $R$ , arbitrary function  $c$  and probability density  $p(\theta)$ ,

$$\int_0^\infty R(c(\theta)) p(\theta) d\theta \geq R\left(\int_0^\infty c(\theta) p(\theta) d\theta\right) \quad (8.70)$$

In terms of the chemical reactor, given a convex production rate expression, the mean production rate is greater than or equal to the production rate evaluated at the mean concentration, i.e., mixing the reactor contents lowers or does not change the overall production rate.

- (a) Derive Jensen's inequality. Hint: Let

$$\alpha = \int_0^\infty c(\theta) p(\theta) d\theta$$

and use Equation 8.69 to show

$$R(c(\theta)) - R(\alpha) \geq m(c(\theta) - \alpha), \quad \text{for every } c(\theta)$$

Multiply by  $p(\theta)$  and integrate to obtain Jensen's inequality.

- (b) What analogous inequality holds when  $R$  is concave?

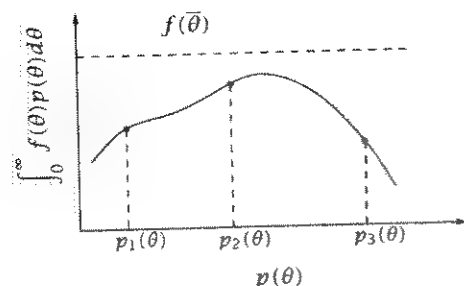


Figure 8.43: Value of  $\int_0^\infty f(\bar{\theta})p(\theta)d\theta$  as the function  $p(\theta)$  varies.

### Exercise 8.15: Optimization of a residence-time distribution function

Let  $f$  be a concave function and  $p(\theta)$  a residence-time distribution with mean

$$\bar{\theta} = \int_0^\infty \theta p(\theta) d\theta \quad (8.71)$$

Consider maximizing  $f$  over all possible RTDs having the specified mean  $\bar{\theta}$

$$\max_{p(\theta)} \int_0^\infty f(\theta) p(\theta) d\theta \quad (8.72)$$

(a) Show the maximum is achieved for

$$p(\theta) = \delta(\theta - \bar{\theta})$$

i.e., the maximum is achieved when the residence-time distribution is arbitrarily narrow at the specified mean value.

Hint: Use Jensen's inequality and choose  $c(\theta) = \theta$  to show

$$f(\bar{\theta}) \geq \int_0^\infty f(\theta) p(\theta) d\theta, \quad \text{for all } p(\theta) \quad (8.73)$$

which provides an upper bound for the maximum that is independent of  $p(\theta)$  as sketched in Figure 8.43.<sup>3</sup> By examining Equation 8.73, for what  $p(\theta)$  is this upper bound achieved? For what  $f$  is this maximum unique? See also Nauman and Buffham [18, p.162].

(b) What analogous optimization problem can you solve when  $f$  is a convex function?

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<sup>3</sup>Notice Figure 8.43 is merely a representation of the true situation because the abscissa is meant to depict a set of functions, not a set of scalar values as in an ordinary graph.

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## 9

### Parameter Estimation for Reactor Models

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Much of this book is devoted to constructing and explaining the fundamental principles governing chemical reactor behavior. As we have seen, these principles are efficiently expressed as mathematical models. Drawing quantitative conclusions from these principles requires not only the models, however, but the values of the parameters in the models as well. In most practical applications, many of the parameters in these models are not known or available beforehand. Evaluating the model parameters therefore usually requires something outside the scope of the theory: experimental data. In this chapter we consider the issues involved in gathering data and estimating parameters from data with the expressed purpose of identifying a chemical reactor model. In the process, we develop convenient and reliable tools that aid us in extracting as much information as possible from the expensive and time-consuming process of building experimental facilities, designing experiments, and making careful measurements.

#### 9.1 Experimental Methods

##### 9.1.1 Analytical Probes for Concentration

Experimental design includes specifying what variables to measure and how best to measure them. Included in the list of variables are reactor volume, inlet flowrates, temperature, inlet (initial) concentrations of one or more components, and effluent (final) concentrations of one or more components. Concentration or molar flowrate are the dependent composition variables in the design equations, and reaction rates are generally specified in terms of component concentrations. Whether the reaction is homogeneous or heterogeneous, solution of the material balance requires knowledge of the fluid-phase concentrations, so

we focus here on the important methods for measuring component concentrations in liquid-phase and gas-phase mixtures. You are likely to encounter these methods in research and analytical services laboratories.

To determine concentration, a unique chemical or physical attribute of the molecule under study is measured. For example, the vibrational frequency can be probed by infrared spectroscopy, the thermal conductivity can be probed by gas chromatography, or the mass/charge ratio for an ionized molecule can be probed by mass spectroscopy. The first example can be applied to the mixture while the second example requires the components to be separated before measuring the thermal conductivity. The final example samples the mixture and performs the mass discrimination as part of the analytical technique.

Analytical methods require calibration and proper selection of conditions to ensure the components of interest can be detected and their amounts quantified. Often only a few components out of an entire mixture can be monitored, and it is necessary to select the ones that provide the most information about the progress of the reactions. In most situations, the composition is measured only at discrete sample times,  $t_i$ . The general problem is to collect enough of the proper information so that the experimental data can be tested against appropriate kinetic and process models.

Infrared spectroscopy exploits the absorption of electromagnetic radiation associated with molecular motion of chemical bonds, in particular stretching, bending and rocking motions. The energies of these vibrational motions are in the infrared region of the electromagnetic spectrum ( $2.5\text{--}20\text{ }\mu\text{m}$ ;  $500\text{--}4000\text{ cm}^{-1}$ ) [19, 3, 23]. Examples include the C—H asymmetric stretching mode of a methyl group ( $\text{CH}_3$ ) at  $2962\text{ cm}^{-1}$ , the C—C stretching mode of ethylene at  $1623\text{ cm}^{-1}$ , the O—H stretching mode of methanol at  $3682\text{ cm}^{-1}$ , the R branch of the C—O stretching mode of carbon monoxide at  $2170\text{ cm}^{-1}$ , the C—O asymmetric stretching mode of carbon dioxide at  $2350\text{ cm}^{-1}$ , and the C—O stretching mode of acetone at  $1718\text{ cm}^{-1}$ . Infrared spectroscopy can be applied to gas, liquid and solid samples, and it is both a qualitative technique to identify the presence of molecules and a quantitative technique that can establish the concentration of the molecules.

The quantitative aspects are based on Beer's law,

$$A = \epsilon bc$$

in which  $A$  is the measured absorbance,  $\epsilon$  is the molar absorptivity,  $b$

is the path length and  $c$  is the concentration of the absorbing species. The liquid-phase or gas-phase sample is placed in a fixed-path-length cell. For gases this cell may be simply a tube with infrared transparent windows affixed to each end. The intensity of a particular wavelength of light passing through the sample is recorded and compared to the intensity recorded when the cell is empty or filled with an inert substance. The infrared spectrometer reports the intensity difference as an absorbance. With proper calibration using standard mixtures, the absorbance intensity can be converted into a concentration using Beer's law because the path length and molar absorptivity are fixed variables. Infrared spectroscopy can be used as a continuous monitoring technique by passing the gas or liquid through the cell as it exits a flow reactor.

Infrared spectrometers vary by the means used to discriminate wavelengths. Dispersive instruments use a monochromator and a grating to select the particular wavelength of interest or range of wavelengths that are scanned. Fourier transform spectrometers use an interferometer with a movable mirror to modulate the light source before it is incident on the sample, and all wavelengths are sampled simultaneously. The signal is recorded as a function of the mirror displacement, and after averaging multiple measurements and transforming the interferograms, the spectrometer reports absorbance versus wavelength.

Because infrared spectroscopy is an optical technique, the sample must be transparent in the spectral region of interest. Infrared spectroscopy can be used for heterogeneous samples and adsorbates on catalyst surfaces [11, 17]. If the sample cannot be made thin enough to be transparent, special probing techniques such as attenuated total internal reflectance can be used to monitor the concentration.

Gas chromatography is the workhorse of routine analysis [14, 18, 21, 16] because it can be used for simple gases, such as separating the constituents in air, as well as complex hydrocarbon mixtures, such as the components in gasoline. Gas chromatography can resolve concentrations in the range of parts per billion to tens of percent. Figure 9.1 presents a simplified schematic of a gas chromatograph. It consists of an injector, column and detector; each section has a separate temperature zone. The gas/liquid sample to be analyzed is injected as a pulse input in the injector and it flows along with the carrier gas through the column, where it is separated into components, and then flows to the detector. Most gases, and liquids that vaporize at temperatures less than about  $350^\circ\text{C}$ , can be analyzed by gas chromatography. More spe-

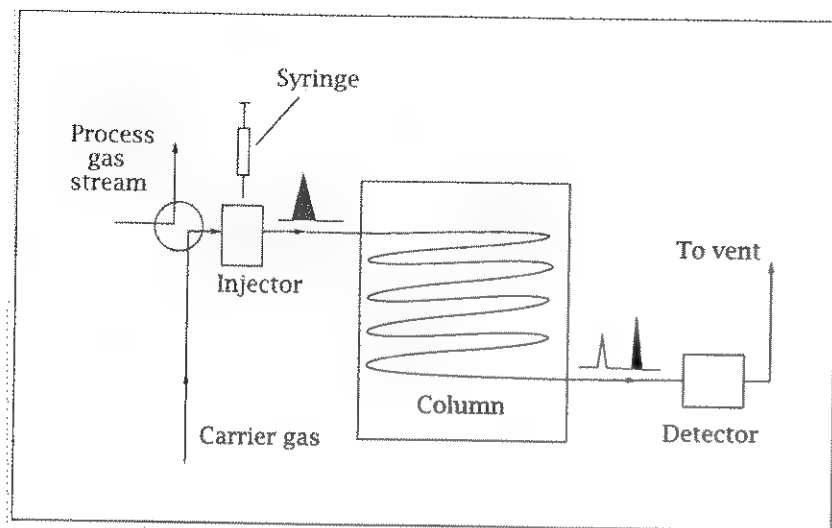


Figure 9.1: Gas chromatograph schematic.

specialized chromatography techniques, such as high-pressure liquid chromatography, extend the basic chromatographic technique to very high molecular weight compounds or to thermally unstable compounds.

The detector is typically a thermal conductivity detector or a flame ionization detector. The thermal conductivity detector is just a heated wire that has a constant current passed through it. The wire cools as a carrier gas flows over it and a steady wire temperature is established as a base-line response. If the gas flowing past the wire changes composition and therefore thermal conductivity, the wire temperature changes. The voltage drop across the wire and the current through the wire are related via Ohm's law,  $V = IR$ , to the wire resistance, which is temperature dependent. The thermal conductivity detector then operates by converting the voltage drop change to a temperature change as the gas mixture flowing past the wire changes from carrier gas only to carrier gas plus sample. A carrier gas is selected, typically He or N<sub>2</sub>, that has a thermal conductivity significantly different from any component in the mixture. The detector identifies the *thermal conductivity* of the gas mixture, not the composition of the gas mixture. The thermal conductivity can be related to the composition by calibrating with carrier gas and samples of known composition.

The flame ionization detector operates by blending the column effluent with H<sub>2</sub> and burning it with air in an ionizing flame. The ions are

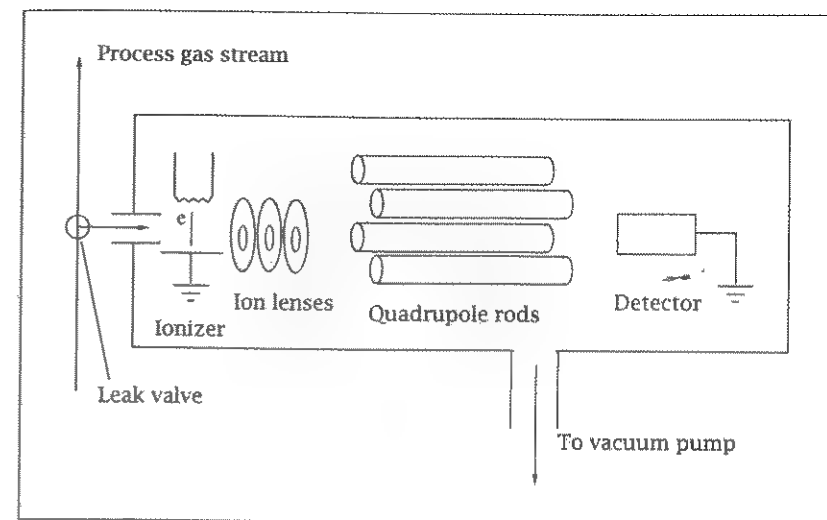


Figure 9.2: Mass spectrometer schematic.

counted with an electrometer. A flame ionization detector works best with hydrocarbons. The electrometer merely records the presence of ions. The greater the concentration of a particular component in the flame, the greater the signal. The flame ionization detector does not identify the composition of the gas in the flame. Separation of the injected sample is accomplished by column selection. The column is a long, narrow tube that is either filled with a packing such as a porous adsorbent, or contains a thin coating of a high-molecular-weight liquid on the inner tube surface that acts as the adsorbent. As the injected pulse travels along the column length, the components interact with the adsorbent to differing degrees and are eventually separated into pulses of almost pure components. The science of gas chromatography lies in designing and selecting a column that separates the constituents, perhaps sorting them by boiling point or polarity. Standard mixtures are used to establish the elution order and time of elution for gases in a particular column.

Mass spectroscopy can be used to determine the partial pressure of gases in a mixture. Figure 9.2 presents a simple schematic of a mass spectrometer for gas monitoring. The mass spectrometer is maintained in a high-vacuum chamber at pressures less than  $10^{-6}$  torr. The gas to be analyzed flows past a special valve, called a leak valve, that permits a small fraction of the flowing gas stream to enter the vacuum chamber.

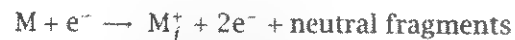
A steady state is established in which the amount of gas leaked into the chamber is pumped away. Pumping speeds of different molecules may vary; a predictable relationship between the relative mole fractions in the flowing gas stream and the relative mole fractions in the vacuum chamber can be established [20]. The mass spectrometer measures the partial pressures of the molecules inside the vacuum chamber. As configured in Figure 9.2, the mass spectrometer generates a continuous measure of the gas-phase composition.

The spectrometer has four major components: ionizer, focusing lens system, quadrupole section and detector. The spectrometer operates on the principal that ions can be steered into the detector or diverted away from the detector. The ionizer and lens system produce, collimate and impart a constant kinetic energy to the ions. The quadrupole rods have a varying direct current (dc) potential and oscillating radio frequency (rf) field applied across pairs of rods that causes all ions except those with a particular mass/charge ( $m/z$ ) ratio to follow an unstable path and collide with the rods. Depending on the instantaneous dc voltage, only ions with a single  $m/z$  follow a stable path to the detector, which is at a large negative potential ( $-2$  to  $-4$  kiloelectronvolt (keV)). The ions received at the detector induce a current to flow in the detector.

Various ionization techniques can be used [14, 13, 9]; the most common is electron impact ionization. Electrons (typically 70 eV) emitted from a very hot filament collide with molecules in the ionizer region. The electron impact ionization process ionizes the molecule or atom



and ionizes and fragments the molecule into a number of ions,  $M_j^+$ , of mass less than  $M$ .



The relative ratios of the  $M_j^+$  fragments can be used to identify the structure of  $M$ . The current generated by  $M^+$  and all  $M_j^+$  fragments at the detector is used to establish the partial pressure of each component in the vacuum chamber.

### 9.1.2 Experimental Reactors for Kinetics Studies

Time and position are the independent variables, and concentration, volumetric flowrate and temperature are the dependent variables. Ultimately, the experimental data are fit against a model for the reactor in

which the data were collected. This process becomes more challenging when the reactor is nonisothermal. Therefore, one often tries to design experimental reactors and select reaction conditions that permit isothermal operation. The effect of temperature is most easily explored by operating the isothermal reactor at different temperatures.

The reactor models presented in Chapters 4-7 were based on specific flow assumptions, such as the well-mixed reactor and plug flow reactors. Experimental reactors must retain these flow attributes if we wish to use the simple material balances for estimating model parameters.

Starting and quenching the reaction can present formidable challenges. The experiment should be designed to ensure that the reaction takes place only in the reactor and under the conditions assumed in the reactor model. For example, if a reaction has a large activation energy, it is necessary to operate the reaction at elevated temperatures to achieve a measurable rate. If using a flow reactor, the reactants must be preheated and injected into the reactor without any reaction occurring prior to the reactor inlet. In an isothermal batch reactor, the reactor contents must be heated rapidly to the desired temperature and held at constant temperature. Unless steps are taken, the reactions may continue in the samples withdrawn from the reactor. If the reaction has a low activation energy, lowering the temperature does not quench the reaction. It may prove necessary to dilute the sample or add components to cause side reactions to quench a reaction.

Most experimental reactors are small to ensure large surface-to-volume ratio for good temperature regulation, and to minimize the chemical inventory needed to conduct the experiments. As illustrated in Figures 1.6 and 1.8, flow and batch reactors with volumes of several hundred nanoliters have been fabricated out of silicon wafers and used for organic synthesis screening studies.

**Liquid-phase reactions.** A batch reactor, CSTR or PFR could be used for liquid-phase studies. Many times, the choice of a batch reactor is made by pumping considerations. Continuous, steady and non-pulsating flow may be difficult to realize for liquids in a laboratory setting. Accurate and large-volume positive displacement pumps are costly, and small centrifugal pumps may not be available to handle the reactants. In the simplest case, a batch reactor may be nothing more than a beaker with a magnet stir bar for agitation. The situation becomes more complex if temperature regulation, efficient mixing, and isolation of the contents from ambient gases are important factors.



Samples are typically withdrawn as aliquots at preset time intervals via syringe and saved for subsequent analysis. Figure 1.2 illustrates the batch reactor internals required to control the temperature and withdraw samples.

**Gas-phase reactions.** The reactor of choice for gas-phase reactions is the plug-flow reactor. The batch reactor and CSTR are rarely used for gas-phase studies because it is difficult to achieve ideal mixing of gases. The isothermal PFR is usually nothing more complicated than an open tube maintained at a constant temperature. Different tube diameters and lengths are used to change the reactor volume as an experimental variable. Gas flows are routinely and reproducibly regulated with mass flow controllers or manually with metering valves and flow meters; this enables a wide range of volumetric flowrates to be examined as an experimental variable. Sampling is best accomplished by directing the gas effluent through a gas sampling valve so that precise samples can be injected automatically into a gas chromatograph.

It is quite common in experimental studies to operate a PFR as a so-called **differential reactor**. A differential reactor is a PFR in which the reactants experience only a small (differential) change in the extents of the reactions. The differential reactor is modeled using a limiting form of the PFR material balance design equation and generates data of the form rate versus concentration. Rate versus concentration data may prove useful in certain kinetic parameter studies. The PFR design equation is

$$\frac{d}{dV} N_j = R_j = \sum_i v_{ij} r_i \quad (9.1)$$

If we assume the molar flows change by only a small amount, we can approximate the derivative with a finite difference formula

$$\frac{N_j - N_{jf}}{V_R} \approx R_{jf} = \sum_i v_{ij} r_{if} \quad (9.2)$$

$$N_j \approx N_{jf} + R_{jf} V_R \quad (9.3)$$

in which  $R_{jf}$  and  $r_{if}$  are the production rates and reaction rates evaluated at the feed conditions. Now we account for the change in volumetric flowrate (Exercise 9.5 explores what happens if we neglect the volumetric flowrate change.) If we assume an ideal gas, we have  $Q = NRT/P$ , in which  $N$  is the total molar flow, and summing Equa-

tion 9.3 produces

$$N = N_f + \sum_i r_{if} \bar{v}_i V_R$$

$$Q = Q_f + \frac{RT}{P} \sum_i \bar{v}_i r_{if} V_R$$

Substituting the  $Q$  relation into Equation 9.3 and rearranging produces

$$\sum_i (v_{ij} - y_j \bar{v}_i) r_{if} = \frac{Q_f}{V_R} (c_j - c_{jf}) \quad (9.4)$$

Experimentally, one measures  $c_j$  in the effluent stream, which determines the right-hand side of Equation 9.4. One then solves a least-squares problem to determine all of the  $r_{if}$ . By varying  $c_{jf}$ , it is possible to measure efficiently the reaction rates over a range of concentrations.

For example, if we have the single reaction  $A \rightarrow 2B$ , and we measure both  $c_A$  and  $c_B$ , using Equation 9.4, we obtain two estimates of the single reaction rate

$$r = \frac{Q_f}{V_R} \frac{(c_{Af} - c_A)}{-(1 + y_A)} \quad r = \frac{Q_f}{V_R} \frac{(c_{Bf} - c_B)}{(2 - y_B)}$$

and the least squares solution is equivalent to taking the average.

**Heterogeneously catalyzed reactions.** Macroscopic fluid models are combined with microscopic transport models in the catalyst particles to describe how concentration changes with time and position in a catalytic reactor. Special considerations must be given to the selection of experimental temperature and catalyst particle size to minimize (and hopefully eliminate) internal transport limitations on the catalytic reaction rate. The next requirement is that the flow pattern in the reactor is accurately represented by the well-mixed or plug-flow assumption. The subsequent discussion applies to gas-phase reactants.

Commercial experimental reactors that achieve ideal mixing are available. These reactors either hold the catalyst in a basket arrangement and circulate the reacting gases through the bed or they spin the bed at a high revolution rate. The design achieves a sufficiently high linear velocity across the catalyst bed and volumetric flowrate in the reactor vessel so that the contents of the fluid phase can be assumed to be well mixed. The high internal circulation rate and linear velocities ensure uniform temperatures in the reactor, and special seals permit operation at pressures exceeding 100 atm. These well-mixed reactors can be treated as batch or CSTR, depending on how they are operated.



A second type of reactor is the plug-flow, fixed-bed reactor. The reacting gas must move through the reactor as a plug, with a flat velocity profile. Previous studies of transport in porous media have led to characteristic groups that gauge how well the fluid flow can be characterized as plug-like [8]. These groups and their values include the particle Reynolds number

$$Re = \frac{D_p m}{S \mu \epsilon_B} \geq 30$$

the axial aspect ratio

$$\frac{L}{D_p} \geq 30$$

and the radial aspect ratio

$$\frac{D_t}{D_p} \geq 10$$

The Reynolds number ensures turbulent flow and with it effective radial mixing. The axial aspect ratio ensures that axial dispersion is minimal. The radial aspect ratio ensures that channeling does not occur. Channeling refers to the situation in which the fluid close to the reactor walls travels faster than the fluid at the center of the tube. When these three dimensionless conditions are satisfied, one can usually model the reactor as a PFR. The velocity profiles are complex, however, and broad generalizations should be used with caution [28].

### 9.1.3 Characterizing Catalysts and Surfaces

A wide variety of analytical probes are used to study, characterize and monitor catalysts and catalyst surfaces. Our intent here is to discuss some of the more common and routine techniques. Much more detail and many more techniques can be found in specialized books [33, 32, 27, 12]. A catalyst functions through the highly specific interactions the active sites have with the reactants. The catalyst might be a metal dispersed on an inert carrier, a polycrystalline or amorphous mixture of metal oxides, or a zeolite (a crystalline and highly porous oxide). The experimentalist is typically interested in the catalyst composition, structure of the catalyst, distribution of active sites, presence of poisons/impurities after the catalyst has been used, and number of active sites — parameters that influence the catalytic activity.

Active site densities can be determined using adsorption isotherms as discussed in Chapter 5. The equipment is straightforward. Figure 9.3 presents a schematic view of the essential components of a

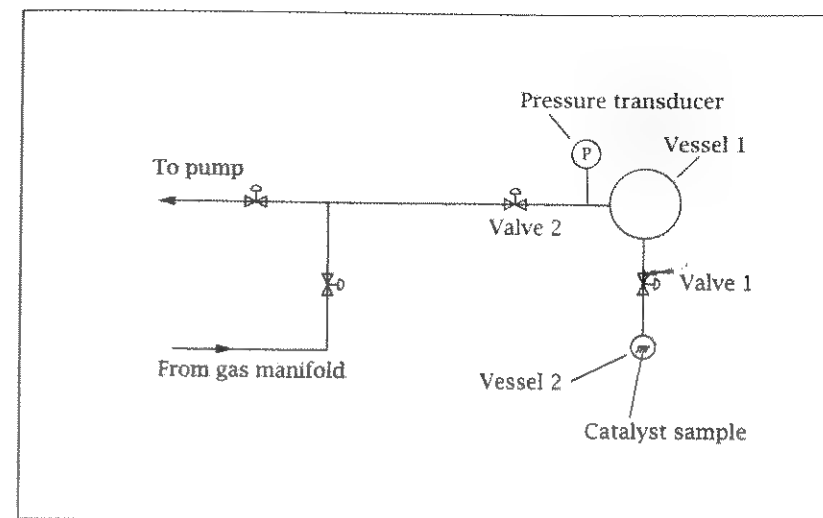


Figure 9.3: Volumetric chemisorption apparatus.

chemisorption apparatus. The apparatus includes a pressure transducer for accurately measuring the pressure, a gas mixing manifold for introducing the adsorbate gas, a vacuum pump for establishing a low base pressure and removing all adsorbates from the catalyst, and two vessels of known volumes ( $V_1$  and  $V_2$ ) separated by a valve. The catalyst is placed in Vessel 2 and this section of the apparatus is maintained at constant temperature. An experiment proceeds by adding the catalyst and evacuating the entire apparatus until nothing is adsorbed on the catalyst and the pressure is below the detection level of the pressure transducer. Valve 1 is closed, Valve 2 is opened, and gas is admitted to Vessel 1. Valve 2 is closed, and the pressure and temperature of Vessel 1 are recorded. Valve 1 is then opened and gas both expands into Vessel 2 and adsorbs on the catalyst. The pressure is noted after reaching a steady value. Using an equation of state, such as the ideal gas law, the temperatures and volumes of Vessels 1 and 2, and the pressure before and after admitting gas to Vessel 2, one can determine the moles of adsorbed gas. The experiment continues by closing Valve 1, opening Valve 2 and refilling Vessel 1, and then closing Valve 2. The pressure in Vessel 1 is noted before and after opening Valve 1. This sequence of filling and expanding is repeated until there is no change in the pressure upon expanding the gas into Vessel 2. This procedure generates the number of moles adsorbed as a function

of pressure, which can be used to determine the number of active sites and the adsorption equilibrium constant.

Crystallinity and particle size can be determined by X-ray diffraction. Supported metals and metal oxides generally adopt their native packing when they form crystallites sufficiently large to define a structure. A diffraction pattern identifies the atomic packing structure, provided the material is crystalline, and is in sufficiently large particles ( $\geq 10$  nm) and has a diffraction pattern that is not obscured by the support. X-ray diffraction is used to assess the presence of amorphous versus crystalline phases and changes in the relative amounts of different crystalline phases with thermal treatment. A given crystal has a unique diffraction pattern. As particle size decreases, the diffraction pattern becomes diffuse and the peaks broaden. This phenomenon is termed **line broadening** and it can be used to determine particle size [33].

Particle size also can be established using transmission electron microscopy. In this technique, a mono-energetic beam of high-energy electrons is directed through a thin, solid specimen ( $\leq 200$  nm). As the electrons travel through the sample, they interact with the atomic constituents of the sample and scatter. This scattering process leads to contrast images being projected on a detector. Generally, it is easy to discern a heavy element from lighter elements, such as Pt dispersed on  $\text{Al}_2\text{O}_3$ . The heavier element produces a different intensity image. It is routine to resolve less than 0.2 nm and image individual atoms. The particle size is determined by measuring the size of the image and working back from the magnification used in the experiment. A sufficient number of particles must be measured to give a statistically meaningful average particle size. This technique can be applied before and after reaction to determine if the particles have changed size during the reaction.

Composition can be determined by dissolving the catalyst in a suitable liquid, injecting the liquid solution into an inductively coupled plasma, and analyzing the ions generated in the plasma with a mass spectrometer. The inductively coupled plasma source is highly efficient at ionizing the injected sample, and this measurement has very high elemental sensitivity and is quantitative. The digestion technique is destructive.

Composition also can be determined in a nondestructive manner using electrons or soft X-ray sources to excite the core levels of the atoms. Secondary or primary electrons are ejected from the sample at

energies that are characteristic of the elements. These ejected electrons have a mean free path that is energy dependent and can only exit the sample if they are formed in the top 0.5–10 nm of the sample surface. The ejected electron techniques identify what is present; it is difficult to make these techniques quantitative.

Auger electron spectroscopy uses a focused electron beam (3–5 keV) to eject an inner atomic level electron and create a singly ionized excited atom. Electrons from other energy states in the atom rapidly fill the vacancy. The energy released in this de-excitation electronic transition can be transferred to a third electron. If the binding energy of this third electron is less than the energy transferred to it, it is ejected as an Auger electron. The energy of this ejected electron is monitored with an Auger electron spectrometer. Elements have discrete electron energy levels with distinguishable binding energies, so the energy of the ejected Auger electrons provide a means of identifying the elements that are present [10].

X-ray photoelectron spectroscopy uses soft X-rays (Mg  $K\alpha$  at 1253.6 eV and Al  $K\alpha$  at 1486.6 eV) to eject core-level electrons from the unknown sample. The kinetic energy of the ejected electrons is monitored and related back to the binding energy. The binding energy is used to identify the elements present [22]. Since the binding energy of inner-shell electrons typically changes with the valence of the element, X-ray photoelectron spectroscopy can be used to follow oxidation or reduction of the catalyst as well as identify the presence of elements.

## 9.2 Data Modeling and Analysis

Having provided the context in which we perform experiments and make measurements, we now turn to the issue of how we extract the information contained in the data. The two large questions we must address are: what model *structure* is appropriate to describe the reacting system of interest, and, having selected a structure, what model *parameters* best represent the data we have collected, and how certain are we about these parameter values. The first question has occupied us up to this point; it has been the central focus of Chapters 1–8. Indeed, understanding the fundamental principles and models that explain the many kinds of chemical reaction and reactor behaviors is one of the essential goals of the chemical engineering education.

In this section, we focus finally on the second question, how to estimate model parameters from data. This problem is also important, and

has a distinguished place in the history of science and engineering. Accurate prediction of the motions of the planets based on astronomical measurements was one of the early motivating problems of parameter estimation. Solving this problem led Gauss to invent the least-squares method in the late 1700s. Gauss's summary of this effort more than 175 years ago remains valid today:

One of the most important problems in the application of mathematics to the natural sciences is to choose the best of these many combinations, i.e., the combination that yields values of the unknowns that are least subject to errors.

Theory of the Combination of Observations Least Subject to Errors. C. F. Gauss, 1821 [15].

### 9.2.1 Review of the Normal Distribution

Probability and statistics provide one useful set of tools to model the uncertainty in experimental data. It is appropriate to start with a brief review of the normal distribution, which plays a central role in analyzing data. The normal or Gaussian distribution is ubiquitous in applications. It is characterized by its mean,  $m$ , and variance,  $\sigma^2$ , and is given by

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \frac{(x-m)^2}{\sigma^2}\right) \quad (9.5)$$

Figure 9.4 shows the univariate normal with mean zero and unit variance. We adopt the following notation to write Equation 9.5 more compactly

$$x \sim N(m, \sigma^2)$$

which is read "the random variable  $x$  is distributed as a normal with mean  $m$  and variance  $\sigma^2$ ." Equivalently, the probability density  $p(x)$  for random variable  $x$  is given by Equation 9.5.

For distributions in more than one variable, we let  $\mathbf{x}$  be an  $n_p$ -vector and the generalization of the normal is

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{n_p/2} |\mathbf{P}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{P}^{-1}(\mathbf{x} - \mathbf{m})\right]$$

in which the  $n_p$ -vector  $\mathbf{m}$  is the mean and the  $n_p \times n_p$ -matrix  $\mathbf{P}$  is called the covariance matrix. The notation  $|\mathbf{P}|$  denotes determinant of  $\mathbf{P}$ . We also can write for the random variable  $\mathbf{x}$  vector

$$\mathbf{x} \sim N(\mathbf{m}, \mathbf{P})$$

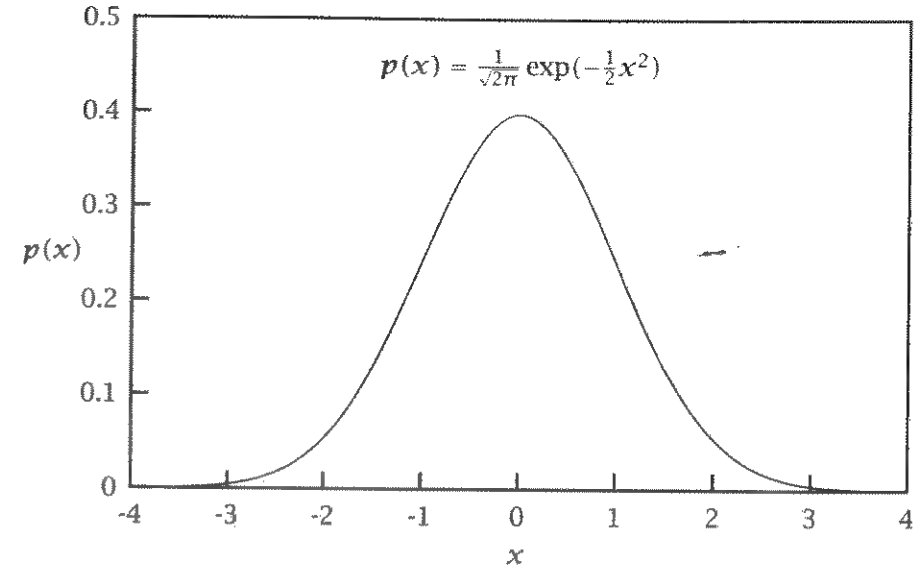


Figure 9.4: Univariate normal with zero mean and unit variance.

The matrix  $\mathbf{P}$  is a real, symmetric matrix. Figure 9.5 displays a multivariate normal for

$$\mathbf{P}^{-1} = \begin{bmatrix} 3.5 & 2.5 \\ 2.5 & 4.0 \end{bmatrix}$$

As displayed in Figure 9.5, lines of constant probability in the multivariate normal are lines of constant

$$(\mathbf{x} - \mathbf{m})^T \mathbf{P}^{-1}(\mathbf{x} - \mathbf{m})$$

To understand the geometry of lines of constant probability (ellipses in two dimensions, ellipsoids or hyperellipsoids in three or more dimensions) we examine the eigenvalues and eigenvectors of the  $\mathbf{P}$  matrix.

### 9.2.2 Eigenvalues and Eigenvectors

An eigenvector of a matrix  $\mathbf{A}$  is a nonzero vector  $\mathbf{v}$  such that when multiplied by  $\mathbf{A}$ , the resulting vector points in the same direction as  $\mathbf{v}$ , and only its magnitude is rescaled. The rescaling factor is known as the corresponding eigenvalue  $\lambda$  of  $\mathbf{A}$ . Therefore the eigenvalues and eigenvectors satisfy the relation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}, \quad \mathbf{v} \neq \mathbf{0}$$

$$p(\mathbf{x}) = \exp\left(-1/2\left(3.5x_1^2 + 2(2.5)x_1x_2 + 4.0x_2^2\right)\right)$$

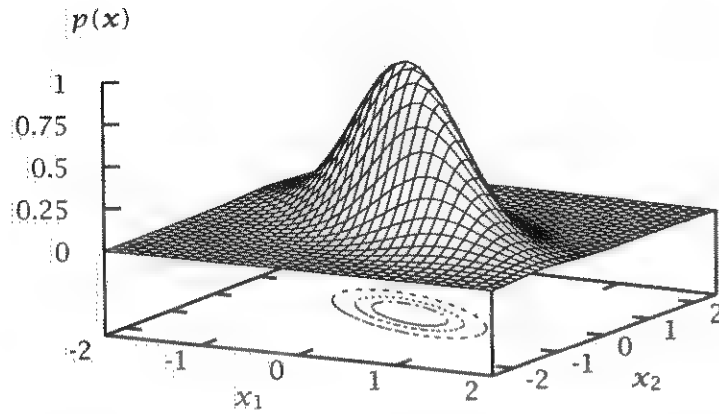


Figure 9.5: Multivariate normal for  $n_p = 2$ .

We normalize the eigenvectors so

$$\mathbf{v}^T \mathbf{v} = \sum_i v_i^2 = 1$$

The eigenvectors show us the orientation of the ellipse given by the normal distribution. Consider the ellipse in the two-dimensional  $\mathbf{x}$  coordinates given by the quadratic

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = b$$

If we march along a vector  $\mathbf{x}$  pointing in the eigenvector  $\mathbf{v}$  direction, we calculate how far we can go in this direction until we hit the ellipse  $\mathbf{x}^T \mathbf{A} \mathbf{x} = b$ . Substituting  $\alpha \mathbf{v}$  for  $\mathbf{x}$  in this expression yields

$$(\alpha \mathbf{v}^T) \mathbf{A} (\alpha \mathbf{v}) = b$$

Using the fact that  $\mathbf{A} \mathbf{v} = \lambda \mathbf{v}$  for the eigenvector gives

$$\alpha^2 \lambda \mathbf{v}^T \mathbf{v} = b$$

because the eigenvectors are of unit length, we solve for  $\alpha$  and obtain

$$\alpha = \sqrt{\frac{b}{\lambda}}$$

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = b$$

$$\mathbf{A} \mathbf{v}_i = \lambda_i \mathbf{v}_i$$

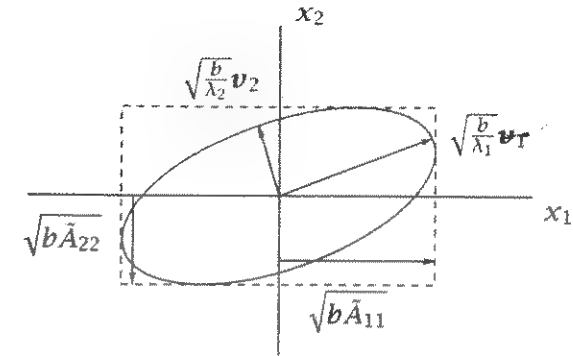


Figure 9.6: The geometry of quadratic form  $\mathbf{x}^T \mathbf{A} \mathbf{x} = b$ .

which is shown in Figure 9.6. Each eigenvector of  $\mathbf{A}$  points along one of the axes of the ellipse. The eigenvalues show us how stretched the ellipse is in each eigenvector direction.

If we want to put simple bounds on the ellipse, then we draw a box around it as shown in Figure 9.6. Notice the box contains much more area than the corresponding ellipse and we have lost the correlation between the elements of  $\mathbf{x}$ . This loss of information means we can put different tangent ellipses of quite different shapes inside the same box. The size of the bounding box is given by

$$\text{length of } i\text{th side} = \sqrt{b \tilde{A}_{ii}}$$

in which

$$\tilde{A}_{ii} = (i, i) \text{ element of } \mathbf{A}^{-1}$$

Figure 9.6 displays these results: the eigenvectors are aligned with the ellipse axes and the eigenvalues scale the lengths. The lengths of the sides of the box that is tangent to the ellipse are proportional to the square root of the diagonal elements of  $\mathbf{A}^{-1}$ .

### 9.2.3 Least-Squares Estimation

Consider again the problem of fitting a straight line to data

$$y_i = m x_i + b$$

in which  $y_i$  is the measurement at  $x_i$ ,  $i = 1, \dots, n_d$  and  $n_d$  is the number of data points. Using matrix vector notation, we can write the equation for all the data as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\theta}$$

in which the parameters to be estimated are placed in the  $\boldsymbol{\theta}$  vector

$$\boldsymbol{\theta} = \begin{bmatrix} m \\ b \end{bmatrix}$$

and the  $\mathbf{y}$  vector and  $\mathbf{X}$  matrix are given by

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n_d} \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_{n_d} & 1 \end{bmatrix}$$

We do not expect the best fit line to pass through all the data points, so we modify the model to account for measurement error<sup>1</sup>

$$\mathbf{y} = \mathbf{X}\boldsymbol{\theta} + \mathbf{e} \quad (9.6)$$

in which  $\mathbf{e}$  is a random variable. We model the measurement error as a normal distribution with mean 0 and variance  $\sigma^2$ .

$$\mathbf{e} \sim N(0, \sigma^2 \mathbf{I}) \quad (9.7)$$

The best estimate of  $\boldsymbol{\theta}$  in a least-squares sense is given by

$$\hat{\boldsymbol{\theta}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} \quad (9.8)$$

a formula that you have used often. However, we also can examine the distribution of parameter values given the observed measurements corrupted by the measurement errors. Imagine we create replicate datasets by drawing measurement errors  $\mathbf{e}$  from the distribution given in Equation 9.7. For each dataset we apply Equation 9.8 and produce a parameter estimate. The distribution of measurement errors creates a

<sup>1</sup>Notice the model *structure* is usually in error also, e.g., the true relationship between  $y$  and  $x$  may be nonlinear, variables other than  $x$  may be required to predict  $y$ , and so on. The procedure outlined here lumps structural error into  $\mathbf{e}$  as well, but structural error is not accounted for correctly in this way. If the structure is in serious doubt, one may pose instead model discrimination tests to choose between competing models with different structures [30, 31].

distribution of parameter estimates. In fact, for models linear in the parameters, we can show the parameters are also distributed as a normal distribution

$$\boldsymbol{\theta} \sim N(\hat{\boldsymbol{\theta}}, \mathbf{P})$$

in which the mean is the least-squares estimate provided above and the covariance is

$$\mathbf{P} = \sigma^2 (\mathbf{X}^T \mathbf{X})^{-1}$$

We also can calculate the parameter "confidence intervals." We merely compute the size of the ellipse containing a given probability of the multivariate normal. That can be shown to be the chi-square probability function [5]. Given the number of estimated parameters,  $n_p$ , and the confidence level,  $\alpha$ , then

$$\frac{(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T \mathbf{X}^T \mathbf{X} (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})}{\sigma^2} \leq \chi^2(n_p, \alpha) \quad (9.9)$$

The  $\chi^2$  distribution is tabulated in many statistics handbooks [4] and is available in many computing environments.

To illustrate the ideas we examine a classic problem: how to estimate the pre-exponential factor and activation energy of a rate constant.

### Example 9.1: Estimating the rate constant and activation energy

Assume a reaction rate has been measured at several different temperatures in the range  $300 \text{ K} \leq T \leq 500 \text{ K}$ . Estimate the pre-exponential factor and activation energy of the rate constant, and quantify your uncertainty in the estimated parameters.

#### Solution

We first model the rate (rate constant) as

$$k = k_0 \exp(-E/T) \quad (9.10)$$

in which  $k_0$  is the pre-exponential factor and  $E$  is the activation energy, scaled by the gas constant.

Figure 9.7 shows a typical experiment. Notice that the measurement of the rate constant is somewhat noisy, a likely outcome if we differentiate the concentration data to obtain the rate. To make the estimation problem linear, we transform the data by taking the logarithm of Equation 9.10

$$\ln k = \ln k_0 - E/T$$

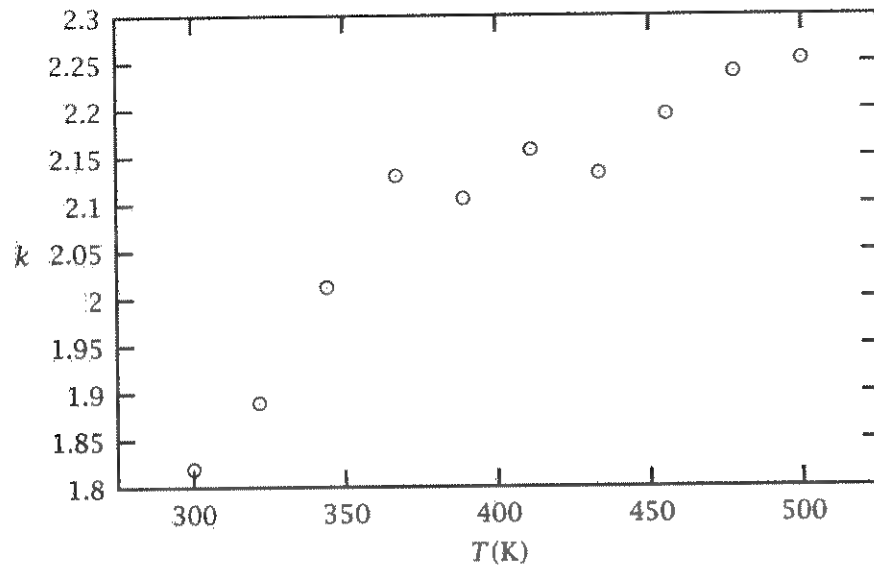


Figure 9.7: Measured rate constant at several temperatures.

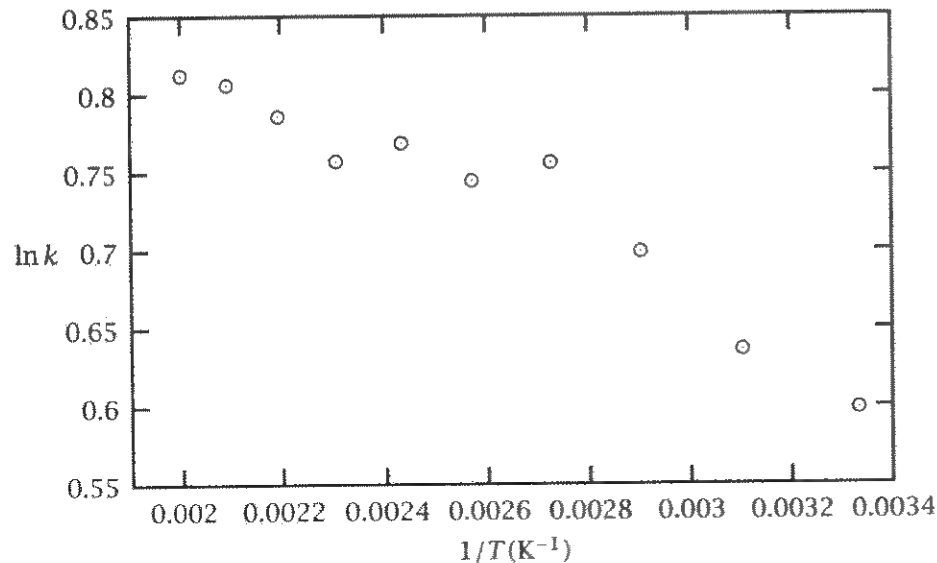


Figure 9.8: Transformed data set,  $\ln k$  versus  $1/T$ .

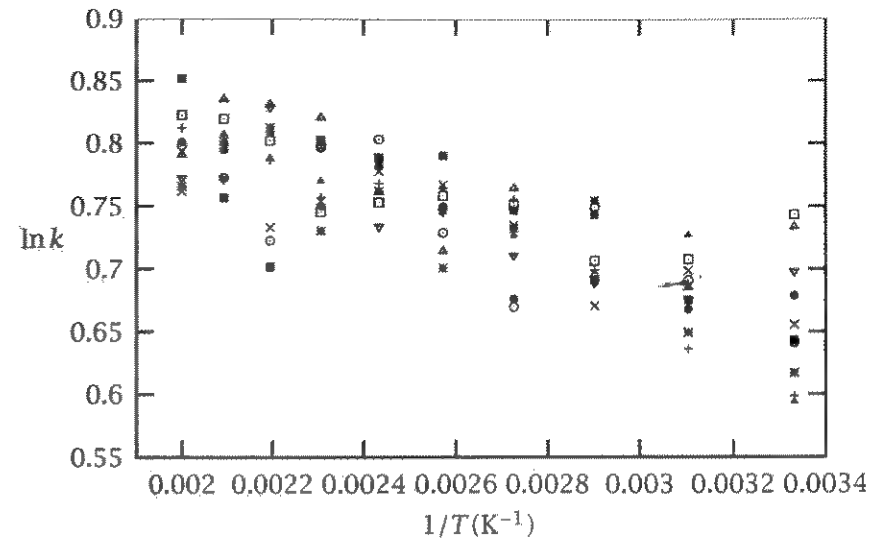


Figure 9.9: Several replicate data sets,  $\ln k$  versus  $1/T$ .

The transformed data are shown in Figure 9.8. To generate these data we assume the model is correct and suppress the units of the parameter values

$$\ln k_0 = 1, \quad E = 100$$

The measurements of  $\ln k$  are corrupted with normally distributed errors having variance 0.001,

$$e \sim N(0, 0.001)$$

We apply Equation 9.8 to estimate the parameter using the transformed model, so  $x_i = 1/T_i$  and  $y_i = \ln k_i$ . To quantify the uncertainty, imagine we replicate the experiment. Figure 9.9 shows several more experiments. Each experiment that we perform allows us to estimate the slope and intercept. Then we can plot the distribution of parameters. Figure 9.10 shows the parameter estimates for 500 replicated experiments. Notice the points are clustering in an elliptical shape. We construct the 95% confidence interval from Equation 9.9. In this problem  $n_p = 2$  and  $\alpha = 0.95$ , and we obtain from a statistics table  $\chi^2(2, 0.95) = 5.99$ , so we plot

$$\frac{(\theta - \hat{\theta})^T X^T X (\theta - \hat{\theta})}{0.001} \leq 5.99 \quad (9.11)$$

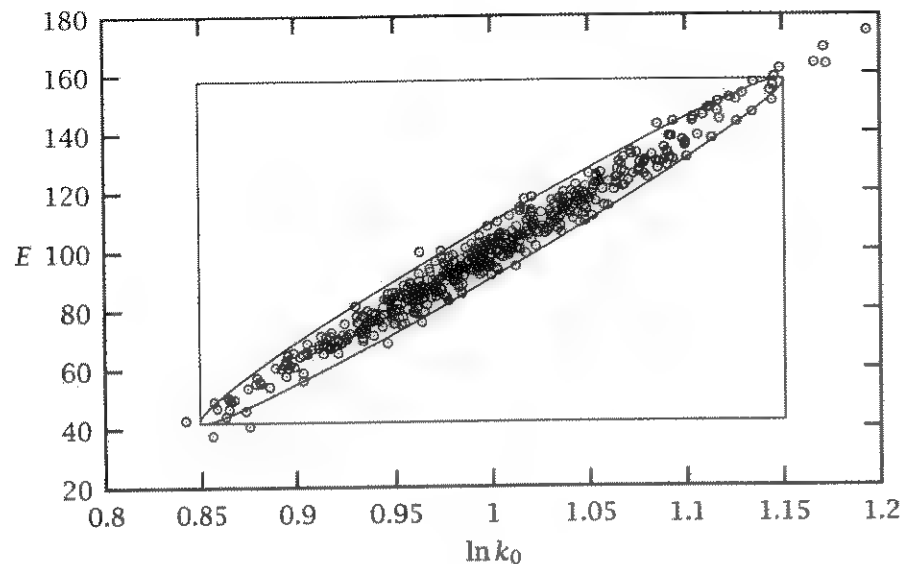


Figure 9.10: Distribution of estimated parameters.

This ellipse is also shown in Figure 9.10. In fact, 24 out of the 500 points, or 4.8% of the estimated parameters, lie outside this ellipse, which indicates Equation 9.11 is fairly accurate with this many random experimental trials. Note one often sees parameters reported with plus/minus limits. For this problem, one might report

$$\begin{bmatrix} \ln k_0 \\ E \end{bmatrix} = \begin{bmatrix} 1 \\ 100 \end{bmatrix} \pm \begin{bmatrix} 0.15 \\ 60 \end{bmatrix}$$

But notice these limits are misleading. The rectangle in Figure 9.10 does *not* indicate the strong correlation between the parameters. The ellipse is much more informative in this case.

Next we show how a simple re-parameterization of the rate constant can reduce the parameter correlation. Consider the mean of the temperatures at which data were collected, and re-parameterize the rate constant as in Chapter 6,

$$k = k_m \exp(-E(1/T - 1/T_m)) \quad (9.12)$$

We may wish to consider the mean of temperature or inverse temper-

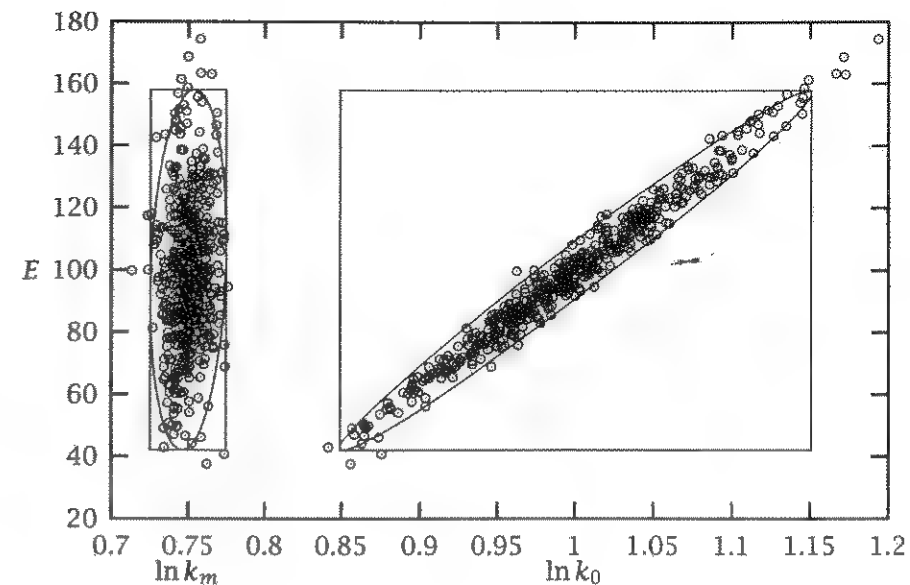


Figure 9.11: Reducing parameter correlation by centering the data.

ature for  $T_m$ . Here we let  $T_m$  be the mean temperature

$$T_m = \frac{300 + 500}{2} = 400 \text{ K}$$

Both Equations 9.10 and 9.12 are two parameter models and we can convert between them using  $k_0 = k_m \exp(E/T_m)$ . But in Equation 9.12 we estimate the rate constant at the mean temperature in contrast to infinite temperature ( $1/T = 0$ ) in Equation 9.10.

If we estimate  $E$  and  $k_m$  in place of  $k_0$  we obtain the results shown in Figure 9.11. Notice that the correlation between  $k_m$  and  $E$  is much reduced compared to  $k_0$  and  $E$ . In fact, reporting confidence limits

$$\begin{bmatrix} \ln k_m \\ E \end{bmatrix} = \begin{bmatrix} 0.75 \\ 100 \end{bmatrix} \pm \begin{bmatrix} 0.05 \\ 60 \end{bmatrix}$$

is an accurate representation of the true 95% confidence interval ellipse.  $\square$

#### 9.2.4 Least Squares with Unknown Variance

In the previous problem, we assumed the variance in the error was known to us. It is often the case that we do not know the measurement



error variance, but must estimate it also from the data. In this case, it can be shown that the distribution of parameters is a multivariate  $t$ -distribution (instead of a normal distribution) with its maximum at the least-squares estimate as before

$$\hat{\theta} = (X^T X)^{-1} X^T y \quad (9.13)$$

For the confidence intervals, we merely compute the size of the ellipse containing a given probability of the multivariate  $t$ -distribution. That can be shown to be an  $F$  probability function [5]. Given the number of estimated parameters,  $n_p$ , the confidence level,  $\alpha$ , and the number of data points,  $n_d$ ,

$$\frac{(\theta - \hat{\theta})^T X^T X (\theta - \hat{\theta})}{s^2} \leq n_p F(n_p, n_d - n_p, \alpha) \quad (9.14)$$

defines the confidence interval ellipse. The  $F$  distribution is also tabulated in statistics handbooks [4] and available in computing environments. The sample variance

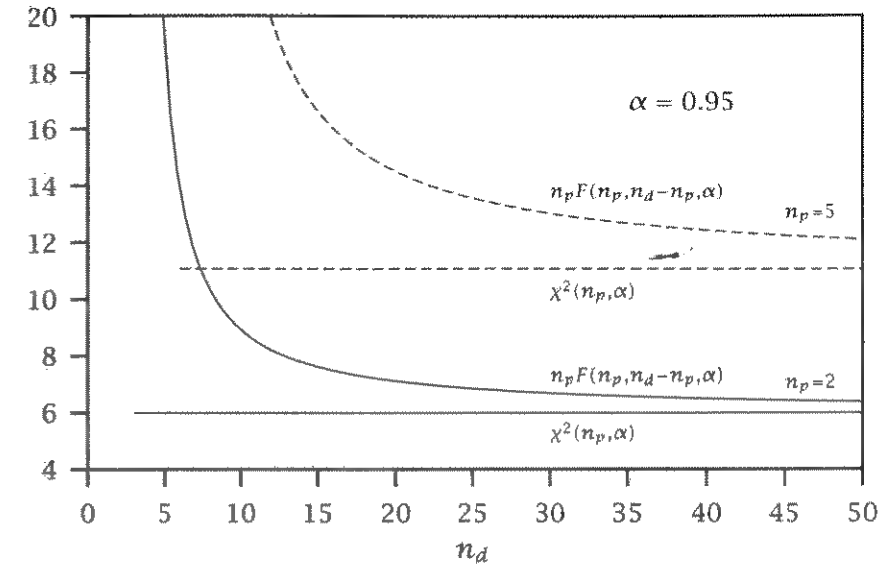
$$s^2 = \frac{1}{n_d - n_p} (y - X\hat{\theta})^T (y - X\hat{\theta}) \quad (9.15)$$

is the estimate of the unknown error variance. Notice the number of data points,  $n_d$ , shows up in the confidence interval when the error variance is unknown. In the limit of large  $n_d$ , the  $F$  distribution converges to the  $\chi^2$ .

$$\lim_{n_d \rightarrow \infty} n_p F(n_p, n_d - n_p, \alpha) = \chi^2(n_p, \alpha)$$

and the confidence intervals given in Equations 9.9 and 9.14 are the same. The sample variance also converges to the error variance in the limit of large number of data points.

On the other hand, in engineering problems we often must contend with few data points, because they are difficult or expensive to obtain. Figure 9.12 shows the values of  $\chi^2$  and  $n_p F$  as a function of the number of data points for 95% confidence limits, and the cases of two and five estimated parameters. Notice that  $n_p F$  is much larger than  $\chi^2$  at small  $n_d$ . With a small number of data points, estimating the variance as well as the parameters from the data inflates significantly the confidence intervals. Figure 9.12 quantifies this effect; we can see that if we have about 10 times as many data points as parameters,  $n_d > 10n_p$ , this effect is rather small.



**Figure 9.12:** Values of  $\chi^2$  and  $F$  versus the number of data points when estimating 2 and 5 parameters; the larger values of  $F$  show information loss (larger parameter uncertainty) when estimating the measurement variance from the data.

Although we should expect larger confidence intervals when  $n_d$  is small, that may not be what happens with a particular dataset as we show with the next example.

### Example 9.2: Unknown measurement variance and few data points

Consider the data shown in Figure 9.13 with two unknown parameters and only 10 data points. The measurement errors are drawn from a normal distributed with zero mean and variance  $\sigma^2 = 10^{-3}$ . Compute the best estimates of activation energy and mean rate constant and the 95% confidence intervals for the cases of known and unknown measurement variance.

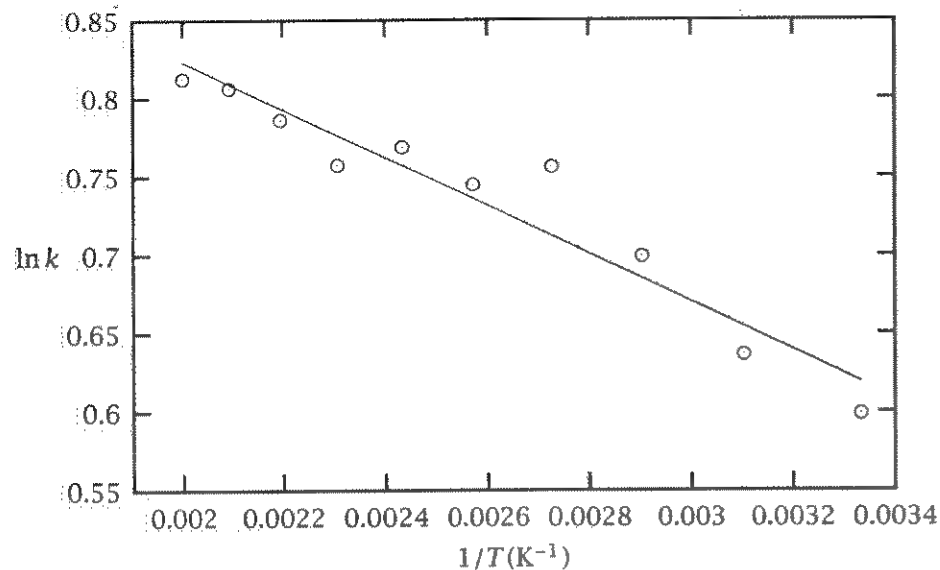


Figure 9.13: Parameter estimates with only 10 data points.

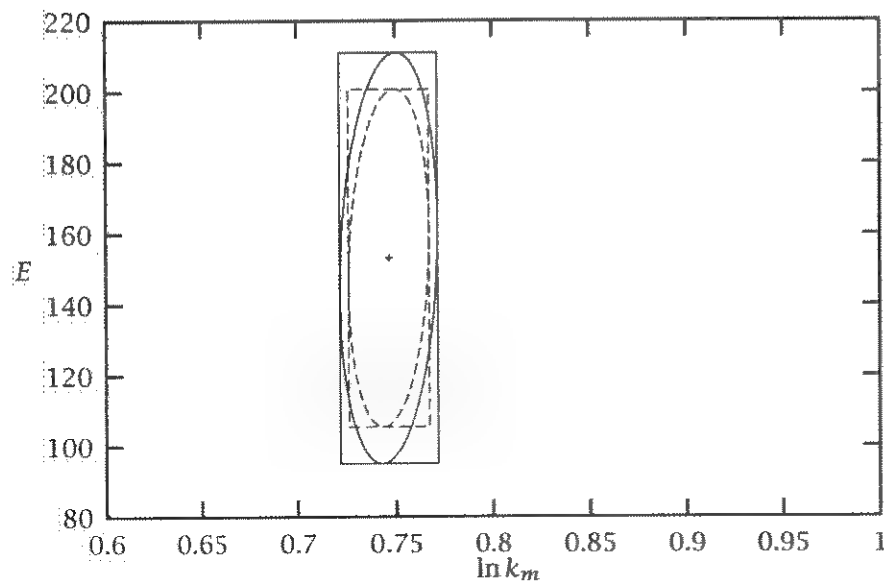


Figure 9.14: Confidence intervals with known (solid line) and unknown (dashed line) error variance.

### Solution

We calculate  $\hat{\theta}$  from Equation 9.13 and the sample variance from Equation 9.15 and obtain

$$\hat{\theta} = \begin{bmatrix} 0.747 \\ 153 \end{bmatrix}, \quad s^2 = 0.000454$$

We construct the 95% confidence interval from Equations 9.9 and 9.14. We obtain from a statistics table  $F(2, 8, 0.95) = 4.46$  so the two confidence intervals are given by

$$(\theta - \hat{\theta})^T X^T X (\theta - \hat{\theta}) \leq (0.001)(5.99), \quad \text{known variance}$$

$$(\theta - \hat{\theta})^T X^T X (\theta - \hat{\theta}) \leq (0.000454)(2)(4.46), \quad \text{unknown variance}$$

The two 95% confidence ellipses are shown in Figure 9.14. Notice that although  $n_p F$  is 50% larger than  $\chi^2$ , our 95% confidence interval for the unknown measurement variance case is *smaller* than the known measurement variance case.

Can you resolve this apparent contradiction? What experiment can you propose in which we would find the confidence interval with estimated measurement variance to be about 50% larger than known measurement variance in agreement with the statistics in Figure 9.12? □

### 9.2.5 Nonlinear Least Squares

In the previous two examples we transformed the model by taking logarithms to obtain a linear estimation problem. In many situations we do not want to make such a transformation, or such transformations simply do not exist. For example, the measurement error may be distributed normally in the original variables and the transformation may distort this distribution and subsequent parameter estimates and confidence intervals.

If we decide to treat the estimation problem using the nonlinear model, the problem becomes more challenging. As we will see, the parameter estimation becomes a nonlinear optimization that must be solved numerically instead of a linear matrix inversion that can be solved analytically as in Equation 9.8. Moreover, the confidence intervals become more difficult to compute, and they lose their strict probabilistic interpretation as  $\alpha$ -level confidence regions. As we will see, however, the “approximate” confidence intervals remain very useful in nonlinear problems. The numerical challenges for nonlinear models

can be addressed successfully in many reaction engineering problems if we employ high-quality numerical software.

Consider the nonlinear model

$$y_i = h(x_i, \theta), \quad i = 1, \dots, n_d \quad (9.16)$$

and the least-squares objective

$$\Phi(\theta) = \sum_{i=1}^{n_d} (\tilde{y}_i - h(x_i, \theta))^2$$

We obtain the parameter estimates by solving the optimization problem

$$\min_{\theta} \Phi(\theta)$$

subject to Equation 9.16. Call the solution to this problem  $\hat{\theta}$ . We now consider the function  $\Phi(\theta)$  near the optimum. Expanding in a second-order multivariable Taylor series gives

$$\Phi(\theta) \approx \Phi(\hat{\theta}) + (\nabla \Phi)^T \Big|_{\theta=\hat{\theta}} (\theta - \hat{\theta}) + \frac{1}{2} (\theta - \hat{\theta})^T H \Big|_{\theta=\hat{\theta}} (\theta - \hat{\theta}) \quad (9.17)$$

in which the gradient of the objective function is the vector of first derivatives of  $\Phi$  with respect to the model parameters

$$(\nabla \Phi)_j = \frac{\partial \Phi}{\partial \theta_j}$$

and the Hessian of the objective function is the matrix of second derivatives

$$H_{kj} = \frac{\partial^2 \Phi}{\partial \theta_k \partial \theta_j}$$

The gradient is zero at the optimum (see also Figure 3.5 of Chapter 3), which allows us to rearrange Equation 9.17 to give

$$\Phi(\theta) - \Phi(\hat{\theta}) \approx \frac{1}{2} (\theta - \hat{\theta})^T H \Big|_{\theta=\hat{\theta}} (\theta - \hat{\theta}) \quad (9.18)$$

Therefore, lines of constant objective function are approximately quadratic functions as shown in Figure 9.6 and we use  $H$  as the  $A$  matrix. This quadratic approximation using the Hessian matrix evaluated at the optimum is accurate if we are in the neighborhood of the optimal parameter values. We also can obtain order-of-magnitude confidence intervals using the relation

$$(\theta - \hat{\theta})^T H \Big|_{\theta=\hat{\theta}} (\theta - \hat{\theta}) \leq 2s^2 n_p F(n_p, n_d - n_p, \alpha) \quad (9.19)$$

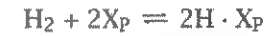
in which  $s^2$  is again the sample variance

$$s^2 = \frac{1}{n_d - n_p} \sum_{i=1}^{n_d} (\tilde{y}_i - h(x_i, \hat{\theta}))^2 = \frac{\Phi(\hat{\theta})}{n_d - n_p} \quad (9.20)$$

These confidence intervals are correct only if the model is linear, in which case  $H = 2X^T X$ . The intervals should be checked occasionally with Monte Carlo simulations when the model is nonlinear. We illustrate this check in Example 9.4.

### Example 9.3: Fitting single and multiple adsorption experiments

To illustrate the use of nonlinear models, we study an adsorption experiment. The system studied was the adsorption of  $H_2$  on a Pd catalyst with  $SiO_2$  support [25, 24]. The adsorption is assumed dissociative. Both the catalyst and the support adsorb  $H_2$  so the adsorption isotherm model is



in which  $X_P$  represents a Pd vacant site and  $X_S$  represents a  $SiO_2$  vacant site. We can apply the methods of Section 5.6 to derive the surface coverage of H atoms

$$\bar{c}_H = \frac{\bar{c}_{mP} \sqrt{K_P} \sqrt{c_{H_2}}}{1 + \sqrt{K_P} \sqrt{c_{H_2}}} + \frac{\bar{c}_{mS} \sqrt{K_S} \sqrt{c_{H_2}}}{1 + \sqrt{K_S} \sqrt{c_{H_2}}} \quad (9.21)$$

It is known that the adsorption constant on the Pd is large, so we may reduce the model to

$$\bar{c}_H = \bar{c}_{mP} + \frac{\bar{c}_{mS} \sqrt{K_S} \sqrt{c_{H_2}}}{1 + \sqrt{K_S} \sqrt{c_{H_2}}} \quad (9.22)$$

and we have three parameters to estimate from data,

$$\theta^T = \left[ \sqrt{K_S} \quad \bar{c}_{mS} \quad \bar{c}_{mP} \right] \quad (9.23)$$

As the measure of fit to the data we choose a least-squares objective,

$$\Phi = \sum_i (\tilde{c}_{Hi} - \bar{c}_{Hi})^2 \quad (9.24)$$

and propose the optimization problem

$$\min_{\theta} \Phi(\theta) \quad (9.25)$$

subject to Equation 9.22 to determine the parameter estimates.

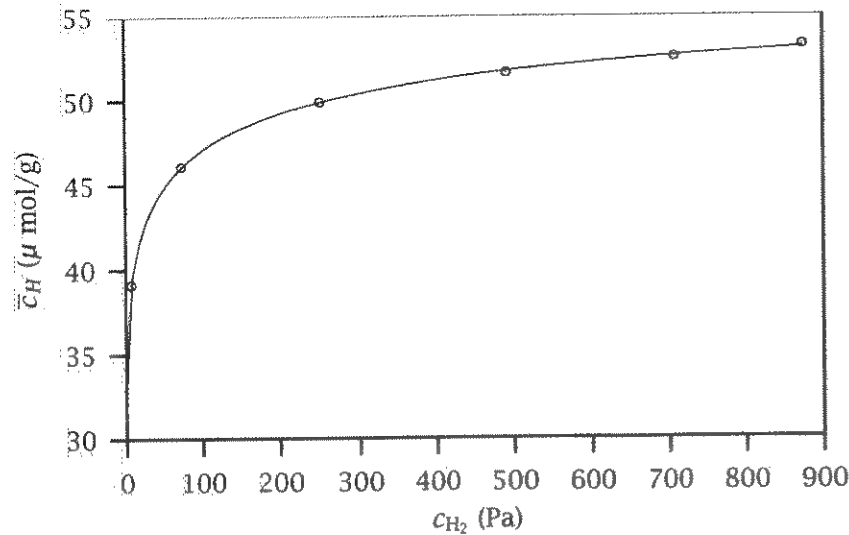


Figure 9.15: Model fit to a single adsorption experiment.

### Solution

Figure 9.15 shows the best fit to a single adsorption experiment. The parameters and the 95% confidence intervals are given by

$$\hat{\theta} = \begin{bmatrix} \sqrt{K_S} \\ \bar{c}_{mS} \\ \bar{c}_{mP} \end{bmatrix} = \begin{bmatrix} 0.127 \\ 26.1 \\ 32.4 \end{bmatrix} \pm \begin{bmatrix} 0.048 \\ 1.5 \\ 2.4 \end{bmatrix} \quad (9.26)$$

We can see that the model fit to the data is excellent, and the parameters are determined with fairly tight confidence intervals.

For this catalyst sample, additional adsorption experiments are available. After the first adsorption experiment, reaction studies were performed with the catalyst sample. Then the catalyst was regenerated with heat treatment, and a second adsorption experiment was performed, followed by an additional reaction study, and so on. The additional adsorption data are shown in Figure 9.16. Notice these are not replicate experiments, because the adsorption experiments are performed after different reaction studies have been performed, and the catalyst regeneration step does not necessarily return the catalyst to exactly the same condition. Estimating the parameters by fitting all of these data simultaneously produces the solid line in Figure 9.16 and

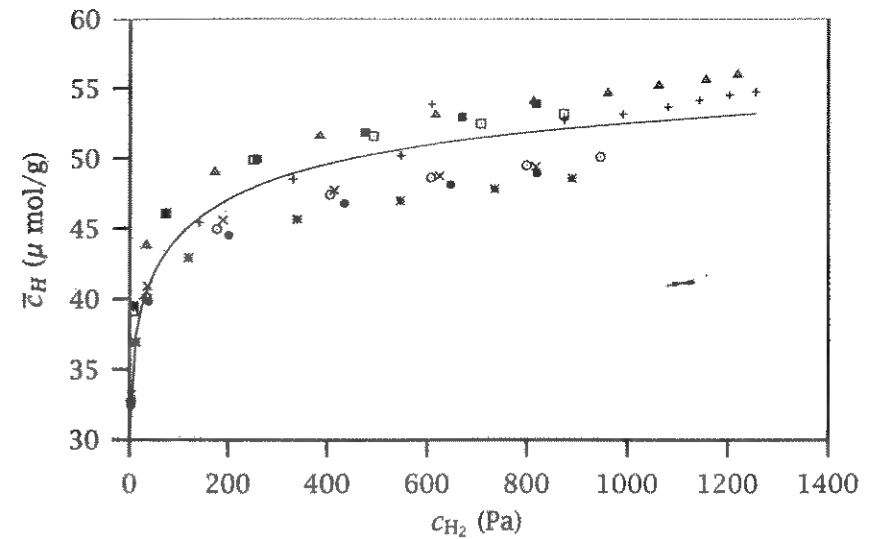


Figure 9.16: Model fit to all adsorption experiments.

the following parameter values and confidence intervals

$$\hat{\theta} = \begin{bmatrix} \sqrt{K_S} \\ \bar{c}_{mS} \\ \bar{c}_{mP} \end{bmatrix} = \begin{bmatrix} 0.0895 \\ 30.4 \\ 30.1 \end{bmatrix} \pm \begin{bmatrix} 0.081 \\ 4.7 \\ 5.4 \end{bmatrix} \quad (9.27)$$

The total coverage estimated here, 60.5  $\mu\text{mol/g}$ , compares favorably to the apparent saturation limit or total uptake value of 55  $\mu\text{mol/g}$  reported by Natal-Santiago et al. [24, p.157].

Notice because the reaction studies have introduced significant variability in the data, the confidence intervals for the parameter values are significantly larger than those for the single adsorption experiment. Both sets of estimated parameters are valid, but they have different meanings. If the model is intended to represent the catalyst in its native state, one might use the estimated values from a single adsorption experiment on a freshly prepared sample. If the model is intended to represent the "average behavior" of the catalyst during the period it is used and regenerated, one would naturally use the estimated values from the many adsorption experiments.  $\square$

### 9.2.6 Design of Experiments

Experimental design is a large topic and we can only mention several of the important issues here. To keep this discussion focused on parameter estimation for reactor models, we must assume the reader has had exposure to a course in basic statistics [4]. We assume the reader understands the source of experimental error or noise, and knows the difference between correlation and causation. The process of estimating parameters in reactor models is part of the classic, iterative scientific method: hypothesize, collect experimental data, compare data and model predictions, modify hypothesis, and repeat. The goal of experimental design is to make this iterative learning process *efficient*.

Our goal at this point can be quite specific because we have already built the causal models of interest to us in Chapters 2-8, and require only the remaining small task of finding best parameter values and quantifying the parameter uncertainty. We are not concerned at this point with embedding this step in the larger task described previously: deciding if our model structure is well chosen and the model assumptions are appropriate given the data.

The main goal of our experimental design then is to choose experiments that make the uncertainty in the parameter estimates small. We also must decide what small means when it comes to parameter uncertainty.

**Optimal experimental design.** *Optimal* experimental design is an experimental design with the express purpose of making the parameter uncertainty *as small as possible*. Before developing precise, quantitative methods of experimental design, let's build up some physical intuition about the problem we are addressing. Imagine we wish to use a straightedge to draw a line on a piece of paper — a task most of us face early in childhood. We mark two points on the paper, lay the straightedge next to the two points, and draw the line. To make the problem more interesting and realistic, consider that no matter how carefully we measure, our placement of the points is subject to small errors. As shown in Figure 9.17, the small circle represents the regions in which we might actually mark a point, with the center of the circle representing the ideal location where we are supposed to mark the point. The size of these circles tells us how much uncontrolled error exists in our experiment. An experienced draftsman or carpenter with steady hands may be able to measure carefully and place points within fairly small circles; a young child may require rather large circles to bound the ex-

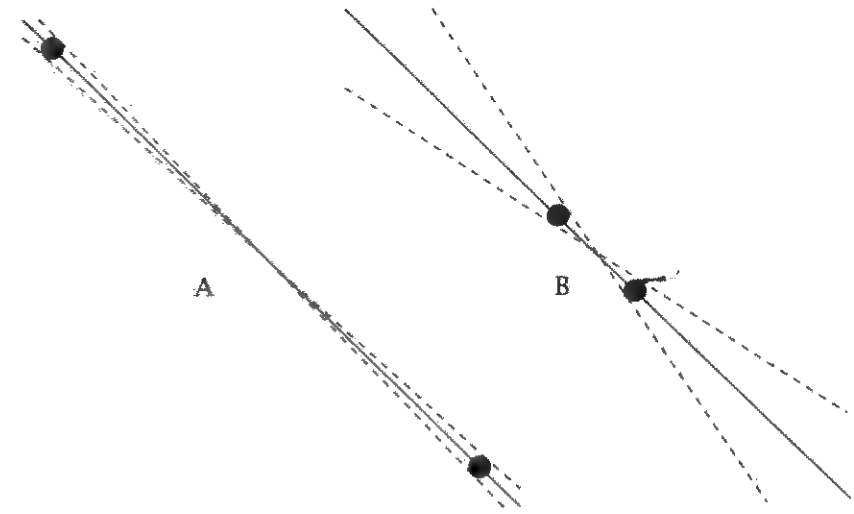


Figure 9.17: Drawing a line through two points under measurement error: points far apart (A); points close together (B).

pected errors. In all cases we face error in point placement and that error produces error in the final line. If we wish to minimize the error in the constructed line, one option is to make the measurement error circles small. Improvements in measurement technology are always an attractive option. A reasonably high-resolution printer, such as the one used to print this page, for example, may be able to locate a point to within 1/1200 inch. But regardless of the current state of technology, the measurement noise persists at some nonzero level.

Experimental design provides another option. Now we ask the question: where on the line should we *locate* the two points to give us the smallest error in the line placement? Our experience and intuition probably lead us to guess rather quickly that we want to spread the two points apart as far as possible. As shown in Figure 9.17, if the points are far apart, the spread in the slopes of the lines is small. If the points are close together, the spread in slopes is large. If the points are close enough that the circles overlap, then a line with *any* slope is possible.

Optimal experimental design generalizes this intuitive notion of how to construct the experiment to minimize the effect of the experimental errors on the estimated parameters. It also can provide exper-

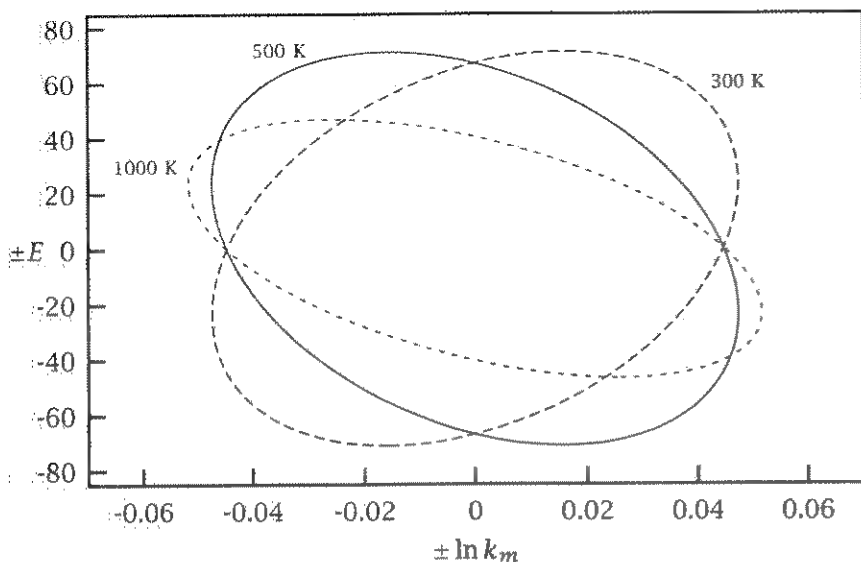


Figure 9.18: Effect of next measurement temperature on parameter confidence intervals.

experimental designs in situations in which our intuition does not provide ready answers. We explore one such nonobvious design issue in an industrial case study in Section 9.3. But for illustrative purposes, consider again the least-squares estimation of the mean rate constant and activation energy from measuring the rate constant at several temperatures. Assume we have temperature constraints in our experimental system, and can only operate the experiment and collect rate data in the temperature range

$$300 \text{ K} \leq T \leq 500 \text{ K}$$

We cannot estimate two parameters unless we have at least two measurements, so, motivated by Figure 9.17, we choose the first two measurements at the extreme limits of the temperatures,  $T = 300 \text{ K}$ ,  $T = 500 \text{ K}$ . We next explore what happens to our confidence intervals as we make different choices for the next experiment. Figure 9.18 shows the confidence intervals for three different choices of the next experiment. The choices of 300 K and 500 K produce two 95% confidence regions of the same size, but with different orientations. This result is consistent

with our physical intuition of drawing straight lines. If we add points at the *right* end of the line shown in Figure 9.13 ( $T = 300 \text{ K}$ ), then that anchors somewhat the right end of the line, and to increase the value of  $\ln k_m$  we must also *increase* the slope. So the 300 K ellipse is stretched in the direction with positive correlation between  $\ln k_m$  and  $E$ . Alternatively, if we add points at 500 K, then we have anchored somewhat the *left* end of the line in Figure 9.13, and to increase the value of  $\ln k_m$ , we must *decrease* the slope. The 500 K ellipse is therefore stretched in the direction with negative correlation between  $\ln k_m$  and  $E$ . If we were able to violate our temperature limits and perform the next experiment at 1000 K, we see that we can achieve more certainty in the activation energy, at the cost of slightly more uncertainty in the intercept. This result is exactly what we observed in Figure 9.17 — to obtain less uncertainty in the slope, spread the points as far apart as is practical.

With this background, we now ask very specific questions and design experiments with precise goals. If experiments are either time consuming or expensive, we ask the natural question: *where should we place the next experiment to obtain as much information about the parameters as possible?* To answer this question we have to decide what information we seek. For example, we may want to know only the activation energy, we may want to know only the intercept, we may want to know both of them, but one is somewhat more important than the other. Figure 9.19 displays the size of the sides of the box tangent to the 95% confidence ellipse as we vary the temperature of the next experiment. If we are interested in the activation energy, we see that the next experiment should be placed at either  $T = 300 \text{ K}$  or  $T = 500 \text{ K}$ . Both values are equally good. In this problem we chose  $T_m = 375 \text{ K}$ , i.e., the mean of inverse temperature,

$$T_m = \frac{1}{1/300 + 1/500} = 375 \text{ K} \quad (9.28)$$

If we are interested in the value of the rate constant at the mean temperature, Figure 9.19 shows we should place the next experiment at the mean temperature  $T = T_m$ .

The results in this case are not too surprising; but that is the main reason we studied this problem, to build some intuition about what to ask and expect of optimal experimental design. One big advantage of the approach is that it is quantitative. In this problem, we can guess immediately to put the next experiment at the upper or lower temperature limit if we want to know the activation energy, but would we know that the uncertainty would change from 82 to 72 when we did so? If

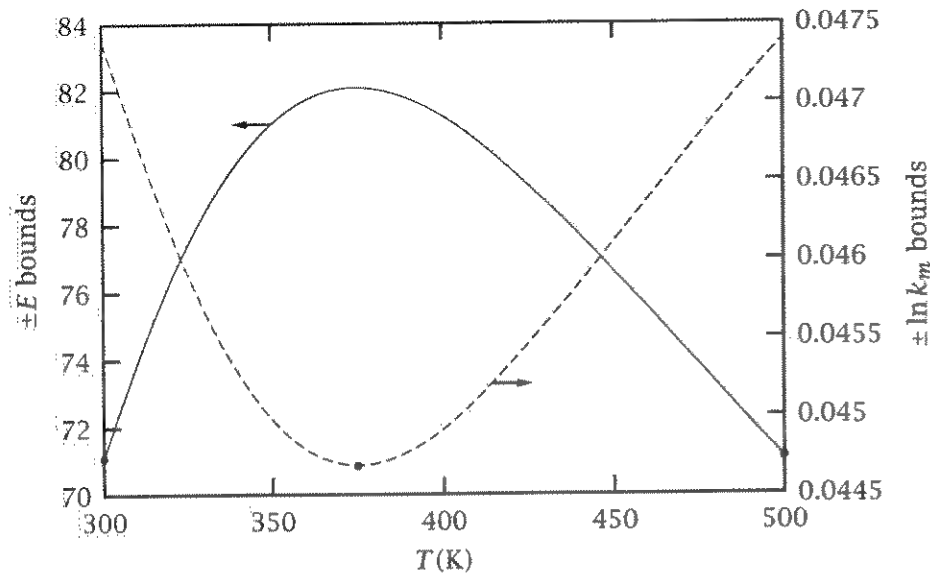


Figure 9.19: Uncertainty in activation energy  $E$  and rate constant  $\ln k_m$  versus next measurement temperature.

that improvement is not large enough for our purposes, we would know further experiments are required. We can answer the question of how many further experiments are required to meet a given bound on uncertainty. The second big advantage of this approach is that it is general and can be applied routinely in even complex experimental situations in which our intuition is not highly developed. As we mentioned previously, experimental design is a large topic, and the interested reader can find a wealth of literature discussing its many aspects [1].

**Replicate experiments.** Finally, we examine the impact of replicating experiments. The main reason one replicates an experiment is to obtain a direct measure of the experimental reproducibility. By replication we are confronted with the simple experimental fact that no matter how much care we take and how much effort we exert, the results of two experiments are never exactly the same. The differences among replicated experiments give us a direct quantitative measure of the experimental error or noise. These errors are partly due to our inability to measure accurately the outputs of interest, but they are also due to our inability to control completely the experimental environment. The

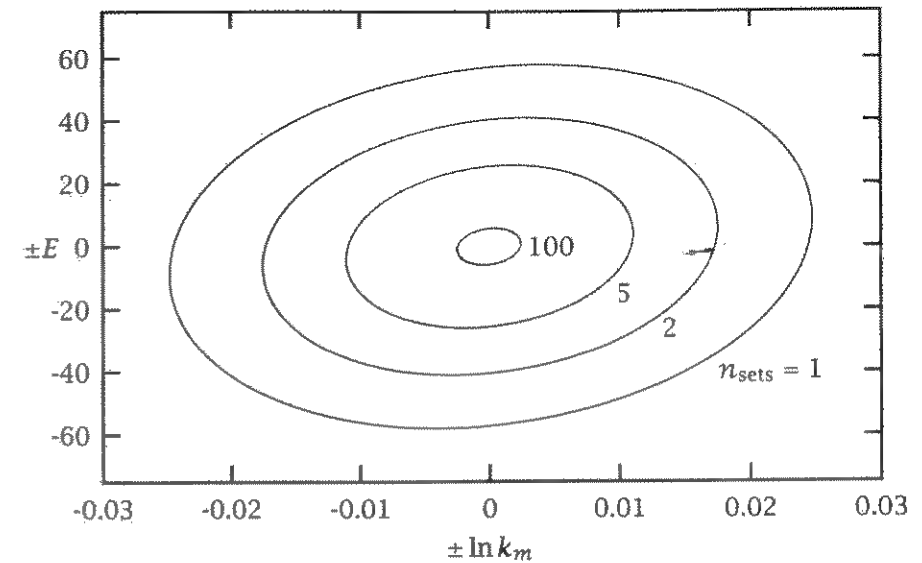


Figure 9.20: Uncertainty in activation energy  $E$  and rate constant  $\ln k_m$  versus number of replicated experiments; 10 equally spaced temperatures per experiment.

irreproducibility of the experiment is one of the key motivations for the random variable description of  $e$  in Equation 9.6.

But replicating an experiment has another interesting and expected benefit. It increases our confidence in our conclusions. To see this benefit, we examine again the least-squares problem for estimating  $k_m, E$ . Assume now the experiments are *inexpensive* and we are not concerned about performing optimal experiments. We simply make measurements at 10 evenly spaced temperatures as shown in Figure 9.13. If we perform one set of 10 measurements we achieve the confidence intervals we have shown previously in Figure 9.14. What happens if we simply repeat the experiment and perform two sets of 10 measurements? The confidence region becomes smaller, as shown in Figure 9.20. We also show the result of using 5 and 100 replicates of the experiment. The confidence intervals continue to shrink as we replicate the experiments. So replicating experiments provides another general avenue for increasing the confidence in the parameter estimates. Be aware, however, as discussed further in Exercise 9.8, that this “brute force”



approach may be time consuming and expensive.

### 9.2.7 Parameter Estimation with Differential Equation Models

Now we turn to the single most important parameter estimation problem in chemical reactor modeling: determining reaction-rate constants given dynamic concentration measurements. We devote the rest of the chapter to developing methods for this problem.

To get started, we consider a simple reactor model consisting of a single differential equation with a single experimentally measured quantity

$$\frac{dx}{dt} = f(x; \theta) \quad (9.29)$$

$$x(0) = g(x_0; \theta) \quad (9.30)$$

$$y = h(x) \quad (9.31)$$

in which  $x$  is the single material balance of interest,  $\theta$  are the unknown model parameters,  $x_0$  is the initial condition, and  $y$  is the experimentally measurable quantity. For simplicity let us assume here that  $x$  itself is measured, in which case  $h(x) = x$ . As we see in several of the examples, it may be necessary to include some of the initial conditions also as unknown parameters. Often  $t$  is time, but in steady-state tubular PFRs, reactor volume or length can take the place of time without changing the structure of the parameter estimation problem.

As before, we define a least-squares objective to measure our fit to the data

$$\Phi(\theta) = \sum_i (\tilde{x}_i - x_i)^2 \quad (9.32)$$

in which  $\tilde{x}_i$  is the experimental measurement at time  $t_i$ , and  $x_i$  is the solution to the model at time  $t_i$ ,  $x_i = x(t_i; \theta)$ . Note  $x_i$  is the only part of the objective function that depends on the model parameters. Again, we minimize this objective function to obtain our parameter estimates

$$\min_{\theta} \Phi(\theta) \quad (9.33)$$

subject to Equations 9.29–9.31.

The major change is that the model constraint consists of nonlinear differential equations rather than linear or nonlinear algebraic equations as in the previous sections. The differential equations make it much more expensive to evaluate the constraints while solving the optimization problem. We can increase the efficiency of the optimizer if

we provide an accurate gradient of the objective function. Recall the gradient of the objective function is the vector of first derivatives of  $\Phi$  with respect to the model parameters. Differentiating Equation 9.32 gives

$$\frac{\partial \Phi}{\partial \theta_j} = -2 \sum_i (\tilde{x}_i - x_i) \frac{\partial x_i}{\partial \theta_j} \quad (9.34)$$

Differentiating a second time gives the Hessian of the objective function, which we again use to construct approximate confidence intervals

$$\frac{\partial^2 \Phi}{\partial \theta_k \partial \theta_j} = 2 \sum_i \left[ \frac{\partial x_i}{\partial \theta_k} \frac{\partial x_i}{\partial \theta_j} - (\tilde{x}_i - x_i) \frac{\partial^2 x_i}{\partial \theta_k \partial \theta_j} \right] \quad (9.35)$$

**Gauss-Newton approximation and sensitivities.** In the Gauss-Newton approximation of the Hessian, we neglect the second term in Equation 9.35 to yield,

$$\frac{\partial^2 \Phi}{\partial \theta_k \partial \theta_j} \approx 2 \sum_i \frac{\partial x_i}{\partial \theta_k} \frac{\partial x_i}{\partial \theta_j} \quad (9.36)$$

Two arguments support the Gauss-Newton approximation. If the model fits the data well at the optimal value of parameters, the residuals are small in magnitude and of different signs. The sum in the second term is then small. Alternatively, the second derivative of the model may be small compared to the first derivative. If the model is linear in the parameters, for example, the second derivatives are identically zero and the Gauss-Newton approximation is exact. The Gauss-Newton approximation is not valid if the model solution is a highly nonlinear function of the parameters, or if the residuals are large and not randomly distributed about zero at the optimal value of parameters. Of course, in the latter case one should question the model structure because the model does not well represent the data. For the highly nonlinear case, one may try numerical finite difference formulas to compute the Hessian. Computing a reliable finite difference approximation for a *second* derivative is not a trivial matter either, however, and the step size should be carefully chosen to avoid amplifying the errors introduced by the finite numerical precision.

The first derivatives of the model solution with respect to the model parameters are known as the model sensitivities,

$$S_{ij} = \frac{\partial x_i}{\partial \theta_j} \quad (9.37)$$

The sensitivities also can be described as the solution to a set of differential equations; these equations are derived in Appendix A. This fact allows us to solve the model and sensitivity equations simultaneously with an ODE solver, rather than use finite difference formulas to obtain the sensitivities. Bard provides a readable account for further study on these issues [2]. Caracotsios, Stewart and Sørensen developed this approach and produced an influential software code (GREG) for parameter estimation problems in chemical reaction engineering [6, 29, 7].

From the sensitivities and model solution, we can then calculate the gradient of the objective function and the Gauss-Newton approximation of the Hessian matrix. Reliable and robust numerical optimization programs are available to find the optimal values of the parameters. These programs are generally more efficient if we provide the gradient in addition to the objective function. The Hessian is normally needed only to calculate the confidence intervals after the optimal parameters are determined. If we define  $\mathbf{e}$  to be the residual vector

$$e_i = \tilde{x}_i - x_i$$

We can express Equation 9.34 in matrix notation as

$$\nabla \Phi = -2\mathbf{S}^T \mathbf{e} \quad (9.38)$$

In terms of the sensitivities, we can express Equation 9.36 as

$$H_{kj} = 2 \sum_i S_{ik} S_{ij} = 2 \sum_i S_{ki}^T S_{ij} \quad (9.39)$$

If we write the last sum as a matrix multiplication, we can summarize this relationship in matrix notation

$$\mathbf{H} = 2\mathbf{S}^T \mathbf{S} \quad (9.40)$$

Given these expressions for the gradient and Hessian, we can construct a fairly efficient parameter estimation method for differential equation models using standard software for solving nonlinear optimization and solving differential equations with sensitivities.

#### Parameter estimation algorithm:

- ① Guess initial parameter values.
- ② Using an appropriate ODE solver, solve the model and sensitivity equations simultaneously given the current parameter values. Compute  $x_i$  and  $S_{ij}$ .

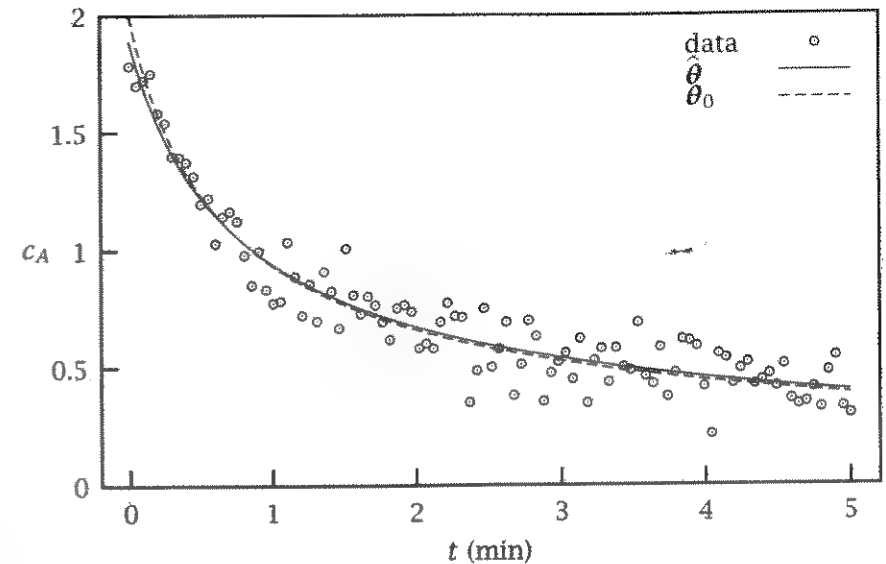


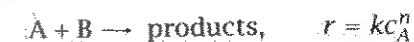
Figure 9.21: Experimental measurement and best parameter fit for  $n$ th-order kinetic model,  $r = kc_A^n$ .

- ③ Evaluate  $\Phi$  and  $\nabla \Phi$  using Equations 9.32 and 9.38.
- ④ Update parameter values to minimize  $\Phi$ . This step and the next are usually controlled by an optimization package.
- ⑤ Check convergence criteria. If not converged, go to ②.
- ⑥ On convergence, set  $\hat{\theta}$  to current parameter values. Calculate  $\mathbf{H}$  using Equation 9.39. Calculate confidence intervals using Equation 9.19.

#### Example 9.4: Fitting reaction-rate constant and order

We illustrate these methods on a classic reactor modeling problem: finding the rate constant and reaction order from isothermal concentration measurements in a batch reactor.

Consider an irreversible,  $n$ th order reaction



taking place in a liquid-phase batch reactor containing a large excess of reactant B. Given the measured concentration of component A shown in Figure 9.21, determine the best values of the model parameters.

### Solution

The material balance for species A is

$$\frac{dc_A}{dt} = -kc_A^n \quad (9.41)$$

$$c_A(0) = c_{A0} \quad (9.42)$$

Given the data shown in Figure 9.21, it does not seem reasonable to assume we know  $c_{A0}$  any more accurately than the other measurements, so we include it as a parameter to be estimated. The model therefore contains three unknown parameters

$$\theta^T = [k \quad c_{A0} \quad n]$$

We can generate a reasonable initial parameter set by guessing values and solving the model until the model simulation is at least on the same scale as the measurements. We provide this as the starting point, and then solve the nonlinear optimization problem in Equation 9.33 using the least-squares objective as shown in Equation 9.32. We then compute the approximate confidence intervals using Equation 9.19 with Equation 9.39 for  $H$ . The solution to the optimization problem and the approximate confidence intervals are given in Equation 9.43.

$$\theta_0 = \begin{bmatrix} k \\ c_{A0} \\ n \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2.0 \\ 2.5 \end{bmatrix} \quad \hat{\theta} = \begin{bmatrix} 0.47 \\ 1.89 \\ 2.50 \end{bmatrix} \pm \begin{bmatrix} 0.052 \\ 0.18 \\ 0.42 \end{bmatrix} \quad (9.43)$$

The parameters that we used to generate the data in Figure 9.21 also are given in Equation 9.43. Notice the estimates are close to the correct values, and we have fairly tight approximate confidence intervals.

Next we examine the quality of these approximate confidence intervals for this problem. Figure 9.22 shows the results of a Monte Carlo simulation study. In this study we generate 500 datasets by adding zero-mean measurement noise with variance  $\sigma^2 = 0.01$  to the model solution with the correct parameters. For each of these 500 datasets, we solve the optimization problem to obtain the parameter estimates. We also produce a value for the Hessian for each dataset, and we use the mean of these for  $H$ . Finally we calculate what fraction of these

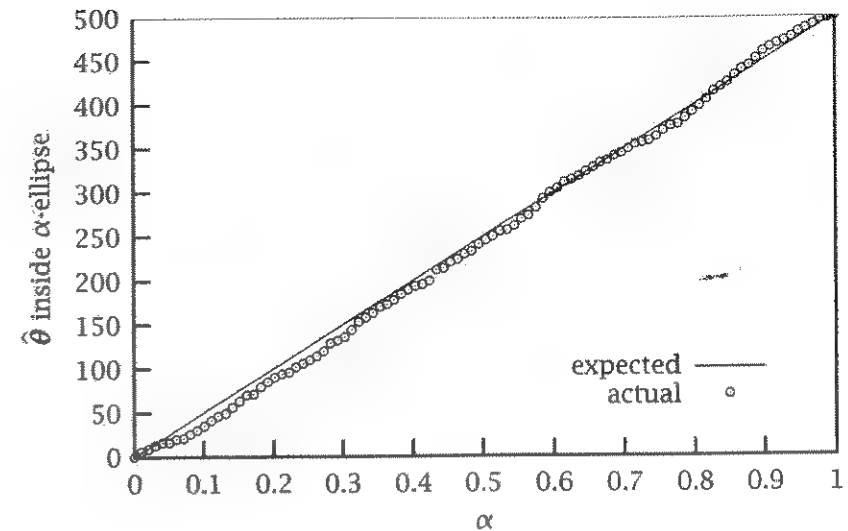


Figure 9.22: Monte Carlo evaluation of confidence intervals; the points indicate the actual number of estimates that fall within each  $\alpha$ -level ellipse for 500 replicated datasets.

500 estimates fall within each  $\alpha$ -confidence region as defined by Equation 9.19 as we vary  $\alpha$ . This result is plotted as the points in Figure 9.22. Notice for all  $\alpha$  values, approximately the correct number of parameter estimates fall within the corresponding ellipse, which indicates the approximate confidence intervals given in Equation 9.19 are fairly reliable for this problem. Be aware that this conclusion may not be true for other nonlinear problems, and should be checked. This computational check is fairly expensive; note that we had to solve 500 optimization problems to produce Figure 9.22, for example. But given the dramatic increase in computational speed, a few Monte Carlo simulations in the final stages of a modeling study are well justified.  $\square$

**Differential equation models with multiple measurements.** Most reactor models of interest contain balances for several species and therefore consist of several differential equations. To determine the parameters in such models, usually the concentrations of several species are measured as well. To treat this case, we consider the general dif-

ferential equation model

$$\frac{dx}{dt} = f(x; \theta) \quad (9.44)$$

$$x(0) = g(x_0; \theta) \quad (9.45)$$

$$y = h(x) \quad (9.46)$$

in which  $x$  is the vector of states that defines the reactor model,  $\theta$  are the unknown model parameters,  $x_0$  are the initial conditions, and  $y$  are the experimentally measurable quantities.

In this case, we again define a scalar objective function that measures our fit to the data. When we have different measured quantities, however, it often does not make sense to sum the squares of the residuals. The measured variables may differ in size from each other by orders of magnitude. The influence a measurement has on the objective also would be influenced by the arbitrary choice of the units of the measurement, which is obviously undesirable. The simplest way to address this issue is to employ *weighted* least squares. The reader should be aware that more general procedures are available for the multiple measurement case, including the maximum likelihood method [29, 30, 31]. For simplicity of presentation, we focus here on weighted least squares.

In weighted least squares, we combine the different measurements by forming the *weighted* sum of the residuals. Let  $e_i$  be the residual vector at the  $i$ th sample time

$$e_i = \hat{y}_i - h(x(t_i; \theta)) \quad (9.47)$$

The objective function is then

$$\Phi(\theta) = \sum_i e_i^T W e_i \quad (9.48)$$

in which  $W$  is a symmetric, positive-definite weighting matrix. Usually  $W$  is chosen to be a diagonal matrix. The elements on the diagonal are the weights assigned to each measurement type. To estimate the parameters we now solve

$$\min_{\theta} \Phi(\theta) \quad (9.49)$$

subject to Equations 9.44–9.46. The sensitivities are now a time-varying matrix,

$$S_{jk}(t_i) = \frac{\partial x_j(t_i; \theta)}{\partial \theta_k}$$

We can compute the gradient as before

$$\nabla \Phi = -2 \sum_i S_i^T \frac{\partial h_i^T}{\partial x_i} W e_i \quad (9.50)$$

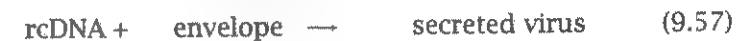
and the Gauss-Newton approximation of the Hessian is

$$H = 2 \sum_i S_i^T \frac{\partial h_i^T}{\partial x_i} W \frac{\partial h_i}{\partial x_i^T} S_i \quad (9.51)$$

in which  $S_i = S(t_i)$ ,  $x_i = x(t_i)$ , and  $h_i = h(x(t_i))$ .

### Example 9.5: Fitting rate constants in hepatitis B virus model

We illustrate using multiple measurements by revisiting the hepatitis B model introduced in Chapter 1



Find rate constants  $k_1$ – $k_6$  from the cccDNA, rcDNA and envelope protein measurements given in Figures 9.23–9.25. As before we assume the nucleotides and amino acids are in large excess, and cccDNA catalyzes Reactions 9.52 and 9.54.

### Solution

The reaction rates and production rates for Reactions 9.52–9.57 are given by

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{bmatrix} = \begin{bmatrix} k_1 C_A \\ k_2 C_B \\ k_3 C_A \\ k_4 C_A \\ k_5 C_C \\ k_6 C_B C_C \end{bmatrix} \quad \begin{bmatrix} R_A \\ R_B \\ R_C \end{bmatrix} = \begin{bmatrix} r_2 - r_4 \\ r_1 - r_2 - r_6 \\ r_3 - r_5 - r_6 \end{bmatrix} \quad (9.58)$$

in which A is cccDNA, B is rcDNA, and C is envelope. We write mass balances for these components to produce three differential equations.

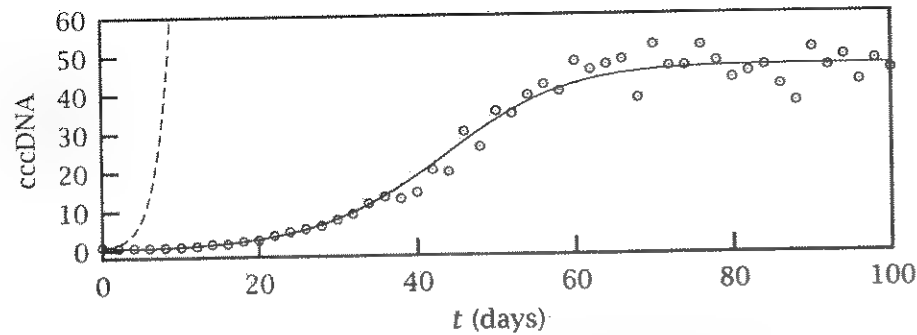


Figure 9.23: Species cccDNA versus time for hepatitis B virus model; initial guess and estimated parameters fit to data.

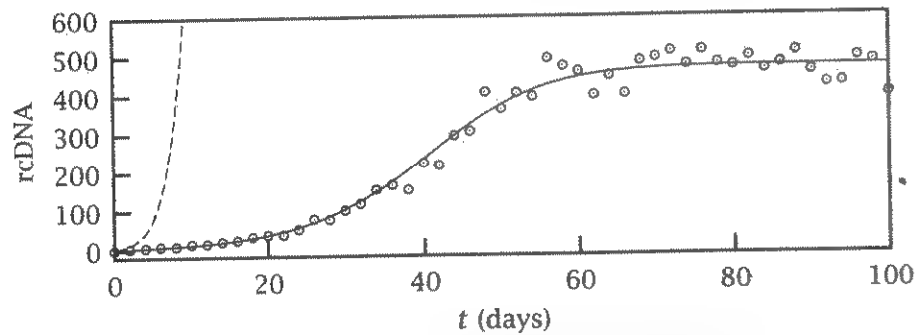


Figure 9.24: Species rcDNA versus time for hepatitis B virus model; initial guess and estimated parameters fit to data.

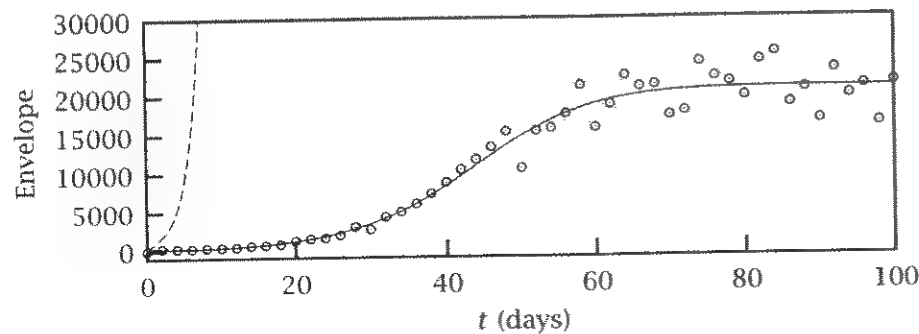


Figure 9.25: Envelope versus time for hepatitis B virus model; initial guess and estimated parameters fit to data.

The initial condition for the experiment is

$$\begin{bmatrix} c_A & c_B & c_C \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

and is assumed to be known precisely. The data are generated by assuming values for the rate constants and adding noise to the model solution. The measurement noise is assumed proportional to the size of the measured value. We choose

$$W = \text{diag} [1, 10^{-2}, 10^{-4}]$$

to reflect the relative sizes of the cccDNA, rcDNA and envelope species. This weight matrix gives each measurement roughly the same relative weight in the objective function.

When estimating several rate constants that may range in values by several orders of magnitude, it often is useful to define the model parameters to be the logarithms of the rate constants.

$$\theta_i = \log_{10}(k_i) \quad (9.59)$$

This transformation often makes it easier for optimization software to find accurate solutions, and automatically enforces the natural constraints that the rate constants are positive. The data in Figures 9.23-9.25 were generated using the following rate constants

$$k_0 = \begin{bmatrix} 2.0 \\ 0.025 \\ 1000 \\ 0.25 \\ 2.0 \\ 7.5 \times 10^{-6} \end{bmatrix} \quad \theta_0 = \begin{bmatrix} 0.30 \\ -1.60 \\ 3.00 \\ -0.60 \\ 0.30 \\ -5.12 \end{bmatrix} \quad (9.60)$$

We generated a random initial guess for the parameters and produced the following optimal set of parameters and approximate confidence interval

$$\theta_{\text{init}} = \begin{bmatrix} 0.80 \\ -1.13 \\ 3.15 \\ -0.77 \\ -0.16 \\ -5.46 \end{bmatrix} \quad \hat{\theta} = \begin{bmatrix} 0.32 \\ -1.50 \\ 3.50 \\ -0.49 \\ 0.85 \\ -5.08 \end{bmatrix} \pm \begin{bmatrix} 0.17 \\ 1.85 \\ 18.3 \\ 1.91 \\ 18.3 \\ 0.32 \end{bmatrix} \quad (9.61)$$

The dashed lines in Figures 9.23–9.25 show the solution of the model with the initial parameter values. As you can see, we are not giving the optimizer a very good initial guess. In spite of the poor initial guess, the optimizer is able to find a good solution. The model solution using the optimal parameter estimates is shown by the solid lines in the figures. Notice we have an excellent fit to the data. But notice also in Equation 9.61 that the confidence intervals are quite large, which means *we have little confidence in the estimated parameters in spite of the good fit to the data*. This outcome is not unusual when estimating many rate constants.

We may be satisfied with fitting the data in this way, but we may wish to reduce the model so that the estimated parameters have tighter confidence intervals. We pursue model reduction next. The Hessian is a  $6 \times 6$  matrix for this problem because we have six parameters. We cannot graphically represent the six-dimensional confidence hyperellipsoid as we did in Figure 9.6 in two dimensions; but we can do the next best thing, which is to examine the eigenvalues and eigenvectors. The eigenvalues are given in Equation 9.62

$$\lambda = \begin{bmatrix} 1.0 \times 10^8 \\ 4.1 \times 10^6 \\ 1.4 \times 10^6 \\ 2.4 \times 10^5 \\ 1.8 \times 10^3 \\ \text{---} \end{bmatrix} \quad v_5 = \begin{bmatrix} 0.03 \\ 0.69 \\ -0.03 \\ 0.71 \\ -0.03 \\ -0.11 \end{bmatrix} \quad v_6 = \begin{bmatrix} 0.00 \\ 0.03 \\ \text{---} \\ 0.03 \\ \text{---} \\ -0.00 \end{bmatrix} \quad (9.62)$$

The two smallest eigenvalues,  $\lambda_5$  and  $\lambda_6$ , are two orders of magnitude smaller than the largest four, which indicates we should remove about two degrees of freedom from the parameter estimation problem. The eigenvectors associated with these two eigenvalues,  $v_5$  and  $v_6$ , also are given in Equation 9.62. Notice that  $v_6$  is aligned almost entirely in the  $(\theta_3, \theta_5)$  coordinate directions; the other elements of this eigenvector are near zero. Similarly,  $v_5$  is aligned almost entirely in the  $(\theta_2, \theta_4)$  coordinate directions. These eigenvectors tell us that we can move the parameters a significant distance in the  $v_5, v_6$  directions without greatly changing the value of the least-squares objective.

To demonstrate this fact, consider the model solutions shown in Figures 9.26–9.28. The solid lines correspond to the estimated parameters in Equation 9.61, and the dashed lines correspond to the following

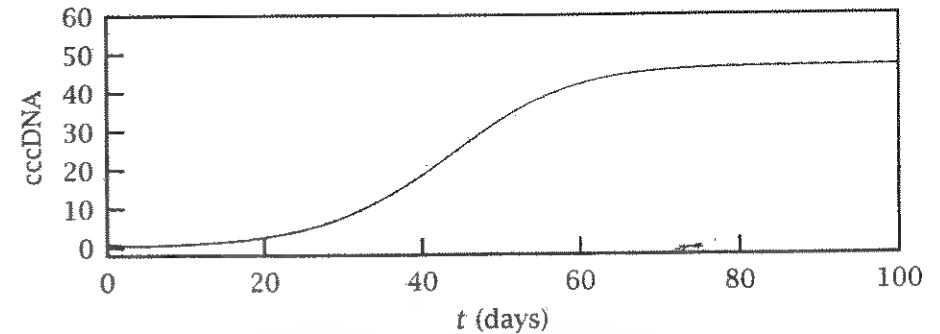


Figure 9.26: Species cccDNA versus time for hepatitis B virus model; estimated and perturbed parameter values.

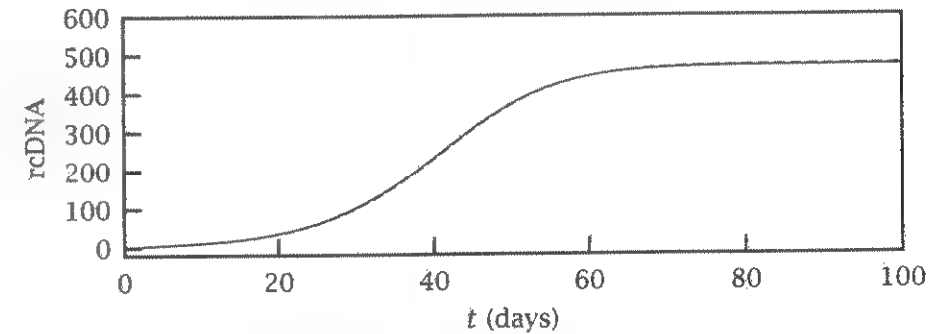


Figure 9.27: Species rcDNA versus time for hepatitis B virus model; estimated and perturbed parameter values.

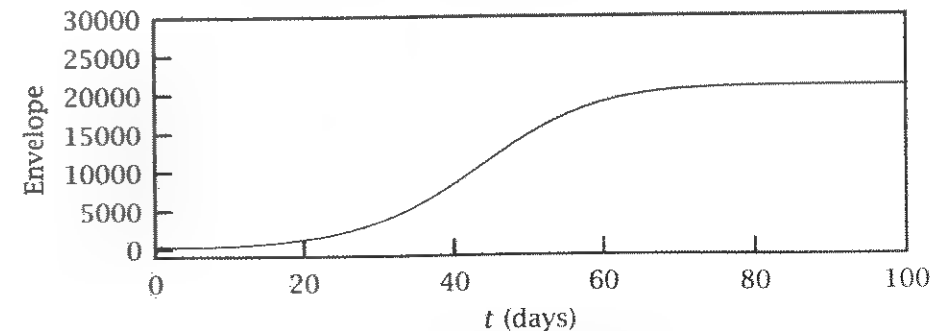


Figure 9.28: Envelope versus time for hepatitis B virus model; estimated and perturbed parameter values.

parameters

$$\theta = \begin{bmatrix} 0.32 & -1.47 & 4.21 & -0.46 & 1.56 & -5.08 \end{bmatrix}^T \quad (9.63)$$

We added  $v_6$  to  $\hat{\theta}$  to obtain Equation 9.63. Notice that even though we have changed the parameters by a large amount, we have not changed the simulations much at all. In other words, because of the logarithmic transformation, if we *multiply* parameters  $k_3$  and  $k_5$  by the same constant, we do not change the fit to the data. Similarly, if we multiply parameters  $k_2$  and  $k_4$  by some other constant, we do not change the fit to the data. We guess immediately that we should find the *ratios* of the rate constants  $k_3/k_5$  and  $k_2/k_4$ .

We also can gain insight by examining the steady-state solution to the model. If we set the production rates given in Equation 9.58 to zero and solve for the steady-state concentrations we obtain<sup>2</sup>

$$c_{As} = \frac{k_1 - k_4}{k_6 \left( \frac{k_4}{k_2} \right) \left( \frac{-k_1 + k_4 + k_3}{k_5} \right)}$$

$$c_{Bs} = \left( \frac{k_4}{k_2} \right) c_{As} \quad c_{Cs} = \left( \frac{-k_1 + k_4 + k_3}{k_5} \right) c_{As}$$

Notice in Equation 9.60 that  $k_3 \gg k_1, k_4$ , and, therefore, to an excellent approximation

$$c_{As} = \frac{k_1 - k_4}{k_6 \left( \frac{k_4}{k_2} \right) \left( \frac{k_3}{k_5} \right)} \quad c_{Bs} = \left( \frac{k_4}{k_2} \right) c_{As} \quad c_{Cs} = \left( \frac{k_3}{k_5} \right) c_{As} \quad (9.64)$$

We see again that the ratios of the rate constants  $k_3/k_5$  and  $k_2/k_4$  appear. Also the difference  $k_1 - k_4$  is the important parameter, not the individual values. We therefore choose the model parameters to be

$$\phi = \log_{10} \begin{bmatrix} k_1 - k_4 & k_2/k_4 & k_3/k_5 & k_6 \end{bmatrix}^T \quad (9.65)$$

and set  $k_3$  and  $k_4$  (or  $k_3$  and  $k_1$ ) to arbitrary values. Note they cannot be completely arbitrary because we require  $k_3 \gg k_1, k_4$  for this analysis to be valid. If we then perform parameter estimation with the reduced set of four parameters we obtain the tight confidence intervals shown in Equation 9.66.

$$\phi_0 = \begin{bmatrix} 0.24 \\ -1.00 \\ 2.70 \\ -5.12 \end{bmatrix} \quad \hat{\phi} = \begin{bmatrix} 0.26 \\ -1.00 \\ 2.65 \\ -5.06 \end{bmatrix} \pm \begin{bmatrix} 0.14 \\ 0.16 \\ 0.12 \\ 0.07 \end{bmatrix} \quad (9.66)$$

<sup>2</sup>Note the zero solution is also a steady state.

We do *not* need to use the correct values of  $k_3$  and  $k_4$  to estimate  $\phi$ ; even if we use values of  $k_3$  and  $k_4$  that are off by an order of magnitude from the values used to generate the data, the estimate of  $\phi$  is the same as in Equation 9.66. In fact, we must design a new experiment to determine the values  $k_3$  and  $k_4$  if they are of interest, because they cannot be found using data shown in Figures 9.23–9.25. Please note that these particular conclusions depend on the particular values of the rate constants we used to generate the data. If very different parameter values are chosen, the method of analysis can be used again but the model reduction may change.  $\square$

The complexity of this model probably precludes us from guessing quickly and intuitively that parameter pair  $k_3$  and  $k_4$  (or  $k_3$  and  $k_1$ ) cannot be determined from the experimental data, and we should estimate certain ratios and differences of rate constants instead of the rate constants themselves. By estimating all parameters, and analyzing the eigenvalues and eigenvectors of the Hessian, we have a systematic approach to develop such understanding fairly quickly when presented with a new problem of interest.

### 9.3 An Industrial Case Study

Reactor modeling in the industrial environment is a challenging task. When successful models are constructed, however, the payoff can be large, resulting in more efficient reactor operation, more consistent and higher quality product, and improved reactor designs. Given the difficulty of the modeling challenge in the best industrial circumstances, the least we should expect is that the computational and numerical procedures being employed are efficient and reliable, and are not compounding the difficulty of extracting models from the available data. To conclude this chapter we examine a small prototypical case study that illustrates some of the challenges we can expect when we model data from industrial facilities. This study was conducted in collaboration with colleagues at Kodak and more details are available [26].

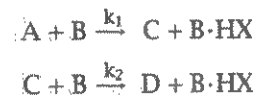
**End-point problems.** In many chemical reactions, two main reactants are combined to yield a primary desired product. A small quantity of one of the reactants often remains at the end of the reaction due to batch-to-batch variability in the purity and reactivity of the starting materials, and variability in the rate of side reactions. In some cases, an excess of one reactant at the final time does not present a problem and the opportunity for improvement with better control is proportional to



the quantity remaining, which is usually small. In these cases, an excess of one reactant is commonly added to ensure complete consumption of the other reactant.

In some cases, however, a small amount of neither starting material is tolerable. These materials may be difficult (impossible for practical purposes) to separate from the products, and their presence even in small quantities may prevent further processing steps from occurring. The excess addition of one reactant could also result in undesired reactions, usually after the limiting reagent is exhausted. In these cases, the cost of having the unreacted materials present in some amount greater than a small threshold is the cost of the entire batch of chemicals. This general problem in which neither reagent can exist at the end of the reaction is called the "end-point control" problem.

The reaction of interest is the dehalogenation of a dihalogenated starting material to form the divinyl product, which is used in photographic film production. It is assumed that the halide groups are removed from the starting material in two consecutive reactions:



- A  $\text{XH}_2\text{CCH}_2\text{RCH}_2\text{CH}_2\text{X}$
- B Organic base
- C  $\text{H}_2\text{C}=\text{CHRCH}_2\text{CH}_2\text{X}$
- D  $\text{H}_2\text{C}=\text{CHRCH}=\text{CH}_2$

The dihalogenated starting material (A) loses HX to the base (B) to form the mono-halogenated intermediate (C), which subsequently loses HX to the base to produce the desired final product (D). Over-addition of base causes polymerization and loss of the batch.

**Modeling.** The reaction is carried out in a well-stirred semi-batch reactor as shown in Figure 9.29. The reactor is initially charged with a weighed amount of component A and component B is added. The ma-

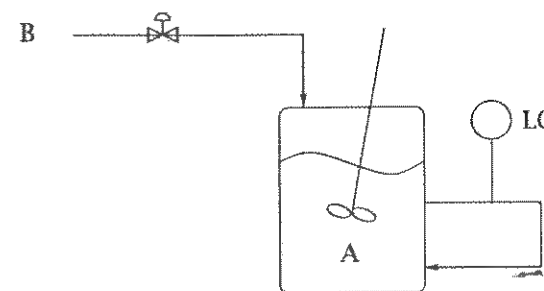


Figure 9.29: Semi-batch reactor addition of component B to starting material A.

terial balances for the semi-batch reactor are

$$\begin{aligned} \frac{dV_R}{dt} &= Q_f(t) \\ \frac{d(c_A V_R)}{dt} &= -k_1 c_A c_B V_R \\ \frac{d(c_B V_R)}{dt} &= Q_f c_{Bf} - (k_1 c_A c_B + k_2 c_C c_B) V_R \\ \frac{d(c_C V_R)}{dt} &= (k_1 c_A c_B - k_2 c_C c_B) V_R \\ \frac{d(c_D V_R)}{dt} &= k_2 c_C c_B V_R \end{aligned} \quad (9.67)$$

in which  $Q_f$  is the volumetric flowrate of base and  $c_{Bf}$  is the feed concentration of B. The primary assumptions in this model are: isothermal operation, negligible volume change upon reaction or liquid chromatograph (LC) sampling, perfect mixing, negligible side reactions such as polymerization of the C and D, and reaction with impurities and inhibitors in the starting material. The initial conditions for the ODEs are:

$$\begin{aligned} V_R(0) &= V_{R0} \\ n_A(0) &= n_{A0} \\ n_B(0) = n_C(0) = n_D(0) &= 0 \end{aligned}$$

It is assumed that  $c_{Bf}$  and  $V_{R0}$  are known precisely, and therefore the unknown model parameters are given by  $\theta = [k_1 \ k_2 \ n_{A0}]^T$ . Although the initial A charged to the reactor is weighed, the parameter  $n_{A0}$  is chosen to be adjustable to account for the unknown level of impurities

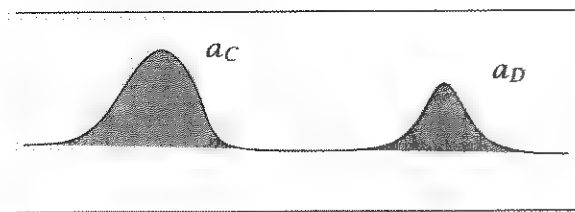


Figure 9.30: Depiction of an LC curve for determining the concentration of intermediate C and product D.

and the neglected side reactions. Notice that making good decisions of this type depend on experience and judgment, and reflect more the art rather than the science of reactor modeling.

The feed flowrate of base is measured by a mass flow meter. For composition analysis there is an on-line LC that draws a sample every 8 to 10 minutes and, after a 7-minute delay, reports the relative amounts of C and D. The unknown parameters,  $\theta$ , are estimated with the dynamic LC data. The LC-detector wavelength is set to detect the R-vinyl bond. Therefore only the C and D species show peaks in the LC output as depicted in Figure 9.30. The areas of the two peaks are related to the molar concentrations by

$$a_D = 2k_{lc}c_D \quad (9.68)$$

$$a_C = k_{lc}c_C \quad (9.69)$$

in which  $k_{lc}$  is the proportionality constant for the LC. By calculating normalized areas this constant can be removed,

$$y(t) = \frac{a_C}{a_C + a_D} = \frac{c_C}{c_C + 2c_D} \quad (9.70)$$

This normalized C peak area,  $y$ , is used for the parameter estimation. The parameter estimation is performed by minimizing the following relative least-squares objective function,

$$\Phi = \sum_i \left( \frac{\hat{y}_i - y_i}{y_i} \right)^2 \quad (9.71)$$

in which  $\hat{y}_i$  is the measured normalized C peak and  $y_i$  is the model prediction at the  $i$ th sampling time.

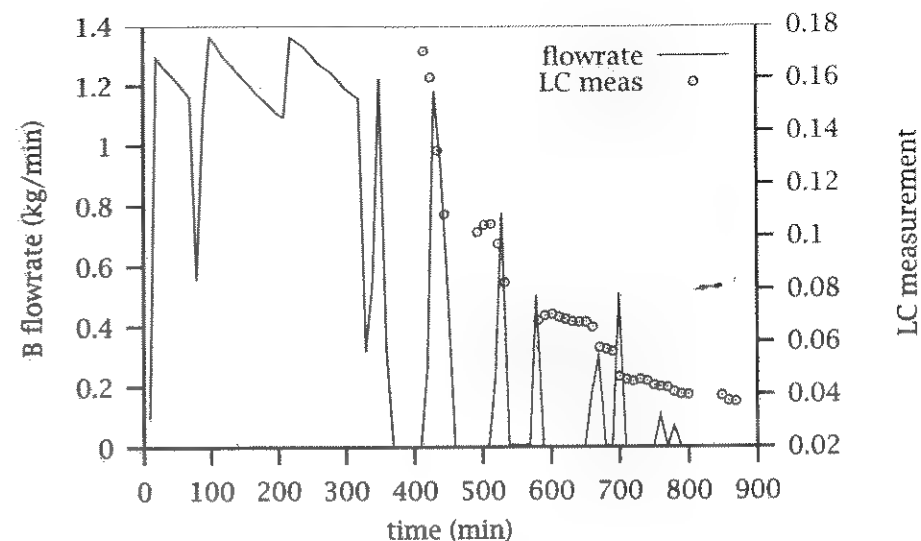


Figure 9.31: Base addition rate and LC measurement versus time.

**Conventional manufacturing procedure.** Normal operating procedure is to charge the reactor with some (only approximately known) amount of A, and add an initial amount of B that is sure not to overshoot the end point. An example of a typical B addition profile is given in Figure 9.31. After the initial B is added, the LC is switched on and the operator waits until the readings stabilize. The operator then sees how much C is remaining, and — based on experience — adds more B. The objective is to add enough B to consume all but 3% of the A. Ideally the operator would like to consume all the A, but the target is set at 3% to allow a margin for error. The penalty for overshoot is so high that only conservative addition steps are generally ever taken. After making the addition, the LC readings are again allowed to stabilize, and the operator again checks to see how close he is to the target before making another addition. This cycle repeats until the operator is satisfied that he is close enough (between 2–4%) to the target. As can be seen in Figure 9.31, the operator required 7 additions and about 500 minutes after first turning on the LC before he had determined how much B was required to reach end point.

**Parameter estimation.** If we perform parameter estimation with these data we achieve the model fit shown in Figure 9.32. The model predictions of all the concentrations are shown in Figures 9.33 and 9.34. Note

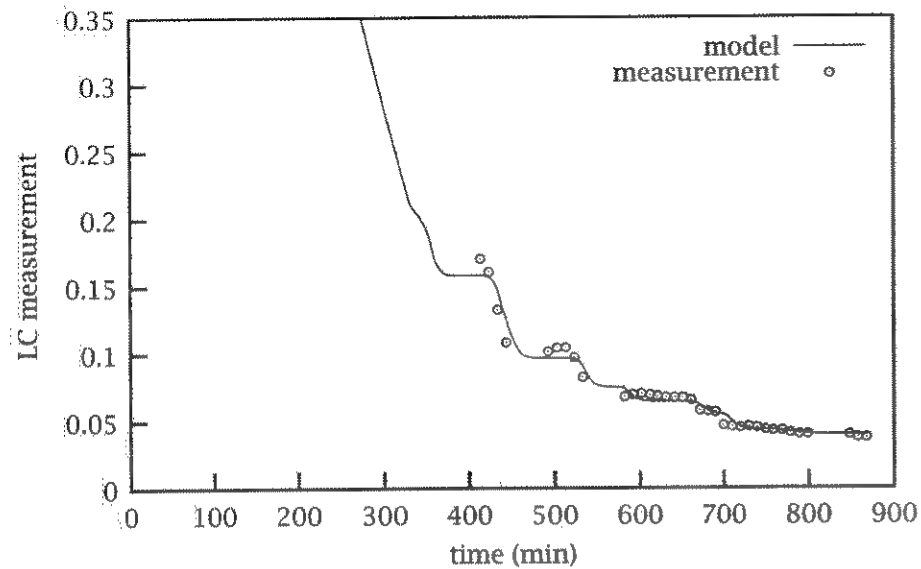


Figure 9.32: Comparison of data to model with optimal parameters.

that the concentration of B provides the most information on the rate constants. Unfortunately this concentration is not measured.

The optimal values of the parameters and their approximate 95% confidence intervals are given by

$$\begin{aligned} n_{A0} &= 2.35 \pm 0.013 \\ k_1 &= 2500 \pm 4800 \\ k_2 &= 1250 \pm 2300 \end{aligned}$$

Notice that the initial amount of A is determined to within 0.6%, but the rate constants have 200% uncertainty. That uncertainty is a direct result of the problem structure. The relative amounts of C and D pin down accurately the amount of starting material. The rate constants determine the speed at which we arrive at these values. After each base addition, we have only rough information about the reaction rates by observing the relative amounts of C and D. If we could measure the B concentration, we could achieve narrow confidence intervals for the rate constants as well. For the purposes of end-point control, however,

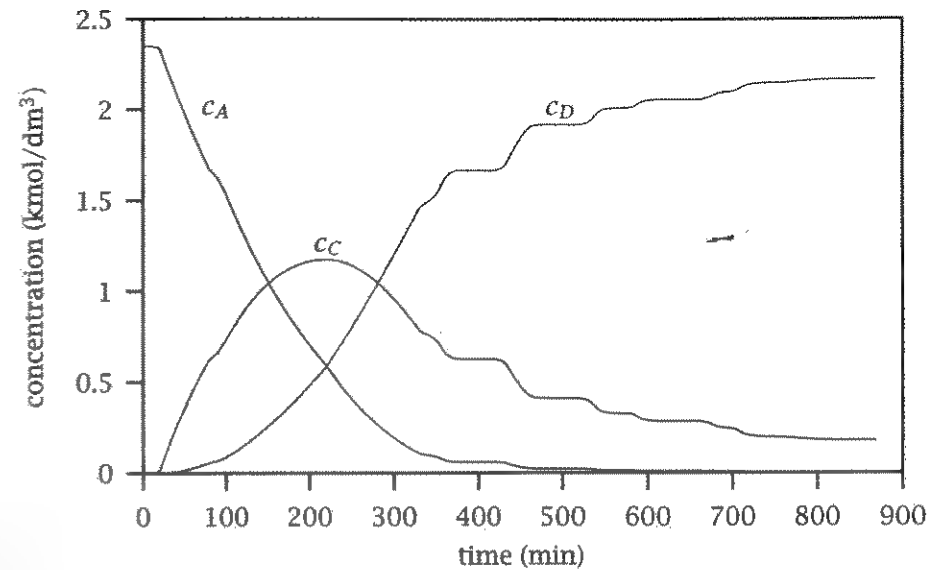


Figure 9.33: Concentrations of species A, C and D versus time.

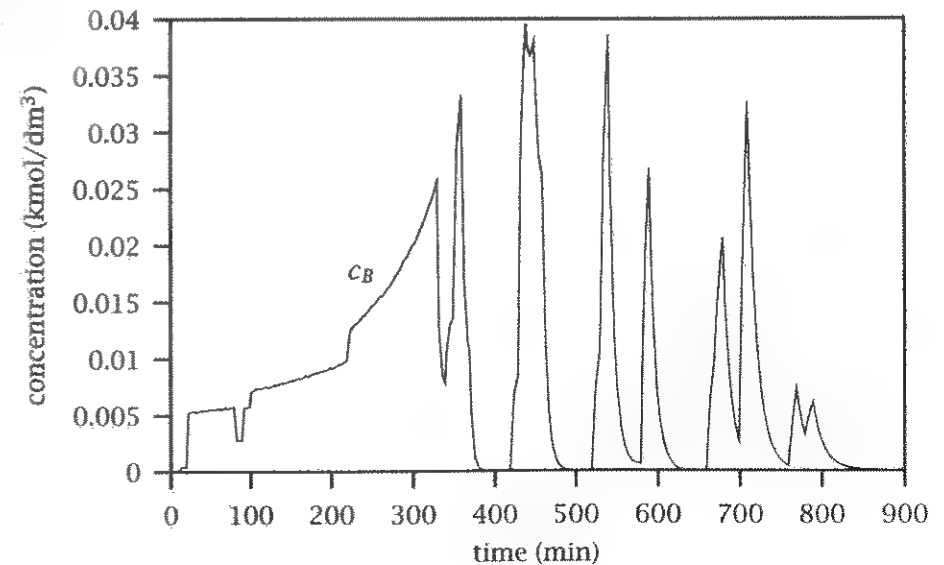


Figure 9.34: Concentration of species B versus time.

the rate constants are irrelevant. We seek to know how much total base to add, not how quickly it will be consumed. In this application, we regard  $k_1$  and  $k_2$  as nuisance parameters. We must estimate them to determine the parameter we care about, but their values are not useful to us.

**Model reduction.** The large parameter uncertainty tells us that the experimental data do not contain sufficient information to determine the rate constants. That diagnosis is essential because it motivates our next step: model reduction.

Although it may not be immediately apparent, the difficulty we face is caused by the presence of large rate constants. We wish to make the equilibrium assumption as described in Chapter 5 to reduce the model. This analysis is clearest if we first rewrite the material balances in terms of extents of the two reactions as in Chapter 5

$$\begin{aligned}\frac{dV_R}{dt} &= Q_f \\ \frac{d\varepsilon_1}{dt} &= r_1 = k_1 c_A c_B V_R = k_1 n_A n_B / V_R \\ \frac{d\varepsilon_2}{dt} &= r_2 = k_2 c_C c_B V_R = k_2 n_C n_B / V_R\end{aligned}\quad (9.72)$$

The initial conditions for this model are

$$\begin{aligned}V_R(0) &= V_{R0} \\ \varepsilon_1(0) &= 0 \\ \varepsilon_2(0) &= 0\end{aligned}$$

We can easily translate from the two reaction extents to total moles of species via

$$n_A = n_{A0} - \varepsilon_1 \quad (9.73)$$

$$n_B = n_{Badd} - (\varepsilon_1 + \varepsilon_2) \quad (9.74)$$

$$n_C = \varepsilon_1 - \varepsilon_2 \quad (9.75)$$

$$n_D = \varepsilon_2 \quad (9.76)$$

in which

$$n_{Badd}(t) = \int_0^t Q_f(t') c_{Bf} dt' \quad (9.77)$$

From the two reaction-rate expressions in Equations 9.72, if we assume equilibrium is established with these irreversible reactions, either the

concentration of B is zero or the concentrations of A and C are zero. Due to the reactor-addition policy, we know B is the limiting reagent, and conclude that under large  $k_1, k_2$ , the B concentration is zero. The material balance for B in Equation 9.76 then provides one algebraic equation for the two extents

$$\varepsilon_1 + \varepsilon_2 = n_{Badd} \quad (9.78)$$

The additional equation comes from examining the two extents' differential equations and noticing

$$\frac{d\varepsilon_1}{dt} = \left(k \frac{n_A}{n_C}\right) \frac{d\varepsilon_2}{dt} \quad k = \frac{k_1}{k_2} \quad (9.79)$$

We see that the concentration of B disappears from this slow time-scale model and the ratio  $k = k_1/k_2$  appears instead of the individual rate constants. If we wish to cast the reduced model in differential equation form, we can differentiate Equation 9.78 and substitute Equation 9.79 to eliminate  $\varepsilon_1$  to obtain

$$\frac{d\varepsilon_2}{dt} = \frac{Q_f c_{Bf}}{1 + k n_A / n_C} \quad (9.80)$$

Substituting Equation 9.76 for the moles of A and C, and using Equation 9.78 again to eliminate  $\varepsilon_1$  produces a differential equation for the second extent

$$\frac{d\varepsilon_2}{dt} = Q_f c_{Bf} \left(1 + k \frac{n_{A0} - n_{Badd} + \varepsilon_2}{n_{Badd} - 2\varepsilon_2}\right)^{-1} \quad (9.81)$$

Solving this reduced model for various values of  $k$  produces the results shown in Figure 9.35. If we perform parameter estimation with this model and these data we obtain the following results.

$n_{A0}$	$= 2.35 \pm 0.0083$
$k = k_1/k_2$	$= 2.00 \pm 0.29$

(9.82)

We have improved the situation compared to the uncertainty in the full model. But we still have more than 10% uncertainty in the ratio of rate constants  $k$ . These results may be adequate, but if we wish to further improve the confidence in  $k = k_1/k_2$  and  $n_{A0}$ , we proceed by reexamining the experimental design.

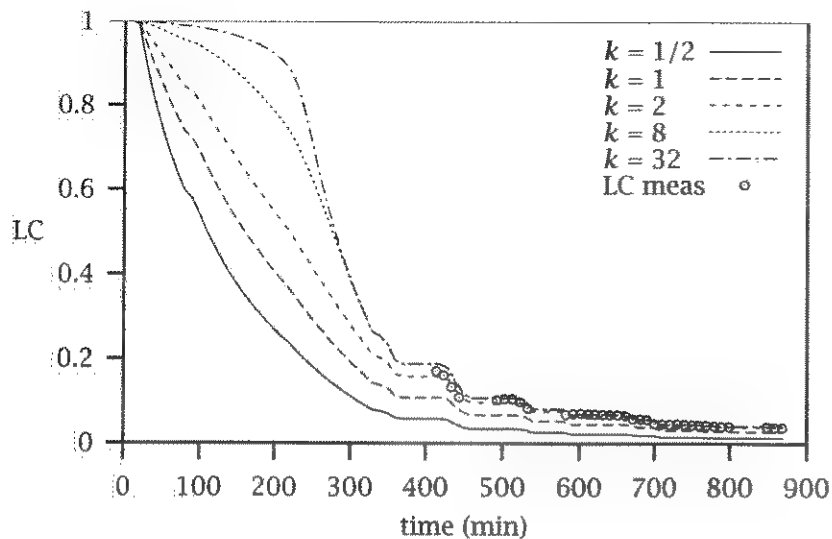


Figure 9.35: Predictions of LC measurement for reduced model.

**Experimental design.** We have removed the rate constants from the estimation problem by assuming their values are arbitrarily large and making the equilibrium assumption. But we still have a reasonably large uncertainty in the ratio of rate constants. Given what we know at this point, we can easily remedy this final, remaining problem. Consider again the results shown in Figure 9.35. We see that the ratio of the rate constants has its primary effect at early times. By the end of the semi-batch addition, there is little information left. For this reason, we have the relatively large uncertainty in  $k$  shown in Equation 9.82. But at early times, when significant amounts of A and C remain in the reactor, the data are much more informative. Although under the original reactor-addition policy, the operator had no reason to turn on the LC until later times, we see that these early measurements are actually the informative ones. We do not have industrial operating data with early LC measurements, but we can simulate the effect with our model. Notice the advantage of modeling. We have the ability to query the model instead of performing expensive experiments to evaluate the impact of a proposed change. Consider Figure 9.36 in which we have simulated early LC measurements by solving the model with

$$k = k_1/k_2 = 2.0, \quad n_{A0} = 2.35$$

and adding measurement noise. Notice this noise is greater than the

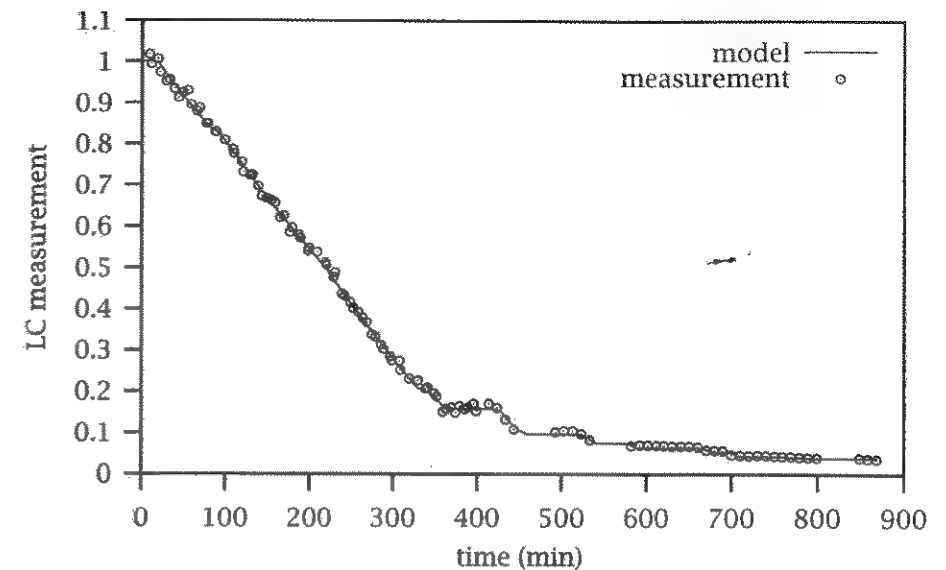


Figure 9.36: Fit of LC measurement versus time for reduced model; early time measurements have been added.

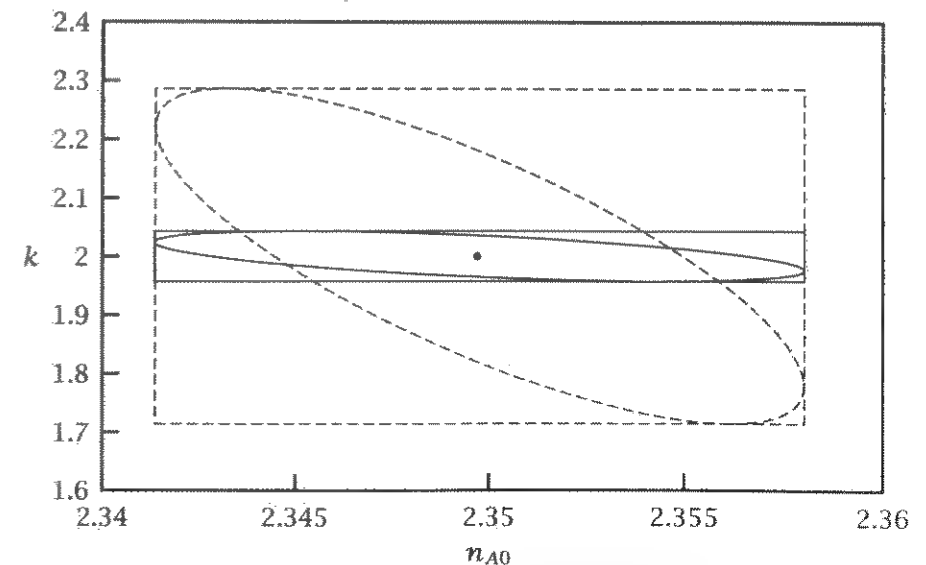


Figure 9.37: Parameter estimates and confidence intervals for reduced model with redesigned experiment.

measurement noise shown in the actual operating data. We next perform parameter estimation on the data shown in Figure 9.36 and obtain the following parameters and approximate confidence intervals

$$\begin{aligned} n_{A0} &= 2.35 \pm 0.0083 \\ k = k_1/k_2 &= 2.00 \pm 0.043 \end{aligned}$$

Figure 9.37 shows the confidence-interval ellipses and boxes when using the original dataset and using the dataset augmented with early-time data. As summarized in Figure 9.37 the primary benefit in adding the early time data is a more precise estimate of  $k$ . The uncertainty in  $k$  is reduced by more than a factor of six when adding the early time LC measurements. The initial number of moles of A is relatively accurately determined by both datasets.

**Improved reactor operation.** Given these modeling results, we can shorten the batch time significantly. First we switch the LC on at time  $t = 0$ . Then as the LC measurements become available, we estimate the initial number of moles  $n_{A0}$  and monitor its confidence interval. As soon as the uncertainty in  $n_{A0}$  reaches a sufficiently low threshold, we are confident how much B is required and can add the remainder in one shot. Testing this approach with many datasets at Kodak allowed us to conclude that by the time the first large addition was completed, we obtained sufficient confidence on  $n_{A0}$  that the rest of the B could be added immediately. Such a procedure reduces the batch time from about 900 minutes with the conventional approach to about half that time with the model-based operation. The new operation essentially doubles the production rate without constructing new reactor facilities, which is significant for this capacity-limited chemical.

## 9.4 Summary

In this chapter we first summarized some of the analytical methods and experimental reactors used to collect reactor data, focusing the discussion on:

- infrared spectroscopy
- gas chromatography
- mass spectrometry

We also discussed how to model differential reactors, which are convenient reactors for measuring reaction rates as a function of species concentrations.

Next we covered analysis of data. We used probability and random variables to model the irreproducible part of the experiment. For models that are linear in the parameters, we can perform parameter estimation and construct exact confidence intervals analytically. For models that are nonlinear in the parameters, we compute parameter estimates and construct approximate confidence intervals using nonlinear optimization methods.

We then covered the important topic of estimating parameters for differential-equation models. We employed computational methods for solving differential equations and sensitivities, and solving nonlinear optimization problems in order to tackle this challenging problem.

The chapter concluded with an industrial case study. In this case study we illustrated the following steps:

- Proposing an initial model given a rough idea of the reaction chemistry.
- Estimating the parameters for the full model given composition measurements.
- Reducing the model based on the parameter confidence intervals.
- Designing new experiments based on the reduced model's confidence intervals.
- Using the final model to find new operating policies to double the production rate.

We hope the examples and methods in this chapter serve to inspire students and practicing engineers to build models as part of understanding new processes and chemistries of interest. The modeling experience often leads to deeper process understanding and produces a compact summary of current knowledge that is easily and efficiently communicated to other colleagues and team members. Process understanding coupled with creativity often leads to process improvement and new discovery.

## Notation

$A$	absorbance
$b$	path length for the absorbance measurement
$c$	concentration of absorbing species
$c_j$	concentration of component $j$
$\hat{c}_j(t_i)$	measured concentration of component $j$ at sampling time $t_i$
$D_p$	catalyst particle diameter
$D_t$	fixed-bed reactor tube diameter
$H_{kj}$	Hessian matrix, $H_{kj} = \partial^2 \Phi / \partial \theta_k \partial \theta_j$
$L$	fixed-bed reactor tube length
$m$	mass velocity
$n_d$	number of data points
$n_p$	number of model parameters
$N_j$	molar flow of component $j$
$p(\mathbf{x})$	probability density function of random variable $\mathbf{x}$
$Q$	volumetric flowrate
$Q_f$	feed volumetric flowrate
$r$	reaction rate of (single) reaction
$r_i$	reaction rate for $i$ th reaction
$\mathbf{r}$	reaction-rate vector
$R_j$	production rate for $j$ th species
$\mathbf{R}$	production-rate vector
$S$	open tube area for flow
$S_{jk}$	sensitivity of state $x_j$ with respect to parameter $\theta_k$
$t_i$	sampling time
$V_R$	reactor volume
$\mathbf{x}$	state vector
$\mathbf{y}$	vector of measured responses
$\epsilon$	molar absorptivity
$\epsilon_B$	bed porosity
$\boldsymbol{\theta}$	parameter vector
$\mu$	fluid viscosity
$\nu_{ij}$	stoichiometric number for the $j$ th species in the $i$ th reaction
$\boldsymbol{\phi}$	transformed parameter vector
$\Phi$	objective function

## 9.5 Exercises

### Exercise 9.1: Bringing it all back home

Given the new tools in this chapter, return to the data displayed in Chapter 2, Figures 2.2 and 2.3. Reaction rates and production rates are related by

$$\mathbf{R} = \mathbf{v}^T \mathbf{r}$$

If we measure production rates and wish to estimate reaction rates, we can model the measurement process with

$$\mathbf{R} = \mathbf{v}^T \mathbf{r} + \mathbf{e}$$

in which  $\mathbf{e}$  is assumed to have a normal distribution.

- What is the formula for the 95% confidence interval on estimated reaction rates  $\hat{\mathbf{r}}$ ?
- Place your 95% confidence interval on the set of points displayed in Figure 2.3. These points are available at [www.che.wisc.edu/~jbrow/chemreacfun](http://www.che.wisc.edu/~jbrow/chemreacfun). Given the Octave code displayed in Chapter 2 that generated  $\mathbf{R}_{\text{meas}}$ , what is the value of the measurement error variance. Of the 500 estimated reaction rates, how many of the values fall outside your 95% confidence interval ellipse?  
  
Notice that if you know (or have some means of estimating) the measurement error variance beforehand, you do not need the experimental measurements to know the confidence intervals. They are computable from the model before you do any experiments.
- Now assume the measured production rates are only the first column in  $\mathbf{R}_{\text{meas}}$  in Chapter 2. Compute the least-squares estimate and 95% confidence interval in which you assume you do not know the variance in the measurement error and must estimate it from these data also. How do your two ellipses compare? Do you obtain a reasonable idea of the parameter uncertainty when you only have one measurement of the six species production rates?

### Exercise 9.2: Batch-reactor parameter estimation

A well-mixed, isothermal, constant-volume batch reactor is initially charged with reactant A and the following reaction takes place





$t$ (min)	0	4	8	12	16	20	24	28	32	36	40
$c_A$ (mol/L)	0.83	0.68	0.59	0.51	0.43	0.38	0.33	0.30	0.26	0.24	0.21

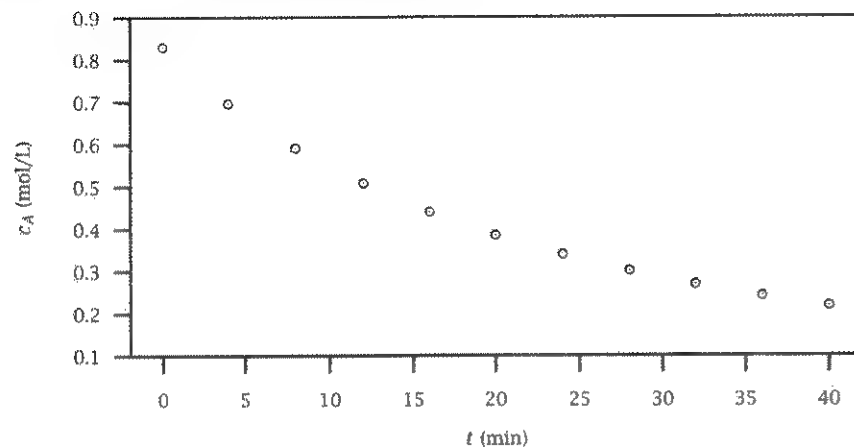


Figure 9.38: Batch reactor data for Exercise 9.2.

The reactor volume is 10 L and the initial charge of A is 8.3 mol. The concentration of A versus time is measured and shown in Figure 9.38. You wish to fit these data using the following kinetic rate-law expression,

$$r = kc_A^n$$

- What is the solution to the model,  $c_A(t)$ , for this rate law?
- Calculate from the data rough estimates of the parameters  $k$  and  $n$ . What are their units?

Hint: consider approximating the time derivative  $dc_A/dt$  by  $\Delta c_A/\Delta t$  and compare those approximate values of the rate using the data to  $kc_A^n$ . A nonlinear transformation of the rate expression can make the estimation problem linear.

- Estimate the parameters by solving a least-squares problem. Plot the fit of both models to the data and compare. Are the model fits comparable?

### Exercise 9.3: The perils of differentiating data

Consider the  $n$ th-order, irreversible reaction



$t$ (min)	0.0	4.0	8.0	12.0	16.0	20.0	24.0	28.0	32.0	36.0	40.0
$c_A$ (mol/L)	0.830	0.696	0.591	0.507	0.440	0.384	0.339	0.301	0.268	0.241	0.217
	0.833	0.683	0.593	0.515	0.446	0.380	0.324	0.303	0.256	0.225	0.205
	0.862	0.699	0.561	0.511	0.412	0.407	0.346	0.276	0.322	0.254	0.215

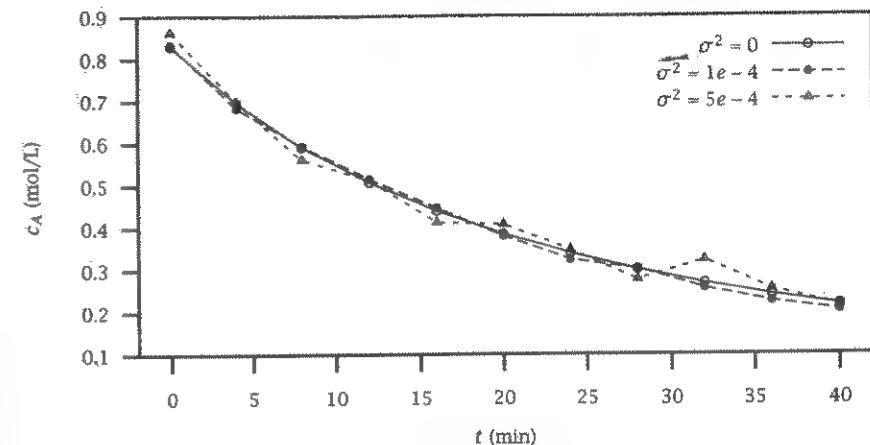


Figure 9.39: Batch-reactor data for Exercise 9.3; 3 runs with different measurement error variance.

taking place in an isothermal, liquid-phase batch reactor. Measurements of  $c_A$  versus time for different measurement accuracies are displayed in Figure 9.39. The following method is sometimes recommended for finding the rate constant and order of this reaction.

- Compute an approximation to the reaction rate by "differentiating" the data

$$r_l = \frac{c_{A,l+1} - c_{A,l}}{t_{l+1} - t_l}, \quad l = 1, \dots, n_d - 1$$

in which  $c_{A,l}$  is the measured concentration at sample time  $t_l$  and  $r_l$  is the "measured" rate at this sample time. Notice the rate is not actually measured; the concentration is measured.

- Take the logarithm of the rate expression to obtain a model linear in the parameters

$$\log(r) = \log(k) + n \log(c_A)$$

- Use linear least squares with this model and the "measured" rate  $r_l$  versus  $c_{A,l}$  to find  $k$  and  $n$ .

Let's examine how well this method performs on the three datasets shown in Figure 9.39.

#### Exercise 9.4: Fitting the RTD step response

Consider the RTD test data given in Exercise 8.11. Use the least squares method of this chapter and find the best value of parameter  $\rho$ , given in Equations 8.68, that fits these data. Solve the model with this value of  $\rho$  and show the fit to the data. Resolve Exercise 8.11 and predict the reactor yield and conversion.

#### Exercise 9.5: The differential reactor and volume change

Your colleague has an idea that, since not much conversion occurs in the differential reactor, we should be able to assume the volumetric flowrate is constant to simplify the model.

Consider the gas-phase reaction  $A \rightarrow 2B$  using a pure A stream and compare analyzing the data from a differential reactor in two ways. First, use Equation 9.4, which accounts for the changing volumetric flowrate. Second, neglect the change in volumetric flowrate, assume  $Q = Q_f$ , and show that

$$\tilde{r} = \frac{Q_f (c_{Af} - c_A)}{V_R - 1} \quad \tilde{r} = \frac{Q_f (c_{Bf} - c_B)}{V_R - 2}$$

Calculate the relative error  $e = (r - \tilde{r})/r$  committed when neglecting the volumetric flowrate change in a differential reactor assuming you are measuring component A. Is this a large or small error? If it is a larger error, can you explain what went wrong with your colleague's logic?

#### Exercise 9.6: Analyzing experimental data

The heterogeneously catalyzed reaction



was studied in a small, research-scale CSTR. The CSTR design eliminates external mass-transfer limitations (i.e., you may assume the bulk fluid concentration equals the surface concentration). The feed consisted of pure A. The reactor pressure was 1.0 atm and the temperature was 573 K. The residence time for the CSTR is 2.0 s. The following table presents effluent-concentration data for different catalyst sizes. The catalyst particles can be assumed spherical. The effective diffusivity of A in the pellet is  $D_A = 0.0008 \text{ cm}^2/\text{s}$ . The pellet density equals the bed density,  $\rho_p = \rho_B$ .

Radius (cm)	$c_A$ (mol/cm <sup>3</sup> )
0.016	$2.39 \times 10^{-6}$
0.071	$4.25 \times 10^{-6}$
0.121	$5.94 \times 10^{-6}$
0.191	$7.84 \times 10^{-6}$
0.246	$9.04 \times 10^{-6}$

Determine the value of the rate constant at 573 K.

#### Exercise 9.7: Oxygen adsorption, revisited

Consider again the adsorption data in Exercise 5.12. The data are listed in Table 5.8 and plotted in Figure 5.25. The adsorption is dissociative



- Write out again the expression for the Langmuir isotherm to model the concentration of adsorbed oxygen in terms of the concentration of gas-phase oxygen. How many unknown parameters does your model contain?
- Estimate these parameters using the nonlinear least-squares technique described in this chapter.
- Consider again the transformation of the model into the linear form, i.e., consider  $1/\bar{c}_O$  as a function of  $1/\sqrt{c_{\text{O}_2}}$ . Estimate the parameters again using linear least squares.
- Compare the parameter estimates obtained with these two approaches by plotting the fit to the original data. Why are the fits so different? Which technique do you recommend and why?

#### Exercise 9.8: Replication and confidence

Imagine you have performed an experiment and set up the least-squares problem

$$\mathbf{y} = \mathbf{X}\boldsymbol{\theta} + \mathbf{e}$$

in which  $\mathbf{y}$  contains the vector of measurements,  $\mathbf{X}$  contains the matrix of independent variables,  $\boldsymbol{\theta}$  is the parameter vector, and  $\mathbf{e}$  is the measurement error, assumed to have known variance  $\sigma^2$ . We developed the following formula for the parameter estimates and  $\alpha$ -level confidence intervals

$$\hat{\boldsymbol{\theta}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}$$

$$(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T \mathbf{X}^T \mathbf{X} (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) \leq \sigma^2 \chi^2(n_p, \alpha) \quad (9.83)$$

If you replicate this experiment  $n_s$  times, the new data matrix is

$$X_s = \begin{bmatrix} X \\ X \\ \vdots \\ X \end{bmatrix} \quad n_s \text{ times}$$

- Calculate  $X_s^T X_s$ . From Equation 9.83, how much smaller is the confidence interval ellipse after 2, 5 and 100 replicate experiments?
- Does the confidence ellipse change shape as well as size upon replication? Does your result agree with Figure 9.20?
- If you want to reduce uncertainty by a factor of  $\beta$ , how many replicate experiments are required? Explain why straight replication is a potentially expensive way to reduce uncertainty.

### Exercise 9.9: Estimating a rate expression involving two reactants

Consider the irreversible reaction



taking place in an isothermal, liquid-phase batch reactor. Measurements of  $c_A$  and  $c_B$  versus time are displayed in Figure 9.40. We wish to determine from the data the rate constant and the order of the reaction with respect to both A and B.

- Estimate rate parameters,  $k$ ,  $n$  and  $m$ , and initial concentrations,  $c_{A0}$  and  $c_{B0}$ , from the data in Figure 9.40 *without* differentiating the concentration data. The data are available at [www.che.wisc.edu/~jbrow/chemreacfun](http://www.che.wisc.edu/~jbrow/chemreacfun). Notice your confidence intervals for  $k$ ,  $n$  and  $m$  are rather large. Why is this model poorly identified from these data?
- Consider a second experiment shown in Figure 9.41. Estimate the parameters using only the second experiment. Are your confidence intervals for  $k$ ,  $n$  and  $m$  any better using the second experiment?
- Estimate the parameters using both experiments simultaneously. Are your confidence intervals any better using both experiments compared to using either experiment alone? Explain why or why not.
- Based on these results, to design experiments to determine the order of the rate expression with respect to two different species, how should you choose the initial conditions of the two species?

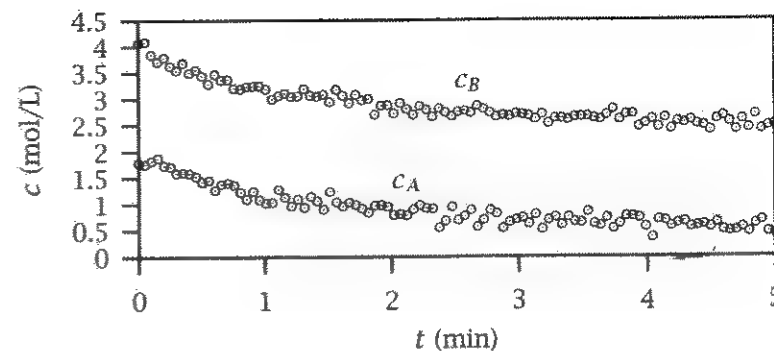


Figure 9.40: Batch-reactor data for Exercise 9.9.

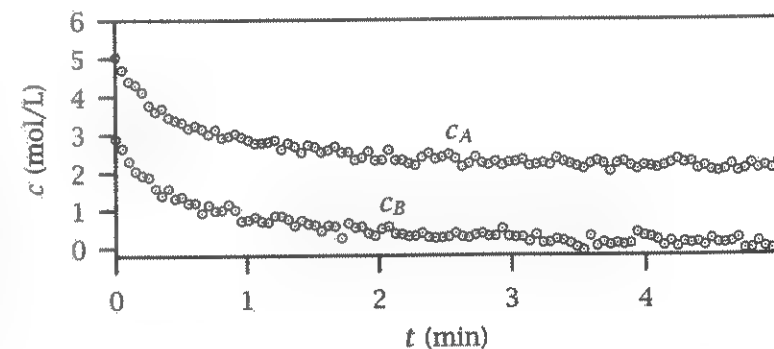


Figure 9.41: A second experiment for Exercise 9.9.

### Exercise 9.10: Living and dying with linear least squares

Consider differentiating the data of Exercise 9.9 as described in Exercise 9.3. Notice differentiating the  $c_A$  data and the  $c_B$  data provide *two* independent estimates of the same reaction rate. You may wish to average these to obtain a good single estimate of the rate. Taking the logarithm of the rate expression again produces a model linear in the parameters

$$\log(r) = \log(k) + n \log(c_A) + m \log(c_B)$$

- To ensure that your method is coded correctly, consider first the “error free” data depicted in Figures 9.42 and 9.43. Estimate  $k$ ,  $n$  and  $m$  using linear least squares and three different datasets: only the data in Figure 9.42, only the data in Figure 9.43, and the data in both figures.

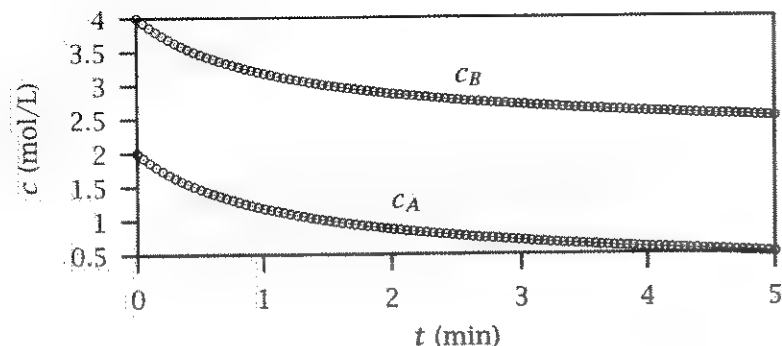


Figure 9.42: Batch-reactor data for Exercise 9.10.

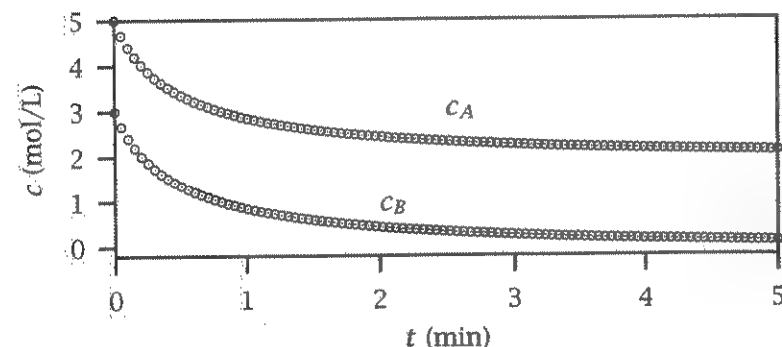


Figure 9.43: A second experiment for Exercise 9.10.

Do you get similar estimates with all three datasets? Are the confidence intervals similar? Explain why or why not.

- (b) Now apply the linear least-squares approach to the data in Figures 9.40 and 9.41. Are your estimates and their uncertainties comparable to the result of Exercise 9.9? Which method of estimating parameters do you recommend and why?

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# A

## Computational Methods

---

In this appendix we summarize many of the fundamental computational procedures required in reactor analysis and design. We illustrate these procedures using the numerical computing language Octave [4, 5] or, equivalently, MATLAB. Octave is freely available for a variety of hardware platforms and can be down-loaded from [www.octave.org](http://www.octave.org). The MATLAB package is commercially available from The MathWorks, Inc., and is becoming a commonly available tool of industrial engineering practice. One is naturally free to use any numerical solution procedure that is convenient, but we have found that these two environments provide reliable and high-quality numerical solutions, and convenient user interfaces.

Many of the figures and tables in this text required numerical solutions of various types of chemical reactor problems. All computations required to produce every figure and table in this text were performed with Octave. The Octave files required to produce these figures and tables are freely available at these two websites:

[www.globaldialog.com/~nobhill](http://www.globaldialog.com/~nobhill)  
[www.che.wisc.edu/~jbrow/chemreacfun](http://www.che.wisc.edu/~jbrow/chemreacfun)

### A.1 Linear Algebra and Least Squares

In this section we briefly introduce Octave and MATLAB, and show how to enter reaction networks quickly and perform linear algebra operations of interest.

After starting an Octave session, type `help -i` introduction at the Octave prompt or `demo` at the MATLAB prompt for information on how to enter matrices and perform matrix multiplication. Consider

again the water gas shift reaction chemistry, for example,

$$\begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \text{H} \\ \text{H}_2 \\ \text{OH} \\ \text{H}_2\text{O} \\ \text{CO} \\ \text{CO}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Say we would like to find the number of linearly independent reactions, and then compute the species production rates from given reaction rates,  $[r_1 \ r_2 \ r_3] = [1 \ 2 \ 3]$ . A typical set of Octave or MATLAB commands to perform these operations would appear as follows

```
octave:1> stoi=[0 1 0 -1 -1 1; -1 1 1 -1 0 0; \
               1 0 -1 0 -1 1];
octave:2> rank(stoi)
ans = 2
octave:3> r=[1;2;3];
octave:4> R=stoi'*r
R =
     1
     3
    -1
    -3
    -4
     4
```

As we explored in Chapters 2 and 9, from measured  $R$ , we can solve for  $r$  as a least-squares problem. Given the production rates and reaction rates are related by

$$R = v^T r \quad (\text{A.1})$$

then the least-squares solution is given by

$$r = (v v^T)^{-1} v R$$

in which the superscript  $-1$  indicates a matrix inverse. The command in Octave and MATLAB for the least-squares solution is

$$r = \text{stoi}' \backslash R$$

which can be recalled with the mnemonic of a “left division” of Equation A.1 by  $v^T$ .

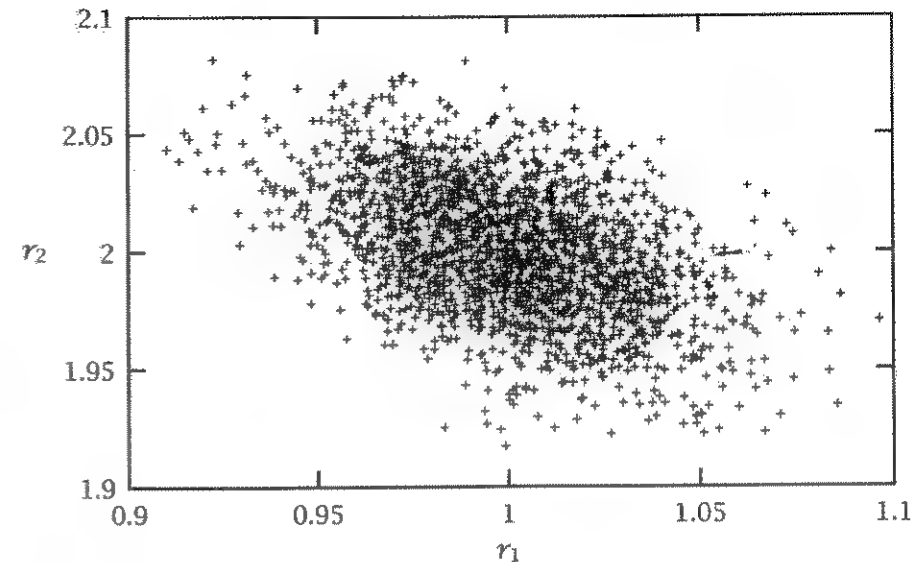


Figure A.1: Estimated reaction rates from 2000 production-rate measurements subject to measurement noise.

### Example A.1: Estimating reaction rates

Consider the first and second water gas shift reaction as an independent set and let  $[r_1 \ r_2] = [1 \ 2]$ . Create 2000 production rate measurements by adding noise to  $R = v^T r$  and estimate the reaction rates from these data. Plot the distribution of estimated reaction rates.

### Solution

The Octave commands to generate these results are

```
stoi=[0 1 0 -1 -1 1; -1 1 1 -1 0 0];
r=[1;2];
R=stoi'*r;
npoints = 2000;
for i=1:npoints R_meas(:,i)=0.05*randn(6,1)+R; endfor
r_est=stoi' \ R_meas;
```

Figure A.1 shows the estimates. We know from Chapter 9 that the distribution of estimates is a multivariate normal. We also know how to calculate the  $\alpha$ -level confidence intervals.  $\square$

## A.2 Nonlinear Algebraic Equations and Optimization

Determining equilibria for reacting systems with multiple phases and reactions can require significant computational effort. As we saw in Chapter 3, the phase and reaction equilibrium conditions generally lead to mathematical problems of two types: solving nonlinear algebraic equations and minimizing a nonlinear function subject to constraints. We explore computing solutions to these types of problems in Octave or MATLAB. In this section it is assumed that the required thermochemical data are available, but finding or measuring these data is often another significant challenge in computing equilibria for systems of industrial interest. See also Section 3.2.1 for a brief discussion of thermochemical databases.

### A.2.1 Defining Functions (function)

Octave and MATLAB provide many convenient built-in functions. Some that we have used in this text include the matrix exponential, `expm`; incomplete gamma function, `gammai`; pseudo-random number generators, `rand` and `randm`; and several others. Type `help -i` and select the menu item **Function Index** for a complete list of Octave's built-in functions.

But often we need to define our own functions. To solve algebraic equations or optimization problems, for example, the user needs to provide a function that can be called by the solver to evaluate the nonlinear equations or the objective function. To define a function in Octave,  $f(x) = x^2$  for example, the command structure is

```
function y = f(x)
    y = x*x;
endfunction
```

The first line states that the value of the variable `y` is returned by invoking the function named `f` with the input value `x`. The body of the function, then, is the calculation of the value of `y` to be returned. As the computational complexity of the function increases, we would like to store our work so that it does not need to be re-entered at the beginning of each subsequent Octave session. Storing, documenting and maintaining the calculations performed in solving complex industrial problems is one of the significant challenges facing practicing chemical engineers. Even for our purposes, we should be able to store and edit our computational problems. Octave commands and function definitions can be stored in text files, called *m-files*; the naming convention

is `filename.m`. Editing these files with a text editor during an interactive Octave session provides a convenient means for debugging new calculations.

### A.2.2 Solving Nonlinear Algebraic Equations (fsolve)

We consider again the problem in Example 3.5. The equilibrium condition requires the solution of two equations in two unknowns,

$$PK_1 y_I y_B - y_{P_1} = 0$$

$$PK_2 y_I y_B - y_{P_2} = 0$$

in which  $y_I, y_B, y_{P_1}, y_{P_2}$  are defined in terms of the two reaction extents in Equations 3.74. An Octave function defining these two equations is

```
function residual = dgdx(x)
    K1 = 108; K2 = 284; P = 2.5;
    yI0 = 0.5; yB0 = 0.5; yP10 = 0; yP20 = 0;
    d = 1 - x(1) - x(2);
    yI = (yI0 - x(1) - x(2)) / d;
    yB = (yB0 - x(1) - x(2)) / d;
    yP1 = (yP10 + x(1)) / d;
    yP2 = (yP20 + x(2)) / d;
    residual(1) = P*K1*yI*yB - yP1;
    residual(2) = P*K2*yI*yB - yP2;
endfunction
```

Notice that `x` is a vector containing the two reaction extents provided to the function named `dgdx` when it is invoked, and `residual` is a vector containing the values of the two equilibrium conditions to be returned by the function. We seek values of `x` such that `residual` is zero. With this function defined, we can use the built-in nonlinear equation solver to find its solution, by entering

```
[x,info] = fsolve("dgdx", x0)
```

The `fsolve` function requires the name of the function, inside quotes, and an initial guess for the unknown extents, provided in the variable `x0`, and returns the solution in `x` and a flag `info` indicating if the calculation was successful. It is highly recommended to examine these information flags after all calculations. Reactor equilibrium problems can be numerically challenging, and even the best software can run into problems. After the function `dgdx` is defined, the following is a typical session to compute the solution given in Example 3.5



```

octave:1> x0=[0.2;0.2];
octave:2> [x,info] = fsolve("dgd",x0)
x =
    0.13317
    0.35087
info = 1

```

The value of `info` = 1 indicates that the solution has converged.

### A.2.3 Solving Nonlinear Optimization Problems (npsol)

The other main approach to finding the reaction equilibrium is to minimize the appropriate energy function, in this case the Gibbs energy. This optimization-based formulation of the problem, as shown in Example 3.3, can be more informative than the algebraic approach discussed above. In Chapter 3, we derived the following Gibbs energy function for an ideal-gas mixture

$$\begin{aligned} \hat{G} = & - \sum_i \epsilon'_i \ln K_i + \left( 1 + \sum_i \bar{v}_i \epsilon'_i \right) \ln P \\ & + \sum_j \left( y_{j0} + \sum_i v_{ij} \epsilon'_i \right) \ln \left[ \frac{y_{j0} + \sum_i v_{ij} \epsilon'_i}{1 + \sum_i \bar{v}_i \epsilon'_i} \right] \end{aligned} \quad (\text{A.2})$$

The minimization of this function of  $\epsilon'_i$  then determines the two equilibrium extents. Figure A.2 shows the lines of constant Gibbs energy determined by Equation A.2 as a function of the two reaction extents. We see immediately that the minimum is unique. Notice that  $\hat{G}$  is not defined for all reaction extents, and the following constraints are required to ensure nonnegative concentrations

$$0 \leq \epsilon'_1 \quad 0 \leq \epsilon'_2 \quad \epsilon'_1 + \epsilon'_2 \leq 0.5$$

We proceed to translate this version of the equilibrium condition into a computational procedure. First we define a function that evaluates  $\hat{G}$  given the two reaction extents.

```

function retval=gibbs(x)
K1 = 108; K2 = 284; P = 2.5;
yI0 = 0.5; yB0 = 0.5; yP10 = 0; yP20 = 0;
d = 1 - x(1) - x(2);
yI = (yI0 - x(1) - x(2)) / d;
yB = (yB0 - x(1) - x(2)) / d;

```

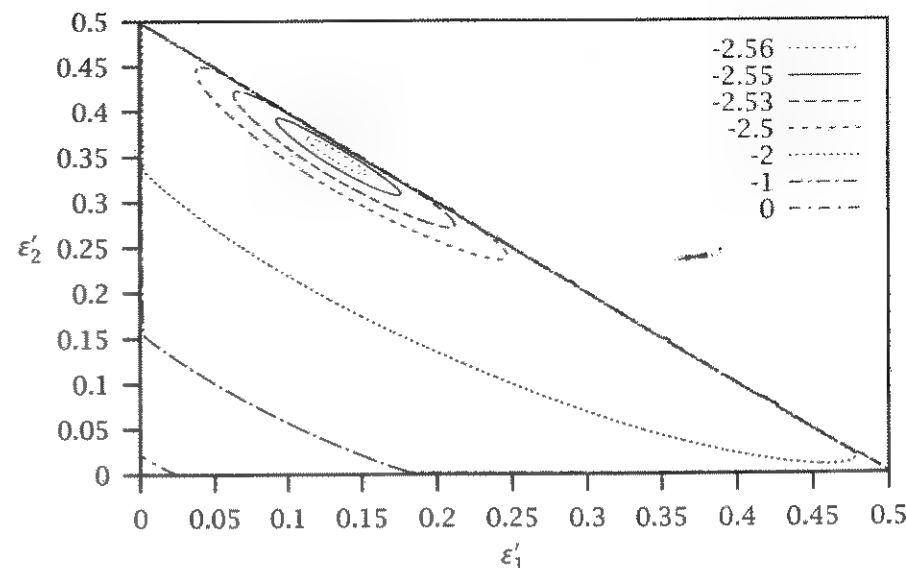


Figure A.2: Gibbs energy contours for the pentane reactions as a function of the two reaction extents.

```

yP1 = (yP10 + x(1)) / d;
yP2 = (yP20 + x(2)) / d;
retval = - (x(1)*log(K1)+x(2)*log(K2)) + \
    (1-x(1)-x(2))*log(P) + yI*d*log(yI) + \
    yB*d*log(yB) + yP1*d*log(yP1) + yP2*d*log(yP2);
endfunction

```

To solve the optimization we use the Octave function `npsol` [6] or MATLAB function `constr`.

```
[x,obj,info]=npsol(x0,"gibbs",lb,ub,alb,a,aub)
```

in which `x0` is the initial guess as before and the new variables are used to define the constraints (if any). The variable `lb` provides lower bounds for the extents and `ub` provides upper bounds for the extents. These are required to prevent the optimizer from “guessing” reaction extent values for which the Gibbs energy function is not defined. For example, consider what the `gibbs` function returns given negative reaction extents. Finally, we specify the sum of extents is less than 0.5 with the last three arguments, which define linear inequality constraints other

than simple bounds,

$$a_{lb} \leq A\varepsilon' \leq a_{ub}$$

For our problem,  $\varepsilon'_1 + \varepsilon'_2 \leq 0.5$ , so  $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$  and  $a_{ub} = 0.5$ . We can set  $a_{lb} = 0$  because both extents are already specified to be positive. Therefore, a typical session to solve for equilibrium composition by minimizing Gibbs energy is

```
octave:1> a=[1 1]; alb=0.001; aub=0.499;
octave:2> lb=[0.001;0.001]; ub=[0.499;0.499];
octave:3> x0=[0.2;0.2];
octave:4> [x,obj,info]=npsol(x0,"gibbs",lb,ub,alb,a,aub)
x =
    0.13317
    0.35087
obj = -2.5593
info = 0
```

In this case, the value of `info` = 0 indicates that the solution has converged, and the results are in good agreement with those computed using the algebraic approach, and the Gibbs energy contours depicted in Figure A.2.

Optimization is a powerful tool for solving many types of engineering modeling and design problems. We also rely heavily on optimization tools in Chapter 9 on parameter estimation.

### A.3 Differential Equations (lsode)

The real workhorse for simulating chemical reactors is the ordinary differential equation (ODE) solver. A high-quality ODE solver is an indispensable tool of the reactor designer because chemical reactions can take place at widely different rates and time scales, presenting difficult numerical challenges. Type

```
help -i differential equations
```

for an overview of Octave's ODE solver, `lsode` [7]. Type `help ode23s` for an overview of MATLAB's ODE solvers. We recommend always using the so-called "stiff" solvers for chemical reactor modeling. The time scales for different reactions may be widely separated, leading to stiff ODEs.

The ODEs required to describe transient-batch, CSTR, and steady-state PFR reactors are of the form

$$\frac{dx}{dt} = f(x)$$

$$x(0) = x_0$$

in which  $x(t)$  is a vector of species concentrations to be found,  $t$  is time in the batch reactor and CSTR, but volume or length in the steady-state PFR, and  $f$  is the function defining the material balances for the species. If the differential equations are nonlinear ( $f$  is not a linear function) and one must keep track of several species simultaneously, then analytical solution is not possible and a numerical procedure is required.

The first task is to formulate a function which defines  $f$ . If we examine the benzene pyrolysis example, Example 4.5, a suitable function is

```
function xdot = benzene_pyrol_rhs(ext,volume)
global k1 k2 K1 K2 R T P NBf
NB = NBf - 2*ext(1) - ext(2);
ND = ext(1) - ext(2);
NH = ext(1) + ext(2);
NT = ext(2);
Q = NBf*R*T/P;
cB = NB/Q; cD = ND/Q; cT = NT/Q; cH = NH/Q;
xdot(1) = k1*(cB*cB - cD*cH/K1);
xdot(2) = k2*(cB*cD - cT*cH/K2);
endfunction
```

in which the meanings of the symbol names should be apparent from the example.

We then need to specify at which  $t$  values, volumes in this case, the solution should be reported, the initial condition (in this case, feed condition),  $x_0$ , and call the ODE solver. A typical Octave session would be

```
NBf = 60e3; R = 0.08205; T = 1033; P = 1;
k1 = 7e5; k2 = 4e5; K1 = 0.31; K2 = 0.48;
x0=[0;0];
vout = linspace(0,1600,50)';
x = lsode("benzene_pyrol_rhs", x0, vout);
conv = ( 2*x(:,1) + x(:,2) ) / NBf;
yB = ( NBf - 2*x(:,1) - x(:,2) ) / NBf;
```

```

yD = ( x(:,1) - x(:,2) ) /NBf;
yT = ( x(:,2) ) /NBf;
yH = ( x(:,1) + x(:,2) ) /NBf;

```

in which the function `linspace` computes 50 linearly spaced points between 0 and 1600 L. Finally, any quantities such as conversions or yields, which are computable from  $x$ , are calculated for plotting. The results of this calculation are shown in Figures 4.20 and 4.21.

### A.3.1 Differential-Algebraic Equations (`dassl`)

Some reactor models require a more general structure than the ODE,  $dx/dt = f(x, t)$ . The nonconstant density, nonconstant volume semi-batch and CSTR reactors in Chapter 4 are more conveniently expressed as differential-algebraic equations (DAEs). To address these models, consider the more general form of implicit ODEs

$$0 = f(dx/dt, x, t)$$

Both DAEs and ODEs can be considered special cases of this structure. Brenan et al. [1] provide further reading on existence and uniqueness of solutions to these models, which are considerably more complex issues than in the case of simple ODEs. Initial conditions are required for  $dx/dt$  as well as  $x$  in this model,

$$\frac{dx}{dt}(t) = \dot{x}_0 \quad x(t) = x_0, \quad \text{at } t = 0$$

Petzold has provided a numerical package, `dassl`, to compute solutions to implicit differential, and differential-algebraic equations. The main difference between using `dassl` and `lsode` is the form of the user-supplied function defining the model. A second difference is that the user must supply  $\dot{x}_0$  as well as  $x_0$ . The reader can consult the software for Example 8.1 at [www.che.wisc.edu/~jbrow/chemreacfun](http://www.che.wisc.edu/~jbrow/chemreacfun) to see the details of how to use `dassl`.

### A.3.2 Automatic Stopping Times (`ddasrt`)

We often need to stop ODE solvers when certain conditions are met. Two examples are when we have reached a certain conversion, and when we have created a new phase and need to change the ODEs governing the system. The program `ddasrt` enables us to stop when specified conditions are met exactly. The following code illustrates how to find

the PFR volume at which 50% conversion of benzene is achieved. The user provides one new function, `stop`, which the ODE solver checks while marching forward until the function reaches the value zero. The PFR volume is found to be  $V_R = 403.25$  L, in good agreement with Figure 4.20.

```

function val = stop(ext, volume)
    global NBf xBstop
    NB = NBf - 2*ext(1) - ext(2);
    xB = 1 - NB/NBf;
    val = xBstop - xB;
endfunction

octave:1> x0 = [0;0]; xdot0 = [0;0];
octave:2> vout = linspace(0,1600,50)'; xBstop = 0.5;
octave:3> [x, xdot, vstop] = ddasrt("benzene_pyro1_rhs", \
                                   "stop", x0, xdot0, vout);

octave:4> vstop(length(vstop))
ans = 403.25

```

## A.4 Parametric Sensitivities of Differential Equations (`ddasac`)

As discussed in Chapter 9, it is often valuable to know not just the solution to a set of ODEs, but how the solution changes with the model parameters. Consider the following vector of ODEs with model parameters  $\theta$ .

$$\frac{dx}{dt} = f(x; \theta) \quad (\text{A.3})$$

$$x(0) = g(x_0; \theta) \quad (\text{A.4})$$

The sensitivities,  $S$ , are defined as the change in the solution with respect to the model parameters

$$S_{ij} = \left( \frac{\partial x_i}{\partial \theta_j} \right)_{\theta_{k*}, j} \quad (\text{A.5})$$

$$S = \frac{\partial x}{\partial \theta^T} \quad (\text{A.6})$$

We can differentiate the  $S_{ij}$  to obtain

$$\frac{dS_{ij}}{dt} = \frac{d}{dt} \left[ \frac{\partial x_i}{\partial \theta_j} \right] = \frac{\partial}{\partial \theta_j} \frac{dx_i}{dt} = \frac{\partial}{\partial \theta_j} f_i(x; \theta) \quad (\text{A.7})$$

Using the chain rule to perform the final differentiation gives

$$\left(\frac{\partial f_i}{\partial \theta_j}\right)_{\theta_{k+j}} = \sum_k \left(\frac{\partial f_i}{\partial x_k}\right)_{x_{l+k}, \theta_j} \left(\frac{\partial x_k}{\partial \theta_j}\right)_{\theta_{k+j}} + \left(\frac{\partial f_i}{\partial \theta_j}\right)_{x_{l+k}, \theta_{k+j}} \quad (\text{A.8})$$

Substituting this back into Equation A.8 and writing the sum as a matrix multiply yields

$$\frac{dS}{dt} = f_x S + f_\theta \quad (\text{A.9})$$

in which the subscript denotes differentiation with respect to that variable. Notice the sensitivity differential equation is linear in the unknown  $S$ . The initial conditions for this matrix differential equation is easily derived from the definition of  $S$

$$S_{ij}(0) = \frac{\partial x_i(0)}{\partial \theta_j} = \frac{\partial g_i}{\partial \theta_j}$$

$$\frac{dS}{dt} = f_x S + f_\theta$$

$$S(0) = g_\theta$$

Notice that if none of the initial conditions are parameters for which one is computing sensitivities, then  $S(0) = 0$ .

#### Example A.2: Two simple sensitivity calculations

Consider the differential equation describing a first-order, isothermal, irreversible reaction in a constant-volume batch reactor

$$\begin{aligned} \frac{dc_A}{dt} &= -kc_A \\ c_A(0) &= c_{A0} \end{aligned} \quad (\text{A.10})$$

1. First write out the solution to the differential equation.
2. Next consider the rate constant to be the parameter in the problem,  $\theta = k$ . Take the partial derivative of the solution directly to produce the sensitivity to this parameter,  $S = \partial c_A / \partial k$ .
3. Next write the differential equation describing  $dS/dt$ . What is  $S(0)$ ? Solve this differential equation and compare to your previous result. Plot  $c_A(t), S(t)$  versus  $t$ .
4. Repeat steps 2 and 3 using the initial concentration as the parameter in the problem,  $\theta = c_{A0}$ .

#### Solution

1. Equation A.10 is our standard, first-order linear equation, whose solution is

$$c_A = c_{A0}e^{-kt}$$

2. Taking the partial derivative of  $c_A(t)$  with respect to  $\theta = k$  produces directly

$$S_1 = -tc_{A0}e^{-kt} \quad (\text{A.11})$$

Evaluating the various partial derivatives gives

$$f_x = -k \quad f_\theta = -c_A \quad g_\theta = 0$$

so the sensitivity differential equation is

$$\begin{aligned} \frac{dS_1}{dt} &= -kS_1 - c_A \\ S_1(0) &= 0 \end{aligned}$$

This equation is linear and can be solved with the method used in the series reactions in Chapter 4, or with Laplace transforms as in Exercise 4.6. The result is

$$S_1(t) = -tkc_{A0}e^{-kt} \quad (\text{A.12})$$

which agrees with Equation A.11. The functions  $c_A(t), S_1(t)$  are shown in Figure A.3.

3. Taking the partial derivative of  $c_A(t)$  with respect to  $\theta = c_{A0}$  produces directly

$$S_2(t) = e^{-kt} \quad (\text{A.13})$$

Evaluating the various partial derivatives for this parameter choice gives

$$f_x = -k \quad f_\theta = 0 \quad g_\theta = 1$$

so the sensitivity differential equation is

$$\begin{aligned} \frac{dS_2}{dt} &= -kS_2 \\ S_2(0) &= 1 \end{aligned}$$

This equation is our standard, first-order linear equation whose solution is

$$S_2(t) = e^{-kt}$$

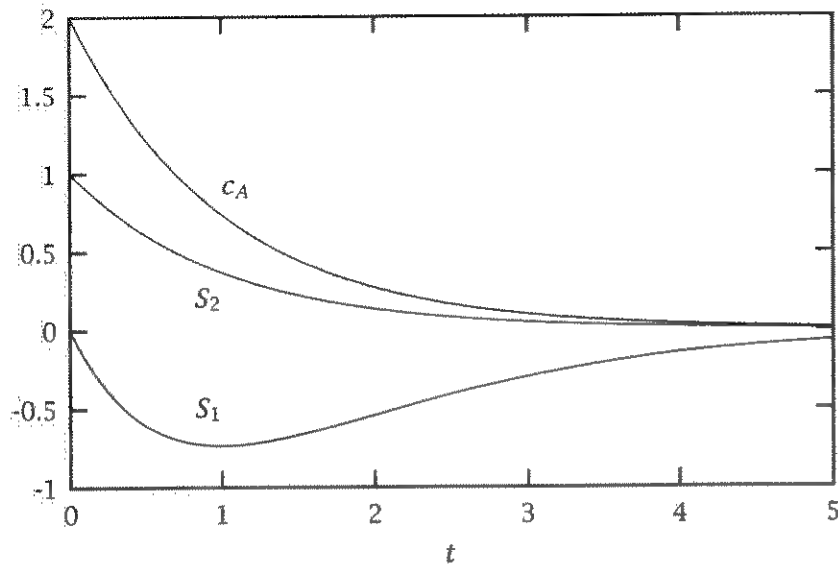


Figure A.3: Solution to first-order differential equation  $dc_A/dt = -kc_A$ , and sensitivities  $S_1 = \partial c_A / \partial k$  and  $S_2 = \partial c_A / \partial c_{A0}$ .

which agrees with Equation A.13. This sensitivity also is plotted in Figure A.3. Notice that only when the initial condition is the parameter is the sensitivity initially nonzero.

□

Stewart and Caracotsios produced the code `ddasac` to perform sensitivity calculations conveniently [3, 2]. The following Octave commands show how to use `ddasac` to solve Example A.2.

```
function xdot = firstorder(t, x, theta)
    global ca0 k
    ca = x(1);
    k = theta(1);
    xdot = -k*ca;
endfunction

x0 = ca0; theta = [k, ca0]; sx0 = [0, 1];
[x, xdot, sx, sxdot] = \
    ddasac("firstorder", x0, theta, sx0, tout);
ca = x;
```

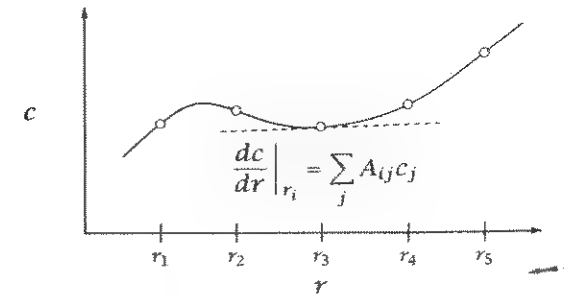


Figure A.4: Function  $c(r)$  and its values at five collocation points. Derivatives and integrals of the polynomial interpolant are linear combinations of the function values at the points.

```
dcadk = nth(sx, 1);
dcadca0 = nth(sx, 2);
```

## A.5 Boundary-Value Problems and Collocation (colloc)

In the collocation method, we approximate a function by passing a polynomial through values of the function at selected points. The selected points are known as collocation points. The locations of the collocation points have a large impact on how well the method works. Evenly spaced points, which seems a natural first choice, turns out to have mediocre properties. Choosing the points as zeros of a member of a family of orthogonal polynomials turns out to have much better properties. This choice is referred to as orthogonal collocation.

Figure A.4 shows values of a function  $c(r)$  at five collocation points. The function is approximated by passing a polynomial through these points. To solve differential equations and boundary-value problems (BVP), we first compute the required derivatives of the polynomial approximation. Then we then find the values of the function such that the differential equation is satisfied at the collocation points. If we increase the number of collocation points,  $n_c$ , we require the differential equation to be satisfied at more locations, and we obtain a more accurate solution.

The derivatives and integrals of the polynomial interpolant can be

computed as linear combinations of the values at the collocation points

$$\begin{aligned}\left. \frac{dc}{dr} \right|_{r_i} &= \sum_{j=1}^{n_c} A_{ij} c_j \\ \left. \frac{d^2c}{dr^2} \right|_{r_i} &= \sum_{j=1}^{n_c} B_{ij} c_j \\ \int_0^1 f(c) dc &= \sum_{j=1}^{n_c} Q_j f(c_j)\end{aligned}$$

To obtain the locations of the collocation points and derivatives and integral weighting matrices and vectors, we use the function `colloc`, based on the methods described by Villadsen and Michelsen [8].

```
[R A B Q] = colloc(npts-2, "left", "right");
```

The strings "left" and "right" specify that we would like to have collocation points at the endpoints of the interval in addition to the zeros of the orthogonal polynomial. We solve for the concentration profile for the reaction-diffusion problem in a catalyst pellet to illustrate the collocation method.

### Example A.3: Single-pellet profile

Consider the isothermal, first-order reaction-diffusion problem in a spherical pellet

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dc}{dr} \right) - \Phi^2 c = 0 \quad (\text{A.14})$$

$$c = 1, \quad r = 3$$

$$\frac{dc}{dr} = 0, \quad r = 0$$

The effectiveness factor is given by

$$\eta = \left. \frac{1}{\Phi^2} \frac{dc}{dr} \right|_{r=3}$$

- Compute the concentration profile for a first-order reaction in a spherical pellet. Solve the problem for  $\Phi = 10$ .
- Plot the concentration profile for  $n_c = 5, 10, 30$  and  $50$ . How many collocation points are required to reach accuracy in the concentration profile for this value of  $\Phi$ .

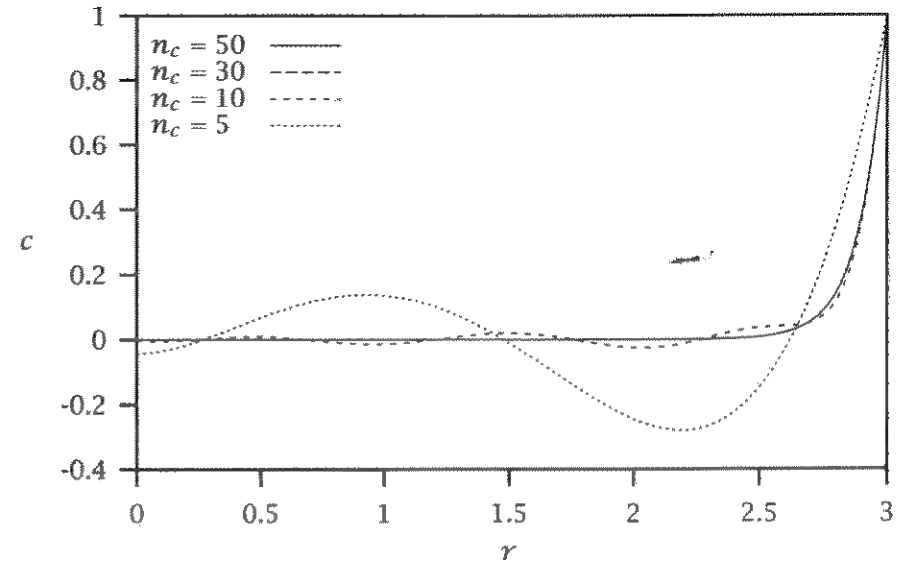


Figure A.5: Dimensionless concentration versus dimensionless radial position for different numbers of collocation points.

- How many collocation points are required to achieve a relative error in the effectiveness factor of less than  $10^{-5}$ ?

### Solution

First we perform the differentiation in Equation A.14 to obtain

$$\frac{d^2c}{dr^2} + \frac{2}{r} \frac{dc}{dr} - \Phi^2 c = 0$$

We define the following Octave function to evaluate this equation at the interior collocation points. At the two collocation endpoints, we satisfy the boundary conditions  $dc/dr = 0$  at  $r = 0$  and  $c = 1$  at  $r = 3$ . The function is therefore

```
function retval = pellet(c)
global Phi A Aint Bint Rint n
npts = length(c);
cint = c(2:npts-1);
retval(1) = A(1,:)*c;
```

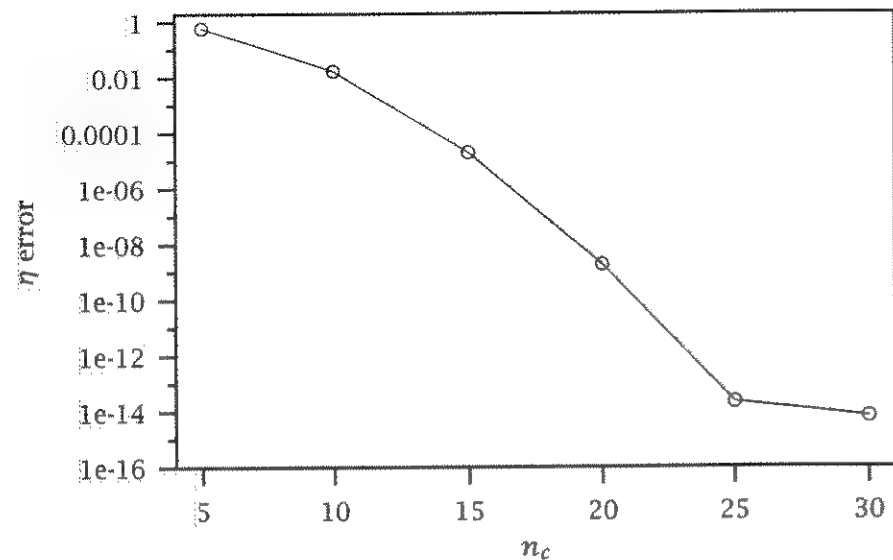


Figure A.6: Relative error in the effectiveness factor versus number of collocation points.

```

retval(2:npts-1) = Bint*c ./ 2*Aint*c./Rint .- \
    PhiA2*cint;
retval(npts) = 1 - c(npts);
endfunction

```

Figure A.5 shows the concentration profiles for different numbers of collocation points. We require about  $n_c = 30$  to obtain a converged concentration profile. Figure A.6 shows the relative error in the effectiveness factor versus number of collocation points. If one is only interested in the pellet reaction rate, about 16 collocation points are required to achieve a relative error of less than  $10^{-6}$ . □

#### Example A.4: A simple fixed-bed reactor problem

Here we set up the collocation approach of this section for the single, second-order reaction



Solve the model exactly and compare the numerical solution to the solution obtained with the *approximate* Thiele modulus and effectiveness factor approaches shown in Figure 7.26.

#### Solution

The model for the fluid and particle phases can be written from Equations 7.84-7.97

$$\frac{dN_A}{dV} = R_A = -(1 - \epsilon_B) \frac{S_p}{V_p} D_A \left. \frac{d\tilde{c}_A}{dr} \right|_{r=R}$$

$$\frac{d^2\tilde{c}_A}{dr^2} + \frac{2}{r} \frac{d\tilde{c}_A}{dr} + \frac{\tilde{R}_A}{D_A} = 0$$

$$\tilde{c}_A = c_A \quad r = R$$

$$\frac{d\tilde{c}_A}{dr} = 0 \quad r = 0$$

The collocation method produces algebraic equations for the pellet profile as for the previous example. We use a DAE solver to integrate the differential equation for  $N_A$  and these coupled algebraic equations.

The results for  $N_A$  versus reactor length using 25 collocation points for the pellet are shown in Figure A.7. Also shown are the simplified effectiveness factor calculations for this problem from Example 7.5. A magnified view is shown in Figure A.8. Notice the effectiveness factor approach gives a good approximation for the bed performance. It is not exact because the reaction is second order. □



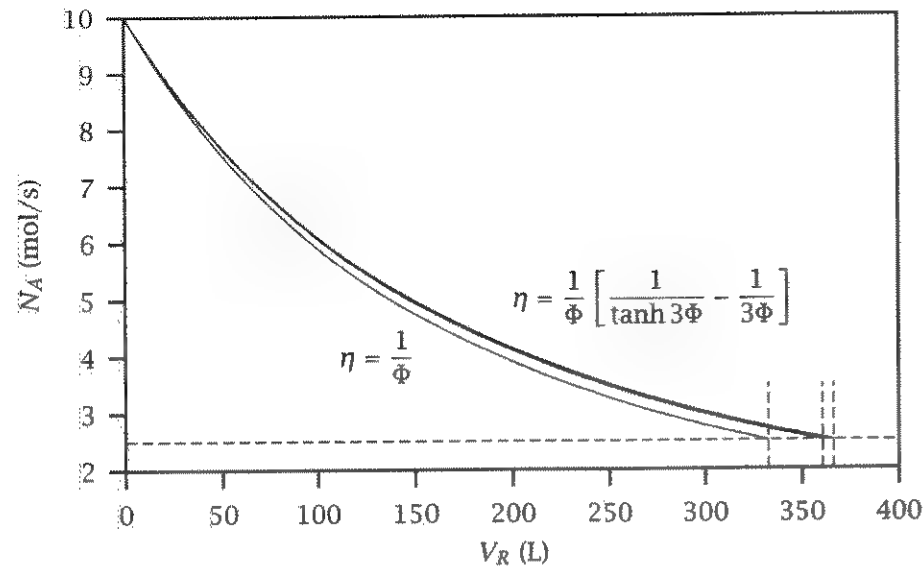


Figure A.7: Molar flow of A versus reactor volume for second-order, isothermal reaction in a fixed-bed reactor; two approximations and exact solution.

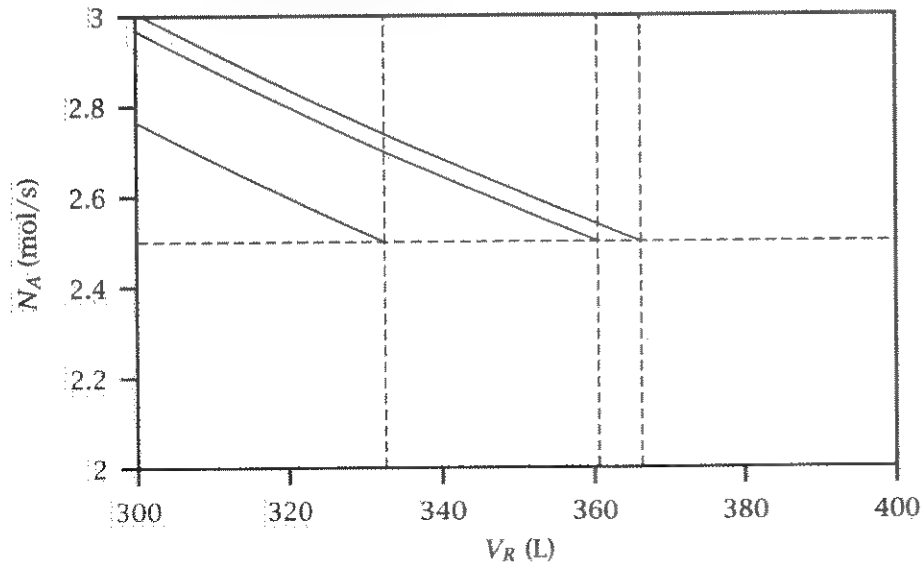


Figure A.8: Magnified view of Figure A.7.

## A.6 Exercises

### Exercise A.1: Water gas shift reaction and production rates

Consider the water gas shift reactions



Take the second and third reactions as an independent set and estimate the reaction rates from the same 2000 production-rate measurements used in Example A.1.

- What value of  $[r_2 \ r_3]$  produces the error-free production rate corresponding to  $[r_1 \ r_2] = [1 \ 2]$
- Create 2000 production-rate data points with measurement noise as in Example A.1.
- Estimate  $[r_2 \ r_3]$  from these data and plot the result. Compare to Figure A.1. Explain the differences and similarities in the two figures.

### Exercise A.2: Semi-batch polymerization reactor

Consider again Example 4.3, in which monomer is converted to polymer in a semi-batch reactor.

- Solve the problem with `ddasrt` and compute the time  $t_1$  at which the reactor fills. Compare your answer with the result  $t_1 = 11.2$  min given in Chapter 4.
- Calculate the solution for both addition policies and compare your numerical result to the analytical result in Figures 4.15-4.18.

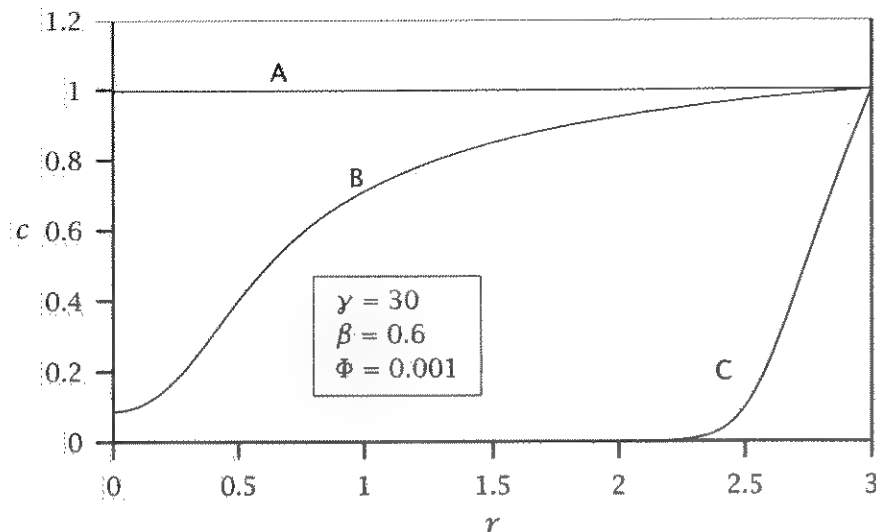
### Exercise A.3: Sensitivity for second-order kinetics

Repeat the analysis in Example A.2 for the differential equation describing a *second-order*, isothermal, irreversible reaction in a constant-volume batch reactor

$$\frac{dc_A}{dt} = -kc_A^2$$

$$c_A(0) = c_{A0}$$

Note: instead of solving the sensitivity differential equation, show only that the sensitivity computed by taking the partial derivative of the solution to the



**Figure A.9:** Dimensionless concentration versus radius for the non-isothermal spherical pellet: lower (A), unstable middle (B), and upper (C) steady states.

differential equation,  $c_A(t)$ , satisfies the sensitivity differential equation and initial condition.

#### Exercise A.4: Multiple solutions to the Weisz-Hicks problem

Using orthogonal collocation, compute the solution  $c(r)$  to the Weisz-Hicks problem, Equations 7.74–7.76, for the following parameter values:  $\beta = 0.6$ ,  $\gamma = 30$ ,  $\Phi = 0.001$ . Compare your result to the calculation presented in Figure A.9.

#### Exercise A.5: Weisz-Hicks problem, increasing $\beta$

Using orthogonal collocation, re-solve the Weisz-Hicks problem for the following parameter values:  $\beta = 0.7$ ,  $\gamma = 30$ ,  $\Phi = 0.001$ . Use initial guesses for the  $c(r)$  profile based on the results in Figure A.9.

- How many solutions are you able to find? Given the results in Figure 7.19, how many solutions do you expect. Explain and justify your answer.
- Compare and contrast your  $c(r)$  solutions to those in Figure A.9.

#### Exercise A.6: Comparing shooting and collocation

Solve the isothermal, first-order reaction-diffusion problem with an ODE solver, an algebraic equation solver, and the shooting method.

$$\frac{1}{r^q} \frac{d}{dr} \left( r^q \frac{dc}{dr} \right) - \frac{2}{n+1} \Phi^2 c^n = 0$$

$$c = 1, \quad r = q + 1$$

$$\frac{dc}{dr} = 0, \quad r = 0$$

- Solve the problem with the shooting method for the first-order reaction in a spherical geometry,  $n = 1$ ,  $q = 2$ , using  $\Phi = 0.1, 1, 10, 50$ .
- Compare  $c(r)$  from the shooting method with the analytical solution.
- Solve the problem using the collocation method. Which method do you prefer and why?

#### Exercise A.7: Transient solution of the dispersed PFR

Revisit the transient PFR example of Chapter 8, Example 8.1. Solve the problem for a range of dispersion numbers. Can you get a CSTR-like solution for large values of the dispersion number? Can you get a PFR-like solution for small values of the dispersion number? Do you have numerical accuracy problems at either of these limits?

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